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A one-phase Stefan problem for a non-classical heat equation with a heat flux condition on the fixed face ☆

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Abstract

We prove the existence and uniqueness, local in time, of the solution of a one-phase Stefan problem for a non-classical heat equation for a semi-infinite material with a heat flux boundary condition at the fixed face $x = 0$. Here the heat source depends on the temperature at the fixed face $x = 0$. We use the Friedman–Rubinstein integral representation method and the Banach contraction theorem in order to solve an equivalent system of two Volterra integral equations.

Keywords: Stefan problem; Non-classical heat equation; Free boundary problem; Similarity solution; Nonlinear heat sources; Volterra integral equation

1. Introduction

The one-phase Stefan problem for a semi-infinite material for the classical heat equation requires the determination of the temperature distribution u of the liquid phase (melting problem) or of the solid phase (solidification problem), and the evolution of the free boundary $x = s(t)$. Phase-change problems appear frequently in industrial processes and other problems of technological interest [1–9]. A large bibliography on the subject was given in [10].

Non-classical heat conduction problem for a semi-infinite material was studied in [11–15], e.g. problems of the type

$$\begin{aligned}
&u_t - u_{xx} = -F(u_x(0,t)), \quad x > 0, \ t > 0, \\
&u(0,t) = 0, \quad t > 0, \\
&u(x,0) = h(x), \quad x > 0,
\end{aligned} \tag{1}$$

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where $h(x)$, $x > 0$, and $F(V)$, $V \in \mathbb{R}$, are continuous functions. In this case, the heat source depends on the heat flux at the boundary $x = 0$. The function F, henceforth referred as control function, is assumed to fulfill the following condition:

$$F(0) = 0.$$

As observed in [14,15] the heat flux $w(x,t) = u_x(x,t)$ for problem (1) satisfies a classical heat conduction problem with a nonlinear convective condition at $x = 0$, which can be written in the form

$$\begin{cases} w_t - w_{xx} = 0, & x > 0, \quad t > 0, \\ w_x(0,t) = F(w(0,t)), & t > 0, \\ w(x,0) = h'(x) \geqslant 0, & x > 0. \end{cases} \tag{2}$$

The literature concerning problem (2) has increased rapidly since the publication of the papers [16–18]. In [19] a one-phase Stefan problem for a non-classical heat equation for a semi-infinite material with a source term which depends on the heat flux at $x = 0$ was presented. In [20] an existence and uniqueness result, local in time, was obtained.

Now, the free boundary problem which we want to consider consists in determining the temperature $u = u(x,t)$ and the free boundary $x = s(t)$ which satisfy the following conditions:

$$\begin{cases} \text{(i)}\ u_t - u_{xx} = -F(u(0,t)), & 0 < x < s(t), \quad 0 < t < T, \\ \text{(ii)}\ u_x(0,t) = -g(t) \leqslant 0, & 0 < t < T, \\ \text{(iii)}\ u(s(t),t) = 0, \quad \text{(iv)}\ u_x(s(t),t) = -\dot{s}(t), & 0 < t < T, \\ \text{(v)}\ u(x,0) = h(x), & 0 \leqslant x \leqslant b = s(0) \ (b > 0). \end{cases} \tag{3}$$

Here, the control function F depends on the evolution of the temperature at the extremum $x = 0$ with a given heat flux. The goal in this paper is to prove in Section 2 the existence and uniqueness local in time of the solution to the one-phase Stefan problem (3) of a non-classical heat equation for a semi-infinite material with a heat flux boundary condition at the fixed face $x = 0$. First, we prove that problem (3) is equivalent to a system of two Volterra integral equations (7) and (8) [21,22] following the Friedman–Rubinstein's method given in [23,24]. Then, we prove that the system (7) and (8) has a unique local solution by using the Banach contraction theorem.

2. Existence and uniqueness of solutions

Let be $g \in C^0[0,T)$, $h \in C^1[0,b]$, $h(0) = b$, $g(0) = -h'(0)$, F is a Lipschitz function over $C^0[0,T]$.

We have the following equivalence for the existence of solutions to the non-classical free boundary problem (3).

Theorem 1. *The solution to the free boundary problem* (3) *is given by*

$$\begin{aligned} u(x,t) = {} & \int_0^b N(x,t;\xi,0)h(\xi)\,\mathrm{d}\xi + \int_0^t N(x,t;0,\tau)g(\tau)\,\mathrm{d}\tau \\ & + \int_0^t N(x,t;s(\tau),\tau)w(\tau)\,\mathrm{d}\tau - \iint_{D(t)} N(x,t;\xi,\tau)F(W(\tau))\,\mathrm{d}\xi\,\mathrm{d}\tau, \end{aligned} \tag{4}$$

$$s(t) = b - \int_0^t w(\tau)\,\mathrm{d}\tau, \tag{5}$$

where $D(t) = \{(x,\tau)/0 < x < s(\tau), 0 < \tau < t\}$, *and the functions* w, W *defined by*

$$w(t) = u_x(s(t),t), \quad W(t) = u(0,t) \tag{6}$$

must satisfy the following system of two Volterra integral equations:

$$w(t) = 2\int_0^b h'(\xi)G(s(t),t,\xi,0)\,d\xi + 2\int_0^t g(t)N_x(s(t),t,0,\tau)\,d\tau + 2\int_0^t w(\tau)N_x(s(t),t,s(\tau),\tau)\,d\tau + 2\int_0^t G(s(t),t,s(\tau),\tau)F(W(\tau))\,d\tau, \quad (7)$$

$$W(t) = \int_0^b h(\xi)N(0,t,\xi,0)\,d\xi + \int_0^t g(t)N(0,t,0,\tau)\,d\tau + \int_0^t w(\tau)N(0,t,s(\tau),\tau)\,d\tau - \iint_{D(t)} N(0,t,\xi,\tau)F(W(\tau))\,d\tau\,d\xi, \quad (8)$$

where G, N are the Green and Neumann functions and K is the fundamental solution of the heat equation, defined respectively by

$$G(x,t,\xi,\tau) = K(x,t,\xi,\tau) - K(-x,t,\xi,\tau), \quad (9)$$

$$N(x,t,\xi,\tau) = K(x,t,\xi,\tau) + K(-x,t,\xi,\tau), \quad (10)$$

$$K(x,t,\xi,\tau) = \begin{cases} \frac{1}{2\sqrt{\pi(t-\tau)}}\exp\left(-\frac{(x-\xi)^2}{4(t-\tau)}\right) & t > \tau, \\ 0 & t \leqslant \tau, \end{cases} \quad (11)$$

where $s(t)$ *is given by* (5).

Proof. Let $u(x,t)$ be the solution to the problem (3) and we integrate on the domain $D_{t,\varepsilon} = \{(\xi,\tau)/0 < \xi < s(\tau), \varepsilon < \tau < t - \varepsilon\}$, the Green identity

$$(Nu_\xi - uN_\xi)_\xi - (Nu)_\tau = NF(u(0,\tau)). \quad (12)$$

Now we let $\varepsilon \to 0$, to obtain the integral representation for $u(x,t)$

$$u(x,t) = \int_0^b N(x,t;\xi,0)h(\xi)\,d\xi + \int_0^t N(x,t;0,\tau)g(\tau)\,d\tau + \int_0^t N(x,t;s(\tau),\tau)u_\xi(s(\tau),\tau)\,d\tau - \iint_{D(t)} N(x,t;\xi,\tau)F(u(0,\tau))\,d\xi\,d\tau.$$

From the definition of $w(t)$ and $W(t)$ by (6), we obtain (4) and (5). If we differentiate in variable x and we let $x \to 0^+$ and $x \to s(t)$, by using the jump relations we obtain the integral equations for w and W.

Conversely the function $u(x,t)$ defined by (4) where w and W are the solutions of (7) and (8) satisfy the conditions (3) (i), (ii), (iv) and (v). In order to prove condition (3) (iii), we define $\psi(t) = u(s(t),t)$. Taking into account that u satisfy the conditions (3) (i), (ii), (iv) and (v), if we integrate the Green identity (12) over the domain $D_{t,\varepsilon}$ ($\varepsilon > 0$) and we let $\varepsilon \to 0$ we obtain that

$$u(x,t) = \int_0^b N(x,t;\xi,0)h(\xi)\,d\xi + \int_0^t N(x,t;s(\tau),\tau)w(\tau)\,d\tau + \int_0^t N(x,t;0,\tau)g(\tau)\,d\tau - \int_0^t \psi(\tau)[N_\xi(x,t;s(\tau),\tau) + N(x,t;s(\tau),\tau)w(\tau)]\,d\tau - \iint_{D(t)} N(x,t;\xi,\tau)F(W(\tau))\,d\xi\,d\tau. \quad (13)$$

Then, if we compare this last expression with (4) we deduce that

$$\int_0^t \psi(\tau)[N_\xi(x,t;s(\tau),\tau) + N(x,t;s(\tau),\tau)w(\tau)]\,d\tau \equiv 0 \quad (14)$$

for $0 < x < s(t)$, $0 < t < \sigma$. We let in (14) $x \to s(t)$ and by using the jump relations we have that ψ satisfy the integral equation

$$\frac{1}{2}\psi(t) + \int_0^t \psi(\tau)[N_\xi(s(t),t;s(\tau),\tau) + N(s(t),t;s(\tau),\tau)w(\tau)]\,d\tau = 0.$$

Then we deduce that

$$|\psi(t)| \leqslant C\int_0^t \frac{|\psi(\tau)|}{\sqrt{t-\tau}}\,d\tau \leqslant C^2\int_0^t \frac{d\tau}{\sqrt{t-\tau}}\int_0^\tau \frac{|\psi(\eta)|}{\sqrt{\tau-\eta}}\,d\eta$$
$$= C^2\int_0^t |\psi(\eta)\,d\eta\int_\eta^t \frac{d\tau}{[(t-\tau)(\tau-\eta)]^{\frac{1}{2}}} = \pi C^2\int_0^t |\psi(\eta)|\,d\eta,$$

where $C = C(t)$, therefore by using the Gronwall inequality we have that $\psi(t) = 0$ over $[0,\sigma]$. □

Next, we use the Banach fixed point theorem in order to prove the local existence and uniqueness of solution w, $W \in C^0[0,\sigma]$ to the system of two Volterra integral equations (7) and (8) where σ is a positive small number ($\sigma \leqslant T$). Consider the Banach Space:

$$C_{R,\sigma} = \left\{\vec{w^*} = \begin{pmatrix} w \\ W \end{pmatrix} \Big/ w, W : [0,\sigma] \to \mathbb{R}, \text{continuous, with} \left\|\vec{w^*}\right\|_\sigma \leqslant R\right\},$$

where

$$\left\|\vec{w^*}\right\|_\sigma := \max_{t\in[0,\sigma]}|w(t)| + \max_{t\in[0,\sigma]}|W(t)|.$$

We define the map $B : C_{R,\sigma} \to C_{R,\sigma}$, such that

$$\widetilde{\vec{w^*}}(t) = B\left(\vec{w^*}(t)\right) = \begin{pmatrix} B_1(w(t),W(t)) \\ B_2(w(t),W(t)) \end{pmatrix},$$

where

$$B_1(w(t),W(t)) = 2\int_0^b h'(\xi)G(s(t),t,\xi,0)\,d\xi + 2\int_0^t g(\tau)N_x(s(t),t,0,\tau)\,d\tau$$
$$+ 2\int_0^t w(\tau)N_x(s(t),t,s(\tau),\tau)\,d\tau + 2\int_0^t G(s(t),t,s(\tau),\tau)F(W(\tau))\,d\tau \tag{15}$$

and

$$B_2(w(t),W(t)) = \int_0^b h(\xi)N(0,t,\xi,0)\,d\xi + \int_0^t g(\tau)N(0,t,0,\tau)\,d + \int_0^t w(\tau)N(0,t,s(\tau),\tau)\,d\tau$$
$$- \iint_{D(t)} N(0,t,\xi,\tau)F(W(\tau))\,d\tau\,d\xi. \tag{16}$$

Lemma 2. *Let* $w \in C^0[0,\sigma]$, $\max_{t\in[0,\sigma]}|w(t)| \leqslant R$ *and* $2R\sigma \leqslant b$ *then* $s(t)$ *defined by* (5) *satisfies*

$$|s(t) - s(\tau)| \leqslant R|t-\tau|, \quad \forall\tau,\ t \in [0,\sigma] \tag{17}$$

$$|s(t) - b| \leqslant \frac{b}{2}, \quad \forall t \in [0,\sigma]. \tag{18}$$

To prove the following lemmas we need the classical inequality:

$$\frac{\exp\left(\frac{-x^2}{\alpha(t-\tau)}\right)}{(t-\tau)^{\frac{n}{2}}} \leqslant \left(\frac{n\alpha}{2ex^2}\right)^{\frac{n}{2}}, \quad \alpha, x > 0,\ \ t > \tau,\ \ n \in \mathbb{N}. \tag{19}$$

Lemma 3. *Let* $\sigma \leqslant 1$, $R \geqslant 1$, $g \in C^0[0,T]$, $h \in C^1[0,b]$, $h(0) = b$, $g(0) = -h'(0)$, *F a Lipschitz function over* $C^0[0,T]$. *Under the hypothesis of Lemma* 2 *we have the following properties*:

$$\int_0^b |h'(\xi)||G(s(t),t,\xi,0)|\,\mathrm{d}\xi \leqslant \|h'\|, \tag{20}$$

$$\int_0^t |g(\tau)||N_x(s(t),t,0,\tau)|\,\mathrm{d}\tau \leqslant \|g\|_t\alpha_1(b)t, \tag{21}$$

$$\int_0^t |w(\tau)||N_x(s(t),t,s(\tau),\tau)|\,\mathrm{d}\tau \leqslant R\alpha_2(b)t + R^2\alpha_3(b)\sqrt{t}, \tag{22}$$

$$\int_0^t |G(s(t),t,s(\tau),\tau)||F(W(\tau))|\,\mathrm{d}\tau \leqslant \frac{2LR\sqrt{t}}{\sqrt{\pi}}, \tag{23}$$

$$\int_0^b |h(\xi)||N(0,t,\xi,0)|\,\mathrm{d}\xi \leqslant \|h\|, \tag{24}$$

$$\int_0^t |g(\tau)||N(0,t,0,\tau)|\,\mathrm{d}\tau \leqslant \frac{\|g\|_t 2\sqrt{t}}{\sqrt{\pi}}, \tag{25}$$

$$\int_0^t |w(\tau)||N(0,t,s(\tau),\tau)|\,\mathrm{d}\tau \leqslant \frac{2R\sqrt{t}}{\sqrt{\pi}}, \tag{26}$$

$$\int\int_{D(t)} |N(0,t,\xi,\tau)||F(W(\tau))|\,\mathrm{d}\xi\,\mathrm{d}\tau \leqslant \frac{3bLR\sqrt{t}}{\sqrt{\pi}}, \tag{27}$$

where L is the Lipschitz constant for F and

$$\alpha_1(b) = \frac{3b}{4\sqrt{\pi}}\left(\frac{24}{eb^2}\right)^{\frac{3}{2}}, \quad \alpha_2(b) = \frac{3b}{2\sqrt{\pi}}\left(\frac{6}{eb^2}\right)^{\frac{3}{2}},$$
$$\alpha_3(b,R) = \frac{1}{2\sqrt{\pi}} + \frac{3b}{4R\sqrt{\pi}}\left(\frac{2}{3eb^2}\right)^{\frac{3}{2}}.$$

Proof. We have

$$\int_0^b |G(s(t),t,\xi,0)||h'(\xi)|\,\mathrm{d}\xi \leqslant \|h'\|\int_0^\infty |G(s(t),t,\xi,0)|\,\mathrm{d}\xi \leqslant \|h'\|$$

because

$$\int_0^\infty |G(s(t),t,\xi,0)|\,\mathrm{d}\xi \leqslant \int_0^\infty |N(s(t),t,\xi,0)|\,\mathrm{d}\xi \leqslant 1,$$

then (20) holds. To prove (21) we have

$$|N_x(s(t),t,0,\tau)| = |K_x(s(t),t,0,\tau) - K_x(-s(t),t,0,\tau)| \leqslant \frac{|s(t)|\exp\left(\frac{-(s(t))^2}{4(t-\tau)}\right)}{2\sqrt{\pi}(t-\tau)^{\frac{3}{2}}}$$
$$\leqslant \frac{|s(t)|\exp\left(\frac{-b^2}{16(t-\tau)}\right)}{2\sqrt{\pi}(t-\tau)^{\frac{3}{2}}} \leqslant \frac{3b}{4\sqrt{\pi}}\left(\frac{24}{eb^2}\right)^{\frac{3}{2}} = \alpha_1(b).$$

Then

$$\int_0^t |g(\tau)||N_x(s(t),t,0,\tau)|\,\mathrm{d}\tau \leqslant \|g\|_t\alpha_1(b)t,$$

which implies (21). To prove (22) we have

$$|N_x(s(t),t,s(\tau),\tau)| = |-2K_x(-s(t),t,s(\tau),\tau) + G_x(s(t),t,s(\tau),\tau)|,$$

$$|-2K_x(-s(t),t,s(\tau),\tau)| = \frac{|s(t)+s(\tau)|}{\sqrt{\pi}(t-\tau)^{\frac{3}{2}}}\exp\left(\frac{-(s(t)+s(\tau))^2}{4(t-\tau)}\right)$$

$$\leqslant \frac{3b\exp\left(\frac{-b^2}{16(t-\tau)}\right)}{4\sqrt{\pi}(t-\tau)^{\frac{3}{2}}} \leqslant \frac{3b}{2\sqrt{\pi}}\left(\frac{6}{eb^2}\right)^{\frac{3}{2}} = \alpha_2(b)$$

and

$$|G_x(s(t),t,s(\tau),\tau)| = |K_x(s(t),t,s(\tau),\tau) + K_x(-s(t),t,s(\tau),\tau)|$$

$$= \frac{(t-\tau)^{-\frac{3}{2}}}{4\sqrt{\pi}}\left|(s(t)-s(\tau))\exp\left(\frac{-(s(t)-s(\tau))^2}{4(t-\tau)}\right) - (s(t)+s(\tau))\exp\left(\frac{-(s(t)+s(\tau))^2}{4(t-\tau)}\right)\right|$$

$$\leqslant \frac{(t-\tau)^{-\frac{3}{2}}}{4\sqrt{\pi}}\left(R(t-\tau) + 3b\exp\left(\frac{-9b^2}{4(t-\tau)}\right)\right) \leqslant \frac{1}{4\sqrt{\pi}}\left(R(t-\tau)^{-\frac{1}{2}} + 3b\left(\frac{2}{3eb^2}\right)^{\frac{3}{2}}\right).$$

Then

$$\int_0^t |w(\tau)||N_x(s(t),t,s(\tau),\tau)|\,d\tau \leqslant \int_0^t |w(\tau)||-2K_x(-s(t),t,s(\tau),\tau) + G_x(s(t),t,s(\tau),\tau)|\,d\tau$$

$$\leqslant R\alpha_2(b)t + R^2\alpha_3(b,R)\sqrt{t}.$$

To prove (23), by taking into account that

$$|G(s(t),t,s(\tau),\tau)| \leqslant \frac{1}{\sqrt{\pi(t-\tau)}}$$

so, we obtain

$$\int_0^t |G(s(t),t,s(\tau),\tau)||F(V(\tau))|\,d\tau \leqslant LR\int_0^t \frac{1}{\sqrt{\pi(t-\tau)}}\,d\tau = \frac{2LR\sqrt{t}}{\sqrt{\pi}}.$$

The inequality (24) is prove in the same way as (20). To prove (25), we have

$$\int_0^t |N(0,t,0,\tau)||g(\tau)|\,d\tau \leqslant \|g\|_t\int_0^t |N(0,t,0,\tau)|\,d\tau = \|g\|_t\int_0^t \frac{1}{\sqrt{\pi(t-\tau)}}\,d\tau = \frac{\|g\|_t}{\sqrt{\pi}}2\sqrt{t}.$$

Eq. (26) holds because

$$|N(0,t,s(\tau),\tau)| \leqslant \frac{1}{\sqrt{\pi(t-\tau)}},$$

$$\int_0^t |w(\tau)||N(0,t,s(\tau),\tau)|\,d\tau \leqslant \frac{2R\sqrt{t}}{\sqrt{\pi}}.$$

To prove (27) we have

$$|N(0,t,\xi,\tau)||F(W(\tau))| \leqslant \frac{L}{\sqrt{\pi(t-\tau)}}\|W\|,$$

then

$$\int\int_{D(t)} |N(0,t,\xi,\tau)||F(W(\tau))|\,d\xi\,d\tau = \int_0^t \left|\int_0^{s(\tau)} |N(0,t,\xi,\tau)||F(W(\tau))|\,d\xi\right| d\tau$$

$$\leqslant LR\int_0^t \frac{|s(\tau)|}{\sqrt{\pi(t-\tau)}}\,d\tau \leqslant \frac{3bLR\sqrt{t}}{\sqrt{\pi}}$$

and therefore the thesis holds. □

Lemma 4. *Let s_1 and s_2 be the functions corresponding to w_1 and w_2 in $C^0[0,\sigma]$, respectively with $\max_{t\in[0,\sigma]}|w_i(t)| \leqslant R$, $i = 1,2$. Then we have*

$$\begin{cases} |s_2(t)-s_1(t)| \leqslant t\|w_2-w_1\|_t, \\ |s_i(t)-s_i(\tau)| \leqslant R|t-\tau|, \quad i=1,2, \\ \frac{b}{2} \leqslant s_i(t) \leqslant \frac{3b}{2}, \quad \forall t\in[0,\sigma], \ \ i=1,2. \end{cases} \tag{28}$$

Lemma 5. *Let be $g \in C^0[0,T]$, $h \in C^1[0,b]$, F a Lipschitz function over $C^0[0,T]$. We have*

$$\int_0^t |w_1(\tau)N(0,t,s_1(\tau),\tau) - w_2(\tau)N(0,t,s_2(\tau),\tau)|\,\mathrm{d}\tau \tag{29}$$
$$\leqslant \|w_1-w_2\|_t\left[\left(\frac{8}{eb^2}\right)^{\frac{1}{2}}\frac{t}{\sqrt{\pi}} + \frac{3bR}{8\sqrt{\pi}}\left(\frac{24}{eb^2}\right)^{\frac{3}{2}}t^2\right];$$

$$\left|\int\int_{D_1(t)} N(0,t,\xi,\tau)F(W_1(\tau))\,\mathrm{d}\xi\,\mathrm{d}\tau - \int\int_{D_2(t)} N(0,t,\xi,\tau)F(W_2(\tau))\,\mathrm{d}\xi\,\mathrm{d}\tau\right|$$
$$\leqslant \frac{LRt^{\frac{3}{2}}}{\sqrt{\pi}}\|w_1-w_2\|_t + \frac{3bL\sqrt{t}}{2\sqrt{\pi}}\|W_1-W_2\|_t, \tag{30}$$

where $D_i(t) = \{(\xi,\tau)/0 < \xi < s_i(\tau),\ 0 < \tau < t\}$, $i = 1,2$;

$$\int_0^b |h'(\xi)||G(s_1(t),t,\xi,0) - G(s_2(t),t,\xi,0)|\,\mathrm{d}\xi \leqslant \frac{2\|h'\|\sqrt{t}}{\sqrt{\pi}}\|w_1-w_2\|_t, \tag{31}$$

$$\int_0^t |g(\tau)||N_x(s_1(t),t,0,\tau) - N_x(s_2(t),t,0,\tau)|\,\mathrm{d}\tau \leqslant \|g\|_t\left[\left(\frac{24}{eb^2}\right)^{\frac{3}{2}}\frac{1}{2\sqrt{\pi}} + \left(\frac{40}{eb^2}\right)^{\frac{5}{2}}\frac{9}{4}b^2\frac{1}{4\sqrt{\pi}}\right]t\|w_1-w_2\|_t, \tag{32}$$

$$\int_0^t |G(s_1(t),t,s_1(\tau),\tau)F(W_1(\tau)) - G(s_2(t),t,s_2(\tau),\tau)F(W_2(\tau))|\,\mathrm{d}\tau$$
$$\leqslant \frac{2L\sqrt{t}}{\sqrt{\pi}}\|W_1-W_2\|_t + \frac{R^3L\sqrt{t}}{\sqrt{\pi}}\|w_1-w_2\|_t + \left(\frac{6}{e}\right)^{\frac{3}{2}}\frac{R^2t}{b^2\sqrt{\pi}}\|w_1-w_2\|_t \tag{33}$$

and

$$\int_0^t |w_1(\tau)N_x(s_1(t),t,s_1(\tau),\tau) - w_2(\tau)N_x(s_2(t),t,s_2(\tau),\tau)|\,\mathrm{d}\tau$$
$$\leqslant \left\{\frac{R\sqrt{t}}{2\sqrt{\pi}} + \left(\frac{6}{eb^2}\right)^{\frac{3}{2}}\frac{3bt}{2\sqrt{\pi}} + \frac{R}{2\sqrt{\pi}}\left[\left(\frac{6}{eb^2}\right)^{\frac{3}{2}} + \frac{9}{2}b^2\left(\frac{10}{eb^2}\right)^{\frac{5}{2}}\right]t + \frac{R(1+R^2t)\sqrt{t}}{2\sqrt{\pi}}\right\}\|w_1-w_2\|_t. \tag{34}$$

Proof. To prove (29) we have

$$|w_1(\tau)N(0,t,s_1(\tau),\tau) - w_2(\tau)N(0,t,s_2(\tau),\tau)|$$
$$\leqslant |w_1(\tau)-w_2(\tau)||N(0,t,s_1(\tau),\tau)| + |w_2(\tau)||N(0,t,s_1(\tau),\tau) - N(0,t,s_2(\tau),\tau)|.$$

Taking into account that

$$|N(0,t,s_1(\tau),\tau)| \leqslant \frac{\exp\left(\frac{-b^2}{16(t-\tau)}\right)}{\sqrt{\pi}(t-\tau)^{\frac{1}{2}}} \leqslant \left(\frac{8}{eb^2}\right)^{\frac{1}{4}}\frac{1}{\sqrt{\pi}}$$

and

$$|N(0,t,s_1(\tau),\tau)-N(0,t,s_2(\tau),\tau)|\leqslant\frac{3b}{4\sqrt{\pi}}\left(\frac{24}{eb^2}\right)^{\frac{3}{2}}t\|w_1-w_2\|_t$$

then

$$\int_0^t|w_1(\tau)N(0,t,s_1(\tau),\tau)-w_2(\tau)N(0,t,s_2(\tau),\tau)|\,\mathrm{d}\tau\leqslant\|w_1-w_2\|_t\left[\left(\frac{8}{eb^2}\right)^{\frac{1}{2}}\frac{t}{\sqrt{\pi}}+\frac{3bR}{8\sqrt{\pi}}\left(\frac{24}{eb^2}\right)^{\frac{3}{2}}t^2\right].$$

To prove (30) we have

$$\begin{aligned}&\int\int_{D_1(t)}N(0,t,\xi,\tau)F(W_1(\tau))\,\mathrm{d}\xi\,\mathrm{d}\tau-\int\int_{D_2(t)}N(0,t,\xi,\tau)F(W_2(\tau))\,\mathrm{d}\xi\,\mathrm{d}\tau\\&\quad=\int\int_{D_1(t)}N(0,t,\xi,\tau)(F(W_1(\tau))-F(W_2(\tau)))\,\mathrm{d}\xi\,\mathrm{d}\tau+\int\int_{D_1(t)}N(0,t,\xi,\tau)F(W_2(\tau))\,\mathrm{d}\xi\,\mathrm{d}\tau\\&\qquad-\int\int_{D_2(t)}N(0,t,\xi,\tau)F(W_2(\tau))\,\mathrm{d}\xi\,\mathrm{d}\tau\end{aligned}$$

Because

$$\begin{aligned}\left|\int\int_{D_1(t)}N(0,t,\xi,\tau)(F(W_1(\tau))-F(W_2(\tau)))\,\mathrm{d}\xi\,\mathrm{d}\tau\right|&\leqslant\int_0^t\int_0^{s_1(\tau)}|N(0,t,\xi,\tau)|L\|W_1-W_2\|_t\,\mathrm{d}\xi\,\mathrm{d}\tau\\&\leqslant\frac{1}{\sqrt{\pi}}\sqrt{t}L|s_1(t)|\|W_1-W_2\|_t\leqslant\frac{3b}{2\sqrt{\pi}}L\sqrt{t}\|W_1-W_2\|_t\end{aligned}$$

and

$$\begin{aligned}&\left|\int\int_{D_1(t)}N(0,t,\xi,\tau)F(W_2(\tau))\,\mathrm{d}\xi\,\mathrm{d}\tau-\int\int_{D_2(t)}N(0,t,\xi,\tau)F(W_2(\tau))\,\mathrm{d}\xi\,\mathrm{d}\tau\right|\\&\quad\leqslant\int_0^t|F(W_2(\tau))|\left|\int_{s_1(\tau)}^{s_2(\tau)}N(0,t,\xi,\tau)\,\mathrm{d}\xi\right|\mathrm{d}\tau\leqslant\frac{LR}{\sqrt{\pi}}t^{\frac{3}{2}}\|w_1-w_2\|_t\end{aligned}$$

then (30) holds.

To prove (31) we have

$$\begin{aligned}&|G(s_1(t),t,\xi,0)-G(s_2(t),t,\xi,0)|\\&\quad\leqslant|K(s_1(t),t,\xi,0)-K(s_2(t),t,\xi,0)|+|K(-s_1(t),t,\xi,0)-K(-s_2(t),t,\xi,0)|\end{aligned}$$

and by the mean value theorem there exists $d=d(t)$ between $s_1(t)$ and $s_2(t)$ such that

$$|K(s_1(t),t,\xi,0)-K(s_2(t),t,\xi,0)|=|s_1(t)-s_2(t)|K(d(t),t,\xi,0)\frac{|d(t)-\xi|}{2t}$$

then

$$\begin{aligned}\int_0^b|s_1(t)-s_2(t)|K(d(t),t,\xi,0)\frac{|d(t)-\xi|}{2t}\,\mathrm{d}\xi&\leqslant t\|w_1-w_2\|_t\int_0^b\frac{|d(t)-\xi|}{\exp\left(\frac{(d(t)-\xi)^2}{4t}\right)4\sqrt{\pi}(t-\tau)^{\frac{3}{2}}}\,\mathrm{d}\xi\\&=\sqrt{t}\frac{\|w_1-w_2\|_t}{2\sqrt{\pi}}\left(\exp\left(\frac{-(d(t)-b)^2}{4t}\right)-\exp\left(\frac{-d^2(t)}{4t}\right)\right)\\&\leqslant\frac{\sqrt{t}\|w_1-w_2\|_t}{\sqrt{\pi}}.\end{aligned}$$

In the same way we have

$$\int_0^b |K(-s_1(t),t,\xi,0)-K(-s_2(t),t,\xi,0)|\,d\xi \leqslant \frac{\sqrt{t}\|w_1-w_2\|_t}{\sqrt{\pi}}.$$

Then

$$\int_0^b |h'(\xi)||G(s_1(t),t,\xi,0)-G(s_2(t),t,\xi,0)|\,d\xi \leqslant 2\|h'\|\frac{\sqrt{t}\|w_1-w_2\|_t}{\sqrt{\pi}}.$$

To prove (32) we apply the mean value theorem and therefore there exists $c=c(t)$ between $s_1(t)$ and $s_2(t)$ such that

$$|N_x(s_1(t),t,0,\tau)-N_x(s_2(t),t,0,\tau)| = |s_1(t)-s_2(t)||N_{xx}(c(t),t,0,\tau)|$$

$$\begin{aligned}|N_{xx}(c(t),t,0,\tau)| &\leqslant \frac{\exp\left(\frac{-c^2(t)}{4(t-\tau)}\right)}{2\sqrt{\pi}(t-\tau)^{\frac{3}{2}}}+\frac{c^2\exp\left(\frac{-c^2(t)}{4(t-\tau)}\right)}{4\sqrt{\pi}(t-\tau)^{\frac{5}{2}}}\\ &\leqslant \frac{\exp\left(\frac{-b^2}{16(t-\tau)}\right)}{2\sqrt{\pi}(t-\tau)^{\frac{3}{2}}}+\frac{9b^2\exp\left(\frac{-b^2}{16(t-\tau)}\right)}{16\sqrt{\pi}(t-\tau)^{\frac{5}{2}}}\\ &\leqslant \left(\frac{24}{eb^2}\right)^{\frac{3}{2}}\frac{1}{2\sqrt{\pi}}+\left(\frac{40}{eb^2}\right)^{\frac{5}{2}}\frac{9}{16\sqrt{\pi}}b^2\end{aligned}$$

and (32) holds. To prove (33) we have

$$\begin{aligned}&|G(s_1(t),t,s_1(\tau),\tau)F(W_1(\tau))-G(s_2(t),t,s_2(\tau),\tau)F(W_2(\tau))|\\ &\quad\leqslant |G(s_1(t),t,s_1(\tau),\tau)||F(W_1(\tau))-F(W_2(\tau))|+|G(s_1(t),t,s_1(\tau),\tau)-G(s_2(t),t,s_2(\tau),\tau)||F(W_2(\tau))|.\end{aligned}$$

We obtain that

$$|G(s_1(t),t,s_1(\tau),\tau)||F(W_1(\tau))-F(W_2(\tau))| \leqslant \frac{L}{\sqrt{\pi(t-\tau)}}\|W_1-W_2\|_t$$

and, following [20] we have:

$$\begin{aligned}&|G(s_1(t),t,s_1(\tau),\tau)-G(s_2(t),t,s_2(\tau),\tau)||F(W_2(\tau))|\\ &\quad\leqslant |K(s_1(t),t,s_1(\tau),\tau)-K(s_2(t),t,s_2(\tau),\tau)||F(W_2(\tau))|+|K(-s_1(t),t,s_1(\tau),\tau)\\ &\qquad -K(-s_2(t),t,s_2(\tau),\tau)||F(W_2(\tau))| \leqslant \frac{R^3L}{\sqrt{\pi(t-\tau)}}\|w_1-w_2\|_t+\left(\frac{6}{e}\right)^{\frac{3}{2}}\frac{R^2L}{b^2\sqrt{\pi}}\|w_1-w_2\|_t.\end{aligned}$$

Then

$$\begin{aligned}&\int_0^t |G(s_1(t),t,s_1(\tau),\tau)F(W_1(\tau))-G(s_2(t),t,s_2(\tau),\tau)F(W_2(\tau))|\,d\tau\\ &\quad\leqslant \frac{2L\sqrt{t}}{\sqrt{\pi}}\|W_1-W_2\|_t+\frac{R^3L\sqrt{t}}{\sqrt{\pi}}\|w_1-w_2\|_t+\left(\frac{6}{e}\right)^{\frac{3}{2}}\frac{R^2t}{b^2\sqrt{\pi}}\|w_1-w_2\|_t.\end{aligned}$$

To finish the thesis, the result (34) can be found in [25]. □

Theorem 6. *The map $B: C_{R,\sigma}\to C_{R,\sigma}$ is well defined and it is a contraction map if σ satisfies the following inequalities*:

$$\sigma \leqslant 1, \quad 2R\sigma \leqslant b \tag{35}$$

$$(2\|g\|_\sigma\alpha_1(b)+2R\alpha_2(b))\sigma+\left(2R^2\alpha_3(b)+\frac{4LR}{\sqrt{\pi}}+\frac{2\|g\|_\sigma+R(2+3bL)}{\sqrt{\pi}}\right)\sqrt{\sigma}\leqslant 1, \tag{36}$$

$$H(\|h'\|,\|g\|_\sigma,b,L,R,\sigma)<1, \tag{37}$$

where R is given by

$$R = 1 + \|h\| + 2\|h'\| \tag{38}$$

and

$$\begin{aligned} &H(\|h'\|, \|g\|_\sigma, b, L, R, \sigma) \\ &\quad = \left\{ \frac{4\|h'\|\sqrt{\sigma}}{\sqrt{\pi}} + 2\|g\|_\sigma N_1(b)\sigma + \frac{4L\sqrt{\sigma}}{\sqrt{\pi}} + N_2(R,L)\sqrt{\sigma} + +N_3(R,b)\sigma + N_4(R)\sqrt{\sigma} \right. \\ &\qquad \left. + N_5(b)\sigma + N_6(R,b)\sigma + \frac{R(1+R^2\sigma)}{\sqrt{\pi}}\sqrt{\sigma} + N_7(b)\sigma + N_8(R,b)\sigma + N_9(b,L)\sqrt{\sigma} + N_{10}(R,L)\sigma^{\frac{3}{2}} \right\}, \end{aligned} \tag{39}$$

where N_1 to N_{10} are given by the expressions

$$N_1(b) = \left(\frac{24}{eb^2}\right)^{\frac{3}{2}} \frac{1}{2\sqrt{\pi}} + \left(\frac{40}{eb^2}\right)^{\frac{5}{2}} \frac{9}{4} b^2 \frac{1}{4\sqrt{\pi}}, \quad N_2(R,L) = \frac{2R^3L}{\sqrt{\pi}},$$

$$N_3(R,b) = \frac{2R^2}{b^2\sqrt{\pi}}\left(\frac{6}{e}\right)^{\frac{3}{2}}, \quad N_4(R) = \frac{R}{\sqrt{\pi}}, \quad N_5(R,b) = \frac{3b}{\sqrt{\pi}}\left(\frac{6}{eb^2}\right)^{\frac{3}{2}},$$

$$N_6(R,b) = \frac{R}{\sqrt{\pi}}\left[\left(\frac{6}{eb^2}\right)^{\frac{3}{2}} + \frac{9}{2}b^2\left(\frac{10}{eb^2}\right)^{\frac{5}{2}}\right], \quad N_7(b) = \left(\frac{8}{eb^2}\right)^{\frac{1}{2}} \frac{1}{\sqrt{\pi}},$$

$$N_8(R,b) = \frac{3Rb}{8\sqrt{\pi}}\left(\frac{24}{eb^2}\right)^{\frac{3}{2}}, \quad N_9(b,L) = \frac{3bL}{2\sqrt{\pi}}, \quad N_{10}(R,L) = \frac{LR}{\sqrt{\pi}}.$$

Then there exists a unique solution on $C_{R,\sigma}$ to the system of integral equations (7) and (8).

Proof. Firstly we demonstrate that B maps $C_{R,\sigma}$ into itself, that is

$$\left\| B\left(\vec{w}^*\right) \right\|_\sigma = \max_{t\in[0,\sigma]} |B_1(w(t), W(t))| + \max_{t\in[0,\sigma]} |B_2(w(t), W(t))| \leqslant R.$$

Using Lemma 3 it results

$$|B_1(w(t), W(t))| \leqslant 2\|h'\| + \left(2R^2\alpha_3(b,R) + \frac{4RL}{\sqrt{\pi}}\right)\sqrt{t} + 2(\|g\|_t\alpha_1(b) + R\alpha_2(b))t,$$

$$|B_2(w(t), W(t))| \leqslant \|h\| + \left(\frac{\|g\|_t + 2R + 3bLR}{\sqrt{\pi}}\right)\sqrt{t}$$

and then

$$\left\| B\left(\vec{w}^*\right) \right\|_\sigma \leqslant 2\|h'\| + \|h\| + 2(\|g\|_\sigma\alpha_1(b) + R\alpha_2(b))\sigma + \left(2R^2\alpha_3(b,R) + \frac{4RL}{\sqrt{\pi}} + \frac{2\|g\|_\sigma + 2R + 3bLR}{\sqrt{\pi}}\right)\sqrt{\sigma}.$$

Selecting R by (38) and σ such that (35) and (36) hold, we obtain $\left\| B\left(\vec{w}^*\right) \right\|_\sigma \leqslant R$. Now, we will prove that

$$\left\| B\left(\vec{w}_2^*\right) - B\left(\vec{w}_1^*\right) \right\|_\sigma \leqslant H(\|h'\|, \|g\|_\sigma, b, L, R, \sigma) \left\| \vec{w}_2^* - \vec{w}_1^* \right\|_\sigma,$$

where $\vec{w}_1^* = \begin{pmatrix} w_1 \\ W_1 \end{pmatrix}$, $\vec{w}_2^* = \begin{pmatrix} w_2 \\ W_2 \end{pmatrix} \in C_{R,\sigma}$. By selecting σ such that (37) holds, B becomes a contracting mapping on $C_{R,\sigma}$ and therefore it has a unique fixed point. Taking into account Lemma 5 we have

$$\begin{aligned} &\left\| B\left(\vec{w}_2^*\right) - B\left(\vec{w}_1^*\right) \right\|_\sigma \\ &\quad = \max_{t\in[0,\sigma]} |B_1(w_2(t), W_2(t)) - B_1(w_1(t), W_1(t))| + \max_{t\in[0,\sigma]} |B_2(w_2(t), W_2(t)) - B_2(w_1(t), W_1(t))| \end{aligned}$$

$$\leqslant \left\{ \frac{4\|h'\|\sqrt{\sigma}}{\sqrt{\pi}} + 2\|g\|_{\sigma}N_1(b)\sigma + \frac{4L\sqrt{\sigma}}{\sqrt{\pi}} + N_2(R,L)\sqrt{\sigma} + N_3(R,b)\sigma + N_4(R)\sqrt{\sigma} + N_5(b)\sigma + N_6(R,b)\sigma \right.$$
$$\left. + \frac{R(1+R^2\sigma)}{\sqrt{\pi}}\sqrt{\sigma} + N_7(b)\sigma + N_8(R,b)\sigma + N_9(b,L)\sqrt{\sigma} + N_{10}(R,L)\sigma^{\frac{3}{2}} \right\} \left\| \vec{w}_2^* - \vec{w}_1^* \right\|_{\sigma}$$
$$= H(\|h'\|, \|g\|_{\sigma}, b, L, R, \sigma) \left\| \vec{w}_2^* - \vec{w}_1^* \right\|_{\sigma}.$$

By hypothesis (37) we have that B is a contraction. □

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