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LETTER TO THE EDITOR

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On the paper D. Burini, S De Lillo, G. Fioriti, Acta Mech., 229 No. 10 (2018), pp 4215–4228

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In the paper [1], a free boundary problem on a finite interval depending on time is formulated and solved for a nonlinear diffusion-convection equation. The authors consider:

$$\theta_t = \theta^2(D\theta_{xx} - \theta_x), \quad \theta = \theta(x,t) \quad t > 0, x \in [s_0(t), s_1(t)], \tag{1}$$
$$\theta(x,0) = \theta_0, \quad x \in [0, L], \tag{2}$$
$$\theta(s_0(t),t) = \alpha, \quad t > 0, \tag{3}$$
$$D\theta_x(s_0(t),t) - \theta(s_0(t),t) = -\dot{s_0}(t), \quad t > 0, \tag{4}$$
$$\theta(s_1(t),t) = \beta, \quad t > 0, \tag{5}$$
$$D\theta_x(s_1(t),t) - \theta(s_1(t),t) = -\dot{s_1}(t), \quad t > 0, \tag{6}$$

and

$$s(0) = 0\,, \qquad s_1(0) = L. \tag{a}$$

In the present case, using dimensionless variables $\theta(x,t)$ denotes the concentration of the drug, which is assumed to be in a percolated phase, that propagates in the arterial wall after it has been released by a drug-eluting stent; D is the coefficient of diffusivity of the drug in the medium.

In order to solve the free boundary problem (1)–(6) and (a), the authors introduce in [1] the following change of the independent variable:

$$\theta(x,t) = \psi(z,t), \quad z = z(x,t), \tag{7}$$
$$\frac{\partial z}{\partial x} = \frac{1}{\theta(x,t)}, \qquad \frac{\partial z}{\partial t} = \theta(x,t) - D\theta_x(x,t), \tag{8}$$

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and they obtain the free boundary problem given by

$$\psi_t = D\psi_{zz} - 2\psi\psi_z, \quad z_0(t) \le z \le z_1(t), \quad t > 0, \tag{9}$$

$$\psi(z,0) = \psi_0 = \theta_0, \tag{10}$$

$$z_0(t) = b_0 + \int_0^{s_0(t)} \frac{\mathrm{d}x'}{\theta(x',t)}, \quad z_1(t) = b_1 + \int_0^{s_1(t)} \frac{\mathrm{d}x'}{\theta(x',t)}, \tag{11}$$

$$\psi(z_0(t),t) = \alpha, \quad t > 0, \tag{12}$$

$$D\psi_z(z_0(t),t) = -\alpha\dot{s}_0(t) + \alpha^2, \quad t > 0, \tag{13}$$

$$\psi(z_1(t),t) = \beta, \quad t > 0, \tag{14}$$

$$D\psi(z_1(t),t) = -\beta\dot{s}_1(t) + \beta^2, \quad t > 0, \tag{15}$$

From the transformations (7) and (8), we can obtain (9)–(10), (12)–(15) but (11) is incorrect in [1] which must be replaced by

$$z_0(t) = A + \frac{\alpha+1}{\alpha}s_0(t), \quad z_1(t) = A + \frac{\alpha+1}{\alpha}s_0(t) + \int_{s_0(t)}^{s_1(t)} \frac{\mathrm{d}x'}{\theta(x',t)}, \tag{11bis}$$

taking into account that the new variable $z(x,t)$, which is defined for (8), is given by

$$z(x,t) = A + \frac{\alpha+1}{\alpha}s_0(t) + \int_{s_0(t)}^{x} \frac{\mathrm{d}x'}{\theta(x',t)}, \tag{b}$$

where A is an arbitrary integration constant, whose expression has not been given in the paper. Moreover, we have

$$z_0(0) = A, \qquad z_1(0) = A + \frac{L}{\theta_0}, \tag{c}$$

where condition (c) was not given in [1].

The inverse function that transforms the problem (9), (10), (11bis), (12)–(15) to the one given by (1)–(6) and (a) is the following:

$$\theta(x,t) = \psi(z,t) \tag{d}$$

with

$$\partial x \partial z = \psi(z,t), \qquad \frac{\partial x}{\partial t} = D\psi_z(z,t) - \psi^2(z,t), \tag{e}$$

then

$$x(z,t) = -\frac{\alpha}{\alpha+1} + \frac{\alpha}{\alpha+1}z_0(t) + \int_{z_0(t)}^{z} \psi(\eta,t)\mathrm{d}\eta. \tag{f}$$

From the transformation (16) in [1], we obtain (9), (17), (19), (21), but (18) and (20) in [1] must be replaced by

$$D\psi_z(F_0(t)) = \alpha^2 - \frac{\alpha^2}{\alpha+1}(2\beta + \dot{F}_0(t)), \tag{18bis}$$

$$D\psi_z(F_1(t)) = \beta^2 - \frac{\alpha^2}{\beta+1}(2\beta + \dot{F}_1(t)), \tag{20bis}$$

and (10) in [1] must be transformed into

$$\phi(z,0) = \theta_0 - \beta = \phi_0 - \beta. \tag{10bis}$$

Then, from the generalized Hopf-Cole transformation given by (23) in [1], we obtain that the inverse transformation is the following:

$$\psi(z,t) = \frac{\varphi(z,t)}{C(t) - \frac{1}{D}\int_{F_1(t)}^{z} \varphi(z',t)\mathrm{d}z'} \tag{22bis}$$

instead of (22) in [1].

Under the above transformation, the Burgers equation (9) is mapped into the linear heat equation (25) in [1] with the compatibility condition

$$\dot{C}(t) = -\varphi_z(F_1(t),t) \tag{26bis}$$

instead of (26). Therefore the free boundary problem (10*bis*), (17), (18*bis*), (19) (20*bis*), (21) must be transformed to the one with the conditions (29) and

$$\varphi(z,0) = (\theta_0 - \beta)\exp\left(-\frac{\theta_0 - \beta}{D}\left[z - \left(A + \frac{L}{\theta_0}\right)\right]\right), \tag{27bis}$$

$$\varphi(F_0(t),t) = (\alpha - \beta)\left[C(t) + \frac{1}{D}\int_{F_0(t)}^{F_1(t)} \varphi(\eta,t)\mathrm{d}\eta\right], \tag{28bis}$$

$$D\varphi_z(F_0(t),t) = \left[C(t) + \frac{1}{D}\int_{F_0(t)}^{F_1(t)} \varphi(\eta,t)\mathrm{d}\eta\right]\frac{\beta(2\alpha - \beta(\alpha+1)) - \alpha^2\dot{F}_0(t)}{\alpha+1}, \tag{30bis}$$

$$D\varphi_z(F_1(t),t) = C(t)\frac{\beta^2}{\beta+1}\left[1 - \beta - \dot{F}_1(t)\right] \tag{31bis}$$

instead of (27), (28), (30) and (31), respectively. The formulae (32) in [1] must be replaced by

$$C(t) = \exp\left[\frac{\beta^2}{D(1+\beta)}\left((\beta-1)t + F_1(t) - A - \frac{L}{\theta_0}\right)\right], \tag{32bis}$$

or

$$C(t) = \exp\left[\frac{\beta^2}{D(1+\beta)}\left(-(\beta+1)t + \frac{\alpha+1}{\alpha}\int_{s_0(t)}^{s_1(t)} \frac{\mathrm{d}\eta}{\theta(\eta,t)}\right)\right]. \tag{32$tris$}$$

Moreover, the explicit solution given in Section 5 of [1] is not a solution because the initial condition is not satisfied since it is not constant.

In conclusion:

The transformation (11) in [1] must be replaced by (11*bis*), and then, the domain of the differential equation (9) is incorrect. Moreover, with the successive transformations we find new incorrect expressions in [1]. The correct expressions are given by our formulae denoted by (*.bis*).

Then, the computations and conclusions given in the [1] are not valid.

Reference

1. Burini, D., De Lillo, S., Fioriti, G.: Nonlinear diffusion in arterial tissues: a free boundary problem. Acta Mech. **229**(10), 4215–4228 (2018)