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A class of moving boundary problems with a source term: application of a reciprocal transformation

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Abstract We consider a new Stefan-type problem for the classical heat equation with a latent heat and phase-change temperature depending of the variable time. We prove the equivalence of this Stefan problem with a class of boundary value problems for the nonlinear canonical evolution equation involving a source term with two free boundaries. This equivalence is obtained by applying a reduction to a Burgers equation and a reciprocal-type transformations. Moreover, for a particular case, we obtain a unique explicit solution for the two different problems.

1 Introduction

In [10], a systematic search was undertaken via Lie–Bäcklund transformations for classes of nonlinear evolution equations which are reducible to a canonical form as originally set down in [17] which incorporates a source term. This nonlinear equation was shown in [17] to admit reduction to Burgers equation and an explicit pulse solution with compact support together with an analytic description of interaction between pulses were thereby derived. In [10], an extension of the Fokas–Yortsos equation with convective term of [15,19] was derived via the Lie–Bäcklund analysis and which incorporates a novel reaction term. This nonlinear evolution equation was shown to be relevant to the modelling of unsaturated flow in a soil with a volumetric extraction mechanism. In [10], a reciprocal transformation was used to solve a nonlinear boundary-value problem incorporating a source term descriptive of transient flow in a finite layer of soil subject to a constant flux boundary condition to compensate for water extraction.

Here by contrast, our concern is with a class of moving boundary problems pertinent to soil mechanics. An inverse procedure is adopted whereby a class of boundary value problems for the canonical nonlinear evolution equation of [17] involving a source term is shown to be amenable to analytic solution by application of a reciprocal link to a Stefan-type problem for the classical heat equation.

It is recalled that moving boundary problems of Stefan-type have their origin in the analysis of the melting of solids and the freezing of liquids (see e.g. [12–14,20,36,37] and the literature cited therein). The standard Stefan problems concern moving boundary problems for the classical linear heat equation where the heat

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balance requirement on the moving boundary separating the phases leads to a nonlinear boundary condition on the temperature.

In [1,2,8,9,11], a novel integral representation version of the Hopf–Cole transformation was used to treat certain classes of Stefan-type problems for Burgers equation. Reciprocal-type transformations on the other hand have been previously applied to solve a wide range of nonlinear moving boundary problems such as arise in nonlinear heat conduction, the analysis of the melting of metals, sedimentation and other physical contexts [16,21–32]. It is remarked that the results in [28–30] obtained via application of reciprocal transformations concern moving boundary problems for certain solitonic equations, namely the Dym, potential mkdV as well as the extended Dym equation derived via geometric considerations in [33]. In the present work, the integral representation of [11] is allied with a reciprocal-type transformation to reduce to canonical form a class of moving boundary problems relevant to the soil mechanics context as described in [10].

Although in classic formulation of Stefan problems the latent heat is considered constant, there are several works in the literature that assume variable latent heat [3–7,18,35,38–41]. In particular, we mention [38] where the latent heat was considered as a linear function of the position of the interface $s = s(t)$ that is $L = \gamma s(t)$ with γ a constant, and when setting the movement of the shoreline to be $s(t) = 2\lambda\sqrt{t}$ it is obtained the time dependence $L = L(t) = L_0\sqrt{t}$.

In following Sect.2 we give a connection between a Stefan problem with variable latent heat term and time-dependent temperature on the free boundary, and a moving boundary problem for the Burgers equation. In Sect. 3 we use the reciprocal-type transformation to prove the equivalence between the moving boundary problem governed by Burgers equation and a free boundary problem governed by the nonlinear evolution equation with source term and two free boundaries. Then we give a parametric expression to solution of this problem through the solution to the Stefan problem. Finally in Sect. 4 we solve the canonical Stefan problem for a particular latent heat and phase change temperature variable in time and we obtain an explicit solution of the similarity type. At last, we express the parametric solution to the nonlinear evolution problem with source and two free boundaries.

2 A canonical connection

Here, a connection is established between a classical Stefan-type problem but with variable latent heat term and a class of moving boundary problems for the nonlinear transport equation incorporating a source term of [17]. The canonical Stefan problem to be considered here adopts the form

$$\begin{aligned}
&T_t = T_{yy}, \quad 0 < y < S(t), \quad t > 0 \\
&-T_y(S(t), t) = L(t)\dot{S}(t), \quad t > 0 \\
&T(S(t), t) = T_m(t), \quad t > 0 \\
&T_y(0, t) = -q, \quad q > 0\,, \quad t > 0 \\
&S(0) = 0\,.
\end{aligned} \tag{2.1}$$

On introduction of the integral representation of [11], namely

$$x^* = -1/\delta \left[\ln |C(t) - \int_{S(t)}^{y} T(y', t) \mathrm{d}y'| \right]_y \,, \tag{2.2}$$

with $C(t) > 0$, $C(0) = 0$, $\delta > 0$ and

$$T = \delta C(t) x^*(y, t) \exp \left[-\delta \int_{S(t)}^{y} x^*(\sigma, t) \mathrm{d}t \right] \tag{2.3}$$

then it may be shown that (see, *in extensive* [27]) the relations (2.2) and (2.3) link the classical heat equation $T_t = T_{yy}$ to the Burgers equation

$$x^*_t = x^*_{yy} - 2\delta x^* x^*_y \tag{2.4}$$

if it is required that

$$\dot{C} + T_y|_{y=S(t)} + \dot{S}T|_{y=S(t)} = 0. \tag{2.5}$$

Thus, in view of the moving boundary conditions in (2.1) it is seen that $C(t)$ is here determined via the relation

$$\dot{C}(t) = [\ L(t) - T_m(t)\ ]\dot{S}(t). \tag{2.6}$$

The boundary requirements on $y = S(t)$ in the Stefan problem (2.1), in turn, become for the associated Burgers equation (2.4)

$$(x^*_y - \delta x^{*2})|_{y=S(t)} = -L(t)\dot{S}(t)/\delta C(t) \tag{2.7}$$

and

$$x^*|_{y=S(t)} = T_m(t)/\delta C(t), \tag{2.8}$$

respectively. The condition on fixed face $y = 0$ becomes

$$x^*_y(0,t) - \delta x^{*2}(0,t) = \frac{-q \exp\left(\delta \int_{S(t)}^{0} x^*(\sigma,t)d\sigma\right)}{\delta C(t)} \tag{2.9}$$

and the initial condition $S(0) = 0$.

3 Application of a reciprocal transformation

Here we will consider the nonlinear evolution equation with source term [17], namely

$$\frac{\partial \Psi}{\partial t} = \frac{\partial}{\partial x^*}\left(\frac{\Psi_{x^*}}{\Psi^2}\right) + 2\delta \tag{3.1}$$

We obtain the following result:

Theorem 3.1 *If $(x^*(y,t), S(t))$ is solution of the problem given by (2.4),(2.6)–(2.9) then the function*

$$\Psi(x^*,t) = \frac{1}{x^*_y(y,t)}$$

satisfies the following problem governed by the nonlinear evolution equation with source term, given by

$$\frac{\partial \Psi}{\partial t} = \frac{\partial}{\partial x^*}\left(\frac{\Psi_{x^*}}{\Psi^2}\right) + 2\delta\ , \quad X^*_0(t) < x^* < X^*_1(t)\ , \quad t > 0 \tag{3.2}$$

with the conditions

$$\begin{array}{ll} (i) & \frac{1}{\Psi(X^*_1(t),t)} - \delta X^{*2}_1(t) = -\frac{L(t)}{\delta C(t)}\left(\Psi(X^*_1(t),t)\dot{X}^*_1(t) + \frac{\Psi_{x^*}(X^*_1(t),t)}{\Psi^2(X^*_1(t),t)} + 2\delta X^*_1(t)\right)\ , \quad t > 0 \\ (ii) & \frac{1}{\Psi(X^*_0(t),t)} - \delta X^{*2}_0(t) = \frac{-q\exp\left(\int_0^t H(\tau)d\tau\right)}{\delta C(t)}\ , \quad q > 0\ , \quad t > 0 \\ (iii) & \dot{C}(t) = [\, L(t) - T_m(t)\,]\left[\Psi(X^*_1(t),t)\dot{X}^*_1(t) + \frac{\Psi_{x^*}(X^*_1(t),t)}{\Psi^2(X^*_1(t),t)} + 2\delta X^*_1(t)\right]\ , \end{array} \tag{3.3}$$

where the moving boundaries X^*_0 and X^*_1 are given by

$$X^*_0(t) = -\int_0^t \left(\frac{\Psi_{x^*}(X^*_0(\tau),\tau)}{\Psi^3(X^*_0(\tau),\tau)} + 2\delta\frac{X^*_0(\tau)}{\Psi(X^*_0(\tau),\tau)}\right)d\tau \quad , \quad X^*_1(t) = T_m(t)/\delta C(t)\ , \quad t > 0 \tag{3.4}$$

and

$$H(t) = -\frac{T^3_m(t)}{L(t)C^2(t)} + \delta\frac{T_m(t)}{L(t)}\frac{1}{\Psi(X^*_1(t),t)} - \frac{\delta}{\Psi(X^*_1(t),t)} + \frac{T^2_m(t)}{C^2(t)} + \frac{\delta}{\Psi(X^*_0(t),t)} - \delta^2 X^{*2}_0(t) \tag{3.5}$$

Proof Let

$$\rho(y,t) = x^*_y(y,t) \tag{3.6}$$

and we define

$$\Psi(x^*,t) = \frac{1}{\rho(y,t)}. \tag{3.7}$$

From (2.4) we known that

$$dx^* = \rho dy + (\rho_y - 2\delta\rho x^*)dt, \tag{3.8}$$

and from (3.7), we have

$$\rho_y = -\frac{\Psi_{x^*}}{\Psi^2} x^*_y = -\frac{\Psi_{x^*}}{\Psi^3} \tag{3.9}$$

then

$$dx^* = \Psi^{-1} dy - (\Psi_{x^*}/\Psi^3 + 2\delta x^*/\Psi)dt \tag{3.10}$$

thus the reciprocal transformation is given by

$$dy = \Psi dx^* + \left(\Psi_{x^*}/\Psi^2 + 2\delta x^*\right) dt. \tag{3.11}$$

and we have

$$\frac{\partial y}{\partial x^*} = \Psi, \quad \frac{\partial y}{\partial t} = \frac{\Psi_{x^*}}{\Psi^2} + 2\delta x^*. \tag{3.12}$$

Therefore (3.2) follows. From (3.6) we can write

$$x^*(y,t) = \int_0^y \rho(\sigma,t)d\sigma + M(t) \tag{3.13}$$

then

$$x_t(y,t) = \int_0^y \rho_t(\sigma,t)d\sigma + M'(t) = \int_0^y \left(x^*_{\sigma\sigma}(\sigma,t) - 2\delta x^*(\sigma,t)x^*_\sigma(\sigma,t)\right)_\sigma d\sigma + M'(t) \tag{3.14}$$

which implies that

$$M'(t) = x^*_{yy}(0,t) - 2\delta x^*(0,t)x^*_y(0,t) \tag{3.15}$$

therefore

$$x^*(y,t) = \int_0^y \rho(\sigma,t)d\sigma + \int_0^t \left[x^*_{yy}(0,\tau) - 2\delta x^*(0,\tau)x^*_y(0,\tau)\right] d\tau. \tag{3.16}$$

For $y = 0$, we obtain

$$x^*(0,t) = \int_0^t \left[x^*_{yy}(0,\tau) - 2\delta x^*(0,\tau)x^*_y(0,\tau)\right] d\tau, \tag{3.17}$$

if we notice $X^*_0(t) := x^*(0,t)$ and taking into account (3.6) -(3.8) we have

$$X^*_0(t) = -\int_0^t \left(\frac{\Psi_{x^*}(X^*_0(\tau),\tau)}{\Psi^3(X^*_0(\tau),\tau)} + 2\delta \frac{X^*_0(\tau)}{\Psi(X^*_0(\tau),\tau)}\right) d\tau. \tag{3.18}$$

From (2.8) we obtain

$$X^*_1(t) := x^*(S(t),t) = T_m(t)/\delta C(t). \tag{3.19}$$

Then the nonlinear evolution equation with source term [17], given by (3.2) is obtained in the domain $X^*_0(t) < x^* < X^*_1(t), t > 0$.

To prove (3.3)(i), we consider condition (2.7) which is equivalent to

$$\frac{1}{\Psi(X^*_1(t),t)} - \delta X^{*2}_1(t) = -L(t)\dot{S}(t)/\delta C(t). \tag{3.20}$$

Now we must write $\dot{S}(t)$ as a function of the new variables and free boundaries. From (3.16) we obtain

$$X^*_1(t) = x^*(S(t),t) = \int_0^{S(t)} \rho(\sigma,t)d\sigma + \int_0^t \left[x^*_{yy}(0,\tau) - 2\delta x^*(0,\tau)x^*_y(0,\tau)\right] d\tau \tag{3.21}$$

then

$$\dot{X}^*_1(t) = \frac{\dot{S}(t)}{\Psi(X^*_1(t),t)} + x^*_{yy}(S(t),t) - 2\delta x^*(S(t),t)x^*_y(S(t),t) \tag{3.22}$$

and taking into account

$$x^*_{yy}(S(t),t) - 2\delta x^*(S(t),t)x^*_y(S(t),t) = -\Psi_{x^*}(X_1^*(t),t)/\Psi^3(X_1^*(t),t) - 2\delta X_1^*(t)/\Psi(X_1^*(t),t)$$

we obtain

$$\dot{S}(t) = \Psi(X_1^*(t),t))\dot{X}_1^*(t) + \Psi_{x^*}(X_1^*(t),t)/\Psi^2(X_1^*(t),t) + 2\delta X_1^*(t). \tag{3.23}$$

We replace $\dot{S}(t)$ in (3.20) to obtain (3.3) (i).

Next, we will deduce (3.3)(ii). From (2.9) we have

$$\frac{1}{\Psi(X_0^*(t),t)} - \delta X_0^{*2}(t) = \frac{-q\exp\left(\delta\int_{S(t)}^0 x^*(\sigma,t)dt\right)}{\delta C(t)}. \tag{3.24}$$

We define

$$R(t) = \exp\left(-\delta\int_0^{S(t)} x^*(\sigma,t)dt\right)$$

then

$$ln(R(t)) = -\delta\int_0^{S(t)} x^*(\sigma,t)dt$$

and

$$\begin{aligned}\frac{R'(t)}{R(t)} &= -\delta x^*(S(t),t)\dot{S}(t) - \delta\int_0^{S(t)} x_t^*(\sigma,t)dt = -\delta x^*(S(t),t)\dot{S}(t) - \delta\int_0^{S(t)}\left(x^*_{\sigma\sigma}(\sigma,t) - 2\delta x^*(\sigma,t)\right)d\sigma\\ &= -\delta x^*(S(t),t)\dot{S}(t) - \delta x_y^*(S(t),t) + \delta^2 x^{*2}(S(t),t) + \delta x_y^*(0,t) - \delta^2 x^{*2}(0,t)\\ &= -\delta X_1^*(t)\dot{S}(t) - \delta x_y^*(S(t),t) + \delta^2 X_1^{*2}(t) + \delta x_y^*(0,t) - \delta^2 X_0^{*2}(t)\\ &= -\delta^2\frac{X_1^*(t)C(t)}{L(t)}\left[\delta X_1^{*2}(t) - \frac{1}{\Psi(X_1^*(t),t)}\right] - \frac{\delta}{\Psi(X_1^*(t),t)} + \delta^2 X_1^{*2}(t) + \frac{\delta}{\Psi(X_0^*(t),t)} - \delta^2 X_0^{*2}(t).\end{aligned}$$

Using (3.4) we obtain

$$\frac{R'(t)}{R(t)} = H(t)$$

with H, given by (3.5), and integrating it results

$$ln(R(t)) - ln(R(0)) = \int_0^t H(\tau)d\tau$$

or equivalently

$$R(t) = exp\left(\int_0^t H(\tau)d\tau\right)$$

because $R(0) = 1$. Therefore (2.9) is equivalent to (3.3)(ii).

The condition (3.3)(iii) yields immediately from (2.6) when $\dot{S}(t)$ is replaced by the expression (3.23). □

Reciprocally through the reciprocal transformation given by

$$dy = \Psi dx^* + (\Psi_{x^*}/\Psi^2 + 2\delta x^*)dt \tag{3.25}$$

we will prove that if $\Psi(x^*,t)$ satisfies (3.2)–(3.5) then the pair $(x^*(y,t), S(t))$ is solution to the problem (2.4),(2.6) -(2.9) where

$$S(t) = \int_{X_0^*(t)}^{X_1^*(t)} \Psi(\sigma,t)d\sigma. \tag{3.26}$$

Theorem 3.2 *If $\Psi(x^*, t)$, $X_0^*(t)$, $X_1^*(t)$ satisfy (3.2)-(3.5) then the pair $(x^* = x^*(y, t), S(t))$ with both components are defined by (3.25) and (3.26), respectively, is solution to the problem (2.4),(2.6)–(2.9).*

Proof From (3.25) we have

$$dx^* = \frac{1}{\Psi} dy - \left(\Psi_{x^*}/\Psi^3 + 2\delta x^*/\Psi\right) dt. \tag{3.27}$$

Taking $\rho(y, t) = \frac{1}{\Psi(x^*,t)}$ we have

$$dx^* = \rho dy + (\rho_y - 2\delta x^* \rho) dt \tag{3.28}$$

which implies

$$x_y^* = \rho\,, \quad x_t^* = \rho_y - 2\delta\rho x^* \tag{3.29}$$

whence

$$x_t^* = x_{yy}^* - 2\delta x^* x_y^*\,.$$

Moreover, from (3.25) we have

$$y = \int_{X_0^*(t)}^{x^*} \Psi(\sigma, t) d\sigma + N(t) \tag{3.30}$$

then

$$\frac{\partial y}{\partial t} = -\Psi(X_0^*(t), t)\dot{X}_0^*(t) + \int_{X_0^*(t)}^{x^*} \Psi_t(\sigma, t) d\sigma + N'(t) \tag{3.31}$$

and taking into account (3.2), (3.4) and the fact that

$$\frac{\partial y}{\partial t} = \Psi_{x^*}/\Psi^2 + 2\delta x^* \tag{3.32}$$

we obtain $N'(t) = 0$ which implies that $N(t)$ is a constant which we take null. Therefore, for $x^* = X_0^*(t)$ results $y = 0$ and for $x^* = X_1^*(t)$ we obtain $y = S(t)$ defined by (3.26).

Derivating in (3.26) we have

$$\dot{S}(t) = \Psi(X_1^*(t), t)\dot{X}_1^*(t) - \Psi(X_0^*(t), t)\dot{X}_0^*(t) + \int_{X_0^*(t)}^{X_1^*(t)} \Psi_t(\sigma, t) d\sigma \tag{3.33}$$

$$= \Psi(X_1^*(t), t)\dot{X}_1^*(t) - \Psi(X_0^*(t), t)\dot{X}_0^*(t) + \frac{\Psi_{x^*}(X_1^*(t), t)}{\Psi^2(X_1^*(t), t)} + 2\delta X_1^*(t) - \frac{\Psi_{x^*}(X_0^*(t), t)}{\Psi^2(X_0^*(t), t)} - 2\delta X_0^*(t) \tag{3.34}$$

and taking into account (3.3)(i) and (3.4) we obtain

$$\dot{S}(t) = \frac{-\delta C(t)}{L(t)} \left(\frac{1}{\Psi(X_1^*(t), t)} - \delta X_1^*(t) \right) \tag{3.35}$$

which is equivalent to (2.7). The condition (2.6) yields immediately from (3.3)(iii) and (3.33), and (2.8) is obtained from (3.4) and the fact that $x^*(S(t), t) = X_1^*(t)$.

To prove (2.9) we define $P(t) = \exp\left(\int_0^t H(\tau) d\tau\right)$ where H is defined by (3.5). We have

$$ln(P(t)) = \int_0^t H(\tau) d\tau \tag{3.36}$$

then

$$\frac{P'(t)}{P(t)} = -\frac{T_m^3(t)}{L(t)C^2(t)} + \delta \frac{T_m(t)}{L(t)} x_y^*(S(t), t) - \delta x_y^*(S(t), t) + \delta^2 X_1^{*2}(t) + \delta x_y^*(0, t) - \delta^2 X_0^{*2}(t) \tag{3.37}$$

and taking into account (2.7) and (2.8) its obtain

$$\frac{P'(t)}{P(t)} = -\delta x^*(S(t),t)\dot{S}(t) - \delta x^*_y(S(t),t) + \delta^2 X_1^{*2}(t) + \delta x^*_y(0,t) - \delta^2 X_0^{*2}(t) = \frac{d}{dt}\left(-\delta \int_0^{S(t)} x^*(\sigma,t)d\sigma\right) \tag{3.38}$$

thus

$$ln(P(t)) - ln(P(0)) = -\delta \int_0^{S(t)} x^*(\sigma,t)d\sigma$$

or equivalently

$$P(t) = exp\left(-\delta \int_0^{S(t)} x^*(\sigma,t)d\sigma\right).$$

Therefore (3.3)(ii) is equivalent to

$$\frac{1}{\Psi(X_0^*(t),t)} - \delta(X_0^*(t))^2 = \frac{-qP(t)}{\delta C(t)}$$

this is

$$x^*_y(0,t) - \delta x^{*2}(0,t) = \frac{-qP(t)}{\delta C(t)}$$

that is to say (2.9). □

In conclusion we have that the solution to the problem (3.2)–(3.5) can be obtained from the solution to the Stefan problem (2.1) $T = T(y,t)$, $S = S(t)$ as established by the following theorem:

Theorem 3.3 *The solution to the problem (3.2)–(3.5) is obtained from the solution to the Stefan problem (2.1) $T = T(y,t)$, $S = S(t)$ and its parametric expression is given by:*

$$\Psi(x^*,t) = \frac{\delta[C(t) - \int_{S(t)}^y T(u,t)du]^2}{T_y(y,t)[C(t) - \int_{S(t)}^y T(u,t)du] + T^2(y,t)} \quad X_0^*(t) < x^* < X_1^*(t)\,, \quad t > 0 \tag{3.39}$$

$$x^* = \frac{T(y,t)}{\delta[C(t) - \int_{S(t)}^y T(u,t)du]}, \quad 0 < y < S(t), \quad t > 0 \tag{3.40}$$

where

$$C(t) = \int_0^t [L(\tau) - T_m(\tau)]\dot{S}(\tau)d\tau, \quad t > 0, \tag{3.41}$$

$$X_0^*(t) = \frac{T(0,t)}{\delta[C(t) - \int_{S(t)}^0 T(u,t)du]}, \quad X_1^*(t) = \frac{T_m(t)}{\delta C(t)}. \tag{3.42}$$

In the next section we will solve the Stefan problem (2.1).

4 Explicit solution for a canonical one-phase Stefan problem with latent heat and phase-change temperature variable in time

In [34], Salva and Tarzia in the context of Stefan problems with variable latent heat introduced a novel similarity solution of the classical heat equation $(2.1)_1$, namely

$$T = 2\sqrt{t}\,\eta(\xi) \tag{4.1}$$

with $\xi = y/2\sqrt{t}$ and

$$\frac{1}{2}\eta''(\xi) + \xi\eta'(\xi) - \eta(\xi) = 0 \tag{4.2}$$

with general solution

$$\eta(\xi) = A[\, e^{-\xi^2} + \sqrt{\pi}\xi \text{ erf } \xi \,] + B\xi \tag{4.3}$$

where A, B are arbitrary constants. Thus,

$$T = 2\sqrt{t}\,[\, A(e^{-\xi^2} + \sqrt{\pi}\; \xi \text{ erf } \xi) + B\xi \,] \tag{4.4}$$

where the boundary condition $(2.1)_4$, on $y = 0$ in the Stefan problem (2.1) requires that

$$\left[A(-2\xi\; e^{-\xi^2} + \sqrt{\pi} \left(\text{erf } \xi + \xi \frac{2}{\sqrt{\pi}} e^{-\xi^2} \right) + B \right] |_{\xi=0} = -q \tag{4.5}$$

so that $B = -q$. Here, the moving boundary in (2.1) is taken as $S(t) = 2\gamma\sqrt{t}$ whence the condition $(2.1)_2$, on reduction, yields

$$A = -\frac{\gamma}{\sqrt{\pi}\text{ erf}\gamma} \left(\frac{1}{\sqrt{t}} \right) L(t) + \frac{q}{\sqrt{\pi}\text{ erf}\gamma} \tag{4.6}$$

so that $L(t) \sim \sqrt{t}$. The condition $(2.1)_3$ on the moving boundary $y = S(t)$ shows that

$$2\sqrt{t} \left[\, A(e^{-\gamma^2} + \sqrt{\pi}\; \gamma \text{ erf } \gamma) - q\gamma \right] = T_m(t) \tag{4.7}$$

so that $T_m(t) \sim \sqrt{t}$, with the result that $L(t)$ and $T_m(t)$ are related according to

$$\frac{T_m(t)/2\sqrt{t} + q\gamma}{e^{-\gamma^2} + \sqrt{\pi}\; \gamma \text{ erf } \gamma} = \frac{-\gamma L(t)/\sqrt{t} + q}{\sqrt{\pi}\text{ erf } \gamma}\,. \tag{4.8}$$

If we take

$$L(t) = L_0\sqrt{t}, \qquad T_m(t) = T_{m_0}\sqrt{t}, \qquad L_0 > T_{m_0} \tag{4.9}$$

then

$$A = \frac{q - L_0\gamma}{\sqrt{\pi}\text{erf}\gamma} \tag{4.10}$$

and the solution to (2.1) is given by

$$T(y,t) = \frac{q - L_0\gamma}{\sqrt{\pi}\text{erf}\gamma} \left(2\sqrt{t}e^{-\frac{y^2}{4t}} + \sqrt{\pi}\; y \text{ erf } \frac{y}{2\sqrt{t}} \right) - qy \tag{4.11}$$

where γ must be solution of

$$\frac{T_{m_0}/2 + q/\gamma}{e^{-\gamma^2} + \sqrt{\pi}\; \gamma \text{ erf } \gamma} = \frac{-\gamma L_0 + q}{\sqrt{\pi}\text{ erf } \gamma}\,. \tag{4.12}$$

which is equivalent to

$$G(\gamma) = F(\gamma) \tag{4.13}$$

where

$$G(x) = q - L_0 x, \quad F(x) = \left(\frac{T_{m_0}}{2} + L_0 x^2 \right) e^{x^2} \sqrt{\pi}\text{ erf}x, \quad x > 0 \tag{4.14}$$

satisfy

$$\begin{aligned} &G(0) = q > 0, \quad G(+\infty) = -\infty, \quad G'(x) < 0, \quad x > 0 \\ &F(0) = 0 > 0, \quad G(+\infty) = +\infty, \quad F'(x) > 0, \quad x > 0. \end{aligned}$$

From properties of functions F and G we obtain that there exists a unique γ, $0 < \gamma < \frac{q}{L_0}$ that (4.13) holds.

We are in a position to establish the following result

Theorem 4.1 *There exists a unique solution to the Stefan problem*

$$\begin{array}{l} T_t = T_{yy}, \quad 0 < y < S(t), \quad t > 0 \\ -T_y(S(t),t) = L_0\sqrt{t}\ \dot{S}(t), \quad t > 0 \\ T(S(t),t) = T_{m0}\sqrt{t}, \quad t > 0 \\ T_y(0,t) = -q, \quad q > 0, \quad t > 0 \\ S(0) = 0 \end{array} \tag{4.15}$$

which is given by

$$T(y,t) = \frac{q - L_0\gamma}{\sqrt{\pi}\mathrm{erf}\gamma}\left(2\sqrt{t}e^{-\frac{y^2}{4t}} + \sqrt{\pi}\ y \ \mathrm{erf}\ \frac{y}{2\sqrt{t}}\right) - qy, \quad 0 < y < s(t), \quad t > 0 \tag{4.16}$$

and

$$S(t) = 2\gamma\sqrt{t} \tag{4.17}$$

where γ is the unique solution to (4.12).

Corollary 4.2 *The coefficient γ which characterizes the free boundary $S(t) = 2\gamma\sqrt{t}$ satisfies the physical condition*

$$q > L_0\gamma + \sqrt{\pi}\frac{T_{m0}}{2}\mathrm{erf}\ \gamma. \tag{4.18}$$

Remark 4.3 It is recalled that in standard Stefan problems wherein L and T_m are constants, γ is determined through the boundary conditions by a transcendental equation.

Remark 4.4 At fixed face $y = 0$ the temperature is time dependent and is given by

$$T(0,t) = \frac{2(q - L_0\gamma)}{\sqrt{\pi}\mathrm{erf}\gamma}\sqrt{t}. \tag{4.19}$$

From theorems (3.3) and (4.1) we can establish the following existence and uniqueness result:

Theorem 4.5 *There exists a unique solution to the problem*

$$\frac{\partial\Psi}{\partial t} = \frac{\partial}{\partial x^*}\left(\frac{\Psi_{x^*}}{\Psi^2}\right) + 2\delta\ , \quad X_0^*(t) < x^* < X_1^*(t)\ , \quad t > 0 \tag{4.20}$$

with the conditions

$$\begin{array}{ll} (i) & \frac{1}{\Psi(X_1^*(t),t)} - \delta X_1^{*2}(t) = -\frac{L_0}{\delta\gamma(L_0 - T_{m0})\sqrt{t}}\left(\Psi(X_1^*(t),t)\dot{X}_1^*(t) + \frac{\Psi_{x^*}(X_1^*(t),t)}{\Psi^2(X_1^*(t),t)} + 2\delta X_1^*(t)\right)\ , \quad t > 0 \\ (ii) & \frac{1}{\Psi(X_0^*(t),t)} - \delta X_0^{*2}(t) = \frac{-q\exp\left(\int_0^t H(\tau)d\tau\right)}{\delta\gamma(L_0 - T_{m0})t}\ , \quad q > 0\ , \quad t > 0 \\ (iii) & \frac{\gamma}{\sqrt{t}} = \Psi(X_1^*(t),t)\dot{X}_1^*(t) + \frac{\Psi_{x^*}(X_1^*(t),t)}{\Psi^2(X_1^*(t),t)} + 2\delta X_1^*(t)\ , \quad t > 0 \end{array} \tag{4.21}$$

where the moving boundaries X_0^* and X_1^* must satisfy

$$X_0^*(t) = -\int_0^t \left(\frac{\Psi_{x^*}(X_0^*(\tau),\tau)}{\Psi^3(X_0^*(\tau),\tau)} + 2\delta\frac{X_0^*(\tau)}{\Psi(X_0^*(\tau),\tau)}\right)d\tau \quad , \quad X_1^*(t) = \frac{T_{m0}}{\delta\gamma(L_0 - T_{m0})\sqrt{t}}\ , \quad t > 0 \tag{4.22}$$

and

$$H(t) = -\frac{T_{m0}^3}{L_0\gamma^2(L_0 - T_{m0})^2} + \delta\frac{T_{m0}}{L_0}\frac{1}{\Psi(X_1^*(t),t)} - \frac{\delta}{\Psi(X_1^*(t),t)} + \frac{T_{m0}^2}{\gamma^2(L_0 - T_{m0})^2 t} + \frac{\delta}{\Psi(X_0^*(t),t)} - \delta^2 X_0^{*2}(t) \tag{4.23}$$

which is given by

$$\Psi(x^*,t)=\frac{\delta\Theta^2(y,t)}{T_y(y,t)\Theta(y,t)+T^2(y,t)},\quad X_0^*(t)<x^*<X_1^*(t)\,,\quad t>0 \tag{4.24}$$

$$x^*=\frac{\frac{q-L_0\gamma}{\sqrt{\pi}\operatorname{erf}\gamma}\left(2\sqrt{t}e^{-\frac{y^2}{4t}}+\sqrt{\pi}\ y\ \operatorname{erf}\frac{y}{2\sqrt{t}}\right)-qy}{\delta\Theta(y,t)},\quad 0<y<S(t),\quad t>0. \tag{4.25}$$

where T is given by (4.16)

$$\Theta(y,t)=\gamma(L_0-T_{m0})t+2q\left(\frac{y^2}{4t}-\gamma^2\right)t \tag{4.26}$$

$$-2\frac{q-L_0\gamma}{\sqrt{\pi}\operatorname{erf}\gamma}\left[\frac{\sqrt{\pi}}{2}\left(\operatorname{erf}\frac{y}{2\sqrt{t}}-\operatorname{erf}\gamma\right)+\sqrt{\pi}\left(\frac{y^2}{4t}\operatorname{erf}\frac{y}{2\sqrt{t}}-\gamma^2\operatorname{erf}\gamma\right)+\frac{y}{2\sqrt{t}}e^{-\frac{y^2}{4t}}-\gamma e^{-\gamma^2}\right]t$$

and the free boundaries are

$$X_0^*(t)=\frac{C_0}{\delta\sqrt{t}},\qquad X_1^*(t)=\frac{C_1}{\delta\sqrt{t}} \tag{4.27}$$

with

$$C_0=\frac{2\frac{q-L_0\gamma}{\sqrt{\pi}\operatorname{erf}\gamma}}{\gamma(L_0-T_{m0})-2q\gamma^2+2\frac{q-L_0\gamma}{\sqrt{\pi}\operatorname{erf}\gamma}\left(\frac{\sqrt{\pi}}{2}\operatorname{erf}\gamma+\sqrt{\pi}\gamma^2\operatorname{erf}\gamma+\gamma e^{-\gamma^2}\right)},\quad C_1=\frac{T_{m0}}{\gamma(L_0-T_{m0})}. \tag{4.28}$$

Proof Here, with $L(t)$ and $T_m(t)$ given by (4.9), on insertion in (2.6), integration shows that $C(t)$ is linear in t is this

$$C(t)=\gamma(L_0-T_{m0})t. \tag{4.29}$$

Moreover, by considering (4.16), (4.17) and

$$\int x\operatorname{erf}(x)dx=\frac{x^2}{2}\operatorname{erf}(x)+\frac{1}{2\sqrt{t}}\exp(-x^2)x-\frac{\operatorname{erf}(x)}{4}$$

we obtain

$$\int_{S(t)}^{y}T(\sigma,t)d\sigma=-2q\left(\frac{y^2}{4t}-\gamma^2\right)t+ \\ +2A\left[\frac{\sqrt{\pi}}{2}\left(\operatorname{erf}\frac{y}{2\sqrt{t}}-\operatorname{erf}\gamma\right)+\sqrt{\pi}\left(\frac{y^2}{4t}\operatorname{erf}\frac{y}{2\sqrt{t}}-\gamma^2\operatorname{erf}\gamma\right)+\frac{y}{2\sqrt{t}}e^{-\frac{y^2}{4t}}-\gamma e^{-\gamma^2}\right]t. \tag{4.30}$$

Taking into account (4.19), (4.29) and (4.30) we obtain (4.24) and (4.25) where function

$$\Theta(y,t):=C(t)-\int_{S(t)}^{y}T(\sigma,t)d\sigma=\gamma(L_0-T_{m0})t+2q\left(\frac{y^2}{4t}-\gamma^2\right)t \tag{4.31}$$

$$-2\frac{q-L_0\gamma}{\sqrt{\pi}\operatorname{erf}\gamma}\left[\frac{\sqrt{\pi}}{2}\left(\operatorname{erf}\frac{y}{2\sqrt{t}}-\operatorname{erf}\gamma\right)+\sqrt{\pi}\left(\frac{y^2}{4t}\operatorname{erf}\frac{y}{2\sqrt{t}}-\gamma^2\operatorname{erf}\gamma\right)+\frac{y}{2\sqrt{t}}e^{-\frac{y^2}{4t}}-\gamma e^{-\gamma^2}\right]t.$$

Rewritten (3.42), we obtain (4.27)

□

5 Conclusions

The equivalence of a new Stefan-problem with latent heat and phase-change temperature with a nonlinear evolution equation with a source term and two free boundaries is obtained. For a particular case a unique explicit solution for both free boundary problems are also obtained.

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Declarations

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