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Existence of an exact solution for a one-phase Stefan problem with nonlinear thermal coefficients from Tirskii's method

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Abstract

The mathematical analysis of a one-phase Lamé–Clapeyron–Stefan problem with nonlinear thermal coefficients following [G.A. Tirskii, Two exact solutions of Stefan's nonlinear problem, Sov. Phys. Dokl. 4 (1959) 288–292] is obtained. Two related cases are considered; one of them has a temperature condition on the fixed face $x = 0$ and the other one has a flux condition of the type $-q_0/\sqrt{t}\,(q_0 > 0)$. We obtain in both cases sufficient conditions for data in order to have the existence of an explicit solution of a similarity type which is given by using a double fixed point.

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1. Introduction

The Lamé–Clapeyron–Stefan problem is nonlinear even in its simplest form due to the free boundary conditions. If the thermal coefficients of the material are temperature dependent, we have a double nonlinear free boundary problem. The present study provides the existence of an exact solution of the similarity type to a one-phase melting problem. We consider the following free boundary problem for a semi-infinite material [1,2]:

$$\rho(T)c(T)T_t = (k(T)T_x)_x\,, \quad 0 < x < s(t) \tag{1}$$

$$T(0,t) = T_b \tag{2}$$

$$T(s(t),t) = T_m \tag{3}$$

$$k\,(T(s(t),t))\,T_x(s(t),t) = -\rho_0\, l\, s'(t) \tag{4}$$

$$s(0) = 0 \tag{5}$$

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where $T = T(x,t)$ is the temperature of the liquid phase; $\rho(T), c(T)$ and $k(T)$ are the body's density, its specific heat, and its thermal conductivity, respectively; T_m is the phase change temperature, $T_b > T_m$ is the temperature on the fixed face $x = 0$; $\rho_0 > 0$ is the constant density of mass at the melting temperature; $l > 0$ is the latent heat of fusion by unit of mass and $s(t)$ is the position of the phase change location.

This problem was first considered in [3] where the integral equation (19) was obtained but no mathematical result is given.

The plan of the paper is the following. In Section 2 we prove the existence of at least one explicit solution of a similarity type for the problem (1)–(5) by using a double fixed point for the integral equation (19) and the transcendental equation (21) under a certain hypothesis for data.

In Section 3 we consider an analogous problem (1) and (3)–(5) and the temperature condition (2) will be replaced by the following flux condition:

$$k(T(0,t))T_x(0,t) = -q_0/\sqrt{t} \tag{6}$$

at the fixed face $x = 0$ where q_0 is a positive constant. Here $-q_0/\sqrt{t}$ denotes the prescribed flux on the boundary $x = 0$ which is of the type imposed in [4]. Furthermore, this kind of heat flux on the fixed boundary has also been considered in several applied problems, e.g. [5–7]. We prove the existence of at least one explicit solution of a similar type for all thermal conditions.

Different methods in order to prove the existence of a solution for the one-phase Stefan problem were considered: integral equation [8–12]; retarding the argument [13]; by the limit of a sequence of approximating solutions [14,15].

2. The one-phase Stefan problem with nonlinear thermal coefficients and temperature boundary condition on the fixed face

If we define the following transformation:

$$\theta(x,t) = (T(x,t) - T_m)/(T_b - T_m) \tag{7}$$

then the problem (1)–(5) becomes

$$N(\theta)\theta_t = \alpha_0 (L(\theta)\theta_x)_x, \quad 0 < x < s(t) \tag{8}$$

$$\theta(0,t) = 1 \tag{9}$$

$$\theta(s(t),t) = 0 \tag{10}$$

$$k(T_m)\theta_x(s(t),t) = -\rho_0 l\, s'(t)/(T_b - T_m) \tag{11}$$

$$s(0) = 0 \tag{12}$$

where $N(T) = \rho(T)c(T)/(\rho_0 c_0)$, $L(T) = k(T)/k_0$ and k_0, ρ_0, c_0 and $\alpha_0 = k_0/(\rho_0 c_0)$ are the reference thermal conductivity, density of mass, specific heat and thermal diffusivity respectively.

Now we assume a similarity solution of the type

$$\theta(x,t) = f(\eta), \quad \eta = x/\left(2\sqrt{\alpha_0 t}\right). \tag{13}$$

Taking into account that problem (8)–(12) is a classical Stefan-like problem with nonlinear thermal coefficient, the free boundary condition (10) implies that the free boundary $s(t)$ must be of the type

$$s(t) = 2\eta_0\sqrt{\alpha_0 t} \tag{14}$$

where η_0 is a positive parameter to be determined later.

Therefore, the conditions (8)–(11) become the following:

$$\left[L(f)f'(\eta)\right]' + 2\eta N(f)f'(\eta) = 0, \quad 0 < \eta < \eta_0 \tag{15}$$

$$f(0) = 1 \tag{16}$$

$$f(\eta_0) = 0 \tag{17}$$

$$f'(\eta_0) = -2\eta_0\alpha_0\rho_0 l/[k(T_m)(T_b - T_m)]. \tag{18}$$

The problem (15)–(17) is equivalent to the following nonlinear integral equation of Volterra type:

$$f(\eta) = 1 - \Phi[\eta, L(f), N(f)] / \Phi[\eta_0, L(f), N(f)] \tag{19}$$

where Φ is given by

$$\Phi[\eta, L(f), N(f)] := (2/\sqrt{\pi}) \int_0^\eta E(t, f)/L(f)(t)\,\mathrm{d}t \tag{20}$$

$$E(x, f) := \exp\left(-2\int_0^x sN(f(s))/L(f(s))\,\mathrm{d}s\right).$$

The condition (18) becomes

$$E(\eta_0, f) \Big/ \Phi[\eta_0, L(f), N(f)] = \eta_0 l\sqrt{\pi} \Big/ [c_0(T_b - T_m)] \tag{21}$$

and then the following theorem holds.

Theorem 1. *The solution of the free boundary problem (1)–(5) is given by (14) and $T(x, t) = T_m + (T_b - T_m) f(\eta)$, with $\eta = x/\left(2\sqrt{\alpha_0 t}\right)$ where the function $f = f(\eta)$ and the coefficient $\eta_0 > 0$ must satisfy the nonlinear integral equation (19) and the condition (21) respectively.* ■

Firstly, in order to prove the existence of the solution to the system (19) and (21) we will obtain some preliminary results. Then we will prove that the integral equation (19) has a unique solution for any given $\eta_0 > 0$ by using a fixed point theorem. Secondly, in order to solve the problem (1)–(5) we will consider Eq. (21).

For convenience of notation, we will define $\Phi[\eta, f] \equiv \Phi[\eta, L(f), N(f)]$.

We suppose that there exists N_m, N_M, L_m, L_M positive constants such as

$$L_m \le L(T) \le L_M, \qquad N_m \le N(T) \le N_M. \tag{22}$$

Furthermore, we assume that the dimensionless thermal conductivity and specific heat are Lipschitz functions, i.e., there exist positive constants $\widetilde{L}$ and $\widetilde{N}$ that verify the following:

$$|L(g) - L(h)| \le \widetilde{L}\,\|g - h\|, \quad \forall g, h \in C^0\left(\mathbb{R}_0^+\right) \cap L^\infty\left(\mathbb{R}_0^+\right) \tag{23}$$

$$|N(g) - N(h)| \le \widetilde{N}\,\|g - h\|, \quad \forall g, h \in C^0\left(\mathbb{R}_0^+\right) \cap L^\infty\left(\mathbb{R}_0^+\right). \tag{24}$$

Then we get:

Lemma 2. *We have*

$$\exp(-N_M x^2/L_m) \le E(x, f) \le \exp(-N_m x^2/L_M), \quad \forall x > 0. \tag{25}$$

Lemma 3. *For $0 < \eta < \eta_0$ we have*

$$\begin{aligned}&\sqrt{L_m/N_M}\,\mathrm{erf}(\sqrt{N_M/L_m}\;\eta)/L_M \le \Phi[\eta, f]\\ &\quad \le \sqrt{L_M/N_m}\,\mathrm{erf}(\sqrt{N_m/L_M}\;\eta)/L_m.\end{aligned} \tag{26}$$

Proof. Taking into account Lemma 2 we have

$$\begin{aligned}\Phi[\eta, f] \le (2/(\sqrt{\pi}L_m)) \int_0^\eta E(v, f)\,\mathrm{d}v &\le (2/\sqrt{\pi}L_m) \int_0^\eta \exp\left(-N_m v^2/L_M\right)\mathrm{d}v\\ &= (2\sqrt{L_M/N_m}/(\sqrt{\pi}L_m)) \int_0^\eta \exp(-t^2)\,\mathrm{d}t\\ &= (\sqrt{L_M/N_m}/L_m)\mathrm{erf}(\sqrt{N_m/L_M}\;\eta).\end{aligned}$$

Analogously we can obtain the other inequality. ■

We consider $C^0[0,\eta_0]$, the space of continuous real functions defined on $[0,\eta_0]$, with its norm $\|f\| = \max_{\eta\in[0,\eta_0]}|f(\eta)|$.

Lemma 4. *Let η_0 be a given positive real number. We suppose that the dimensionless thermal conductivity and specific heat verify conditions (22)–(24). Then, for all $f, f^* \in C^0[0,\eta_0]$ we have*

$$\left|E[\eta,f]-E\left[\eta,f^*\right]\right| \le (\eta^2/L_m)\left(\widetilde{N}+N_M\widetilde{L}/L_m\right)\left\|f^*-f\right\|, \quad \forall\eta\in(0,\eta_0). \tag{27}$$

Proof. If we consider the following inequality:

$$|\exp(-x)-\exp(-y)| \le |x-y|, \quad \forall x,y\ge 0,$$

then we get

$$\begin{aligned}
&\left|E[\eta,f]-E\left[\eta,f^*\right]\right| \\
&\quad= \left|\exp\left(-2\int_0^\eta uN(f(u))/L(f(u))\,\mathrm{d}u\right)-\exp\left(-2\int_0^\eta uN(f^*(u))/L(f^*(u))\mathrm{d}u\right)\right| \\
&\quad\le 2\left|\int_0^\eta uN(f(u))/L(f(u))\,\mathrm{d}u-\int_0^\eta uN(f^*(u))/L(f^*(u))\,\mathrm{d}u\right| \\
&\quad\le 2\int_0^\eta \left|N(f(u))/L(f(u))-N(f^*(u))/L(f^*(u))\right|u\,\mathrm{d}u \\
&\quad\le (\eta^2/L_m)\left(\widetilde{N}+N_M\widetilde{L}/L_m\right)\left\|f^*-f\right\|. \quad\blacksquare
\end{aligned}$$

Lemma 5. *Let η_0 be a given positive real number. We suppose that (22)–(24) holds. For all $f, f^*\in C^0[0,\eta_0]$, $0<\eta<\eta_0$ we have*

$$\begin{aligned}
&\left|\Phi[\eta,f]-\Phi\left[\eta,f^*\right]\right| \\
&\quad\le (2\eta/(L_m^2\sqrt{\pi}))((\widetilde{N}+N_M\widetilde{L}/L_m)\eta^2/3+\widetilde{L})\left\|f^*-f\right\|.
\end{aligned}\tag{28}$$

Proof. (i) We have

$$\begin{aligned}
&\left|\Phi[\eta,f]-\Phi\left[\eta,f^*\right]\right| \\
&\quad\le \left(2/\sqrt{\pi}\right)\int_0^\eta \left|\exp\left(-2\int_0^v uN(f(u))/L(f(u))\,\mathrm{d}u\right)\right. \\
&\qquad \left.-\exp\left(-2\int_0^v uN(f^*(u))/L(f^*(u))u\,\mathrm{d}u\right)\right| \Big/ L(f(v))\ \mathrm{d}v \\
&\qquad +\left(2/\sqrt{\pi}\right)\int_0^\eta \left|1\big/L(f(v))-1/L(f^*(v))\right|\exp\left(-2\int_0^v uN(f^*(s))/L(f^*(s))\,\mathrm{d}u\right)\mathrm{d}v \\
&\quad\equiv T_1(\eta)+T_2(\eta).
\end{aligned}$$

It follows from (23) that

$$\begin{aligned}
T_2(\eta) &\le \left(2/\sqrt{\pi}\right)\int_0^\eta \left|1/L(f(v))-1/L(f^*(v))\right|\,\mathrm{d}v \\
&\le \left(2/\sqrt{\pi}\right)\int_0^\eta \left|\left(L(f^*(v))-L(f(v))\right)/\left(L(f(v))L(f^*(v))\right)\right|\,\mathrm{d}v \\
&\le [2\widetilde{L}\ \eta/(\sqrt{\pi}L_m^2)]\left\|f^*-f\right\|.
\end{aligned}\tag{29}$$

Taking into account Lemma 4 we have that the term $T_1(\eta)$ can also be bounded in the following way:

$$
\begin{aligned}
T_1(\eta) &\le \left(2/\sqrt{\pi}\right)\int_0^\eta \left|E\,[v,f]-E\left[v,f^*\right]\right|/L_m\,\mathrm{d}v\\
&\le \left(4/\left(\sqrt{\pi}L_m\right)\right)\int_0^\eta \left(v^2/L_m\right)\left(\widetilde{N}+N_M\widetilde{L}/L_m\right)\left\|f^*-f\right\|\,\mathrm{d}v\\
&\le (2/(\sqrt{\pi}L_m^2))\left\|f^*-f\right\|\left(\widetilde{N}+N_M\widetilde{L}/L_m\right)\int_0^\eta v^2\,\mathrm{d}v\\
&= (2\eta^3/(3\sqrt{\pi}L_m^2))\left(\widetilde{N}+N_M\widetilde{L}/L_m\right)\left\|f^*-f\right\|. \qquad (30)
\end{aligned}
$$

Therefore, we obtain (28) by using (29) and (30). ■

Theorem 6. *Let η_0 be a given positive real number. We suppose that (22)–(24) holds. If η_0 satisfies the following inequality:*

$$
\beta(\eta_0) := \frac{4}{\sqrt{N_m\pi}}\,\frac{\eta_0 L_M^{5/2} N_M \operatorname{erf}\left(\sqrt{\frac{N_m}{L_M}}\eta_0\right)}{L_m^4\operatorname{erf}^2\left(\sqrt{\frac{N_M}{L_m}}\eta_0\right)}\left(\left(\widetilde{N}+\frac{N_M\widetilde{L}}{L_m}\right)\frac{\eta_0^2}{3}+\widetilde{L}\right) < 1 \qquad (31)
$$

then there exists a unique solution $f \in C^0\,[0,\eta_0]$ of the integral equation (19).

Proof. Let $W : C^0[0,\eta_0] \longrightarrow C^0[0,\eta_0]$ be the operator defined by

$$
W(f)_{(\eta)} = 1 - \Phi\,[\eta, L(f)]\,/\,\Phi\,[\eta_0, L(f)]\,, \quad f \in C^0[0,\eta_0]. \qquad (32)
$$

The solution to the equation (19) is the fixed point of the operator W, that is

$$
W(f(\eta)) = f(\eta), \quad 0 < \eta < \eta_0. \qquad (33)
$$

Let us have $f, f^* \in C^0[0,\eta_0]$; then we obtain

$$
\begin{aligned}
\left\|W(f)-W(f^*)\right\| &= \mathrm{Max}_{\eta\in[0,\eta_0]}\left|W(f(\eta))-W(f^*(\eta))\right|\\
&\le \operatorname*{Max}_{\eta\in[0,\eta_0]}\left|\left(\Phi\left[\eta,f^*\right]\Phi\,[\eta_0,f]-\Phi\left[\eta_0,f^*\right]\Phi\,[\eta,f]\right)/\left(\Phi\,[\eta_0,f]\,\Phi\left[\eta_0,f^*\right]\right)\right|\\
&\le A\operatorname*{Max}_{\eta\in[0,\eta_0]}\left|\Phi\left[\eta,f^*\right]\Phi\,[\eta_0,f]-\Phi\left[\eta_0,f^*\right]\Phi\,[\eta,f]\right|\\
&\le A\operatorname*{Max}_{\eta\in[0,\eta_0]}\left(\left|\Phi\left[\eta,f^*\right]\right|\left|\Phi\,[\eta_0,f]-\Phi\left[\eta_0,f^*\right]\right|\right.\\
&\quad \left.+\left|\Phi\left[\eta_0,f^*\right]\right|\left|\Phi\left[\eta,f^*\right]-\Phi\,[\eta,f]\right|\right)
\end{aligned}
$$

where

$$
A = N_M L_M^2/\left(L_m\operatorname{erf}^2\left(\eta_0\sqrt{N_M/L_m}\right)\right) > 0. \qquad (34)
$$

Finally, for Lemmas 3 and 5 and taking into account that $0 < \eta < \eta_0$, we have

$$
\left\|W(f)-W(f^*)\right\| \le \beta\,(\eta_0)\left\|f^*-f\right\|.
$$

Then W is a contraction operator and therefore there exists a unique solution of the integral equation (19) if the condition (31) is satisfied. ■

Remark 1. The solution f to the integral equation (19), given by Theorem 6, depends on the real number $\eta_0 > 0$. For convenience in the notation from now on we take

$$
f(\eta) = f_{\eta_0}(\eta) = f(\eta_0,\eta), \quad 0 < \eta < \eta_0,\ \eta_0 > 0. \quad ■ \qquad (35)
$$

Let Ω be the set defined by

$$
\begin{aligned}
\Omega &= \left\{\eta_0 \in \mathbb{R}^+/\beta\,(\eta_0) < 1\right\}\\
&= \left\{\eta_0 \in \mathbb{R}^+/\text{there exists a solution of Eq. (19)}\right\}.
\end{aligned}
$$

Lemma 7. *If*

$$2L_M^2\widetilde{L}/L_m^3 < 1 \tag{36}$$

there exists a positive number η_0^* *such that*

$$\beta(\eta_0) < 1 \quad \textit{if } 0 < \eta_0 < \eta_0^*, \qquad \beta(\eta_0) \geq 1 \quad \textit{if } \eta_0 \geq \eta_0^*.$$

Proof. We have $\beta(0) = 2L_M^2\widetilde{L}/L_m^3, \beta(+\infty) = +\infty$ and $\beta'(\eta_0) > 0\ \forall \eta_0 > 0$. Then $\Omega = \left(0, \eta_0^*\right)$ where $\beta(\eta_0^*) = 1$. ■

To prove the existence of the solution to the Eq. (21), we define the real function

$$H(x) := E(x, f)/\Phi[x, f], \quad x > 0$$

where f is the solution to Eq. (19) given by Theorem 6.

Lemma 8. *If* (22) *holds, then function* $H(x)$ *verifies:*

(i) $H_2(x) \leq H(x) \leq H_1(x)$ *where*

$$H_1(x) := L_M\sqrt{N_M}\exp(-N_m x^2/L_M)/(\sqrt{L_m}\operatorname{erf}(x\sqrt{N_M/L_m})),$$
$$H_2(x) := L_m\sqrt{N_m}\exp(-N_M x^2/L_m)/(\sqrt{L_M}\operatorname{erf}(x\sqrt{N_m/L_M}));$$

(ii) $H(0) = +\infty, H(+\infty) = 0$.

Proof. By Lemmas 2 and 3 we have (i). Moreover H_1 and H_2 are decreasing functions which satisfy $H_i(0) = +\infty, H_i(+\infty) = 0\ (i = 1, 2)$; therefore (ii) holds. ■

Theorem 9. *Eq.* (21) *has at least one solution* η_0*. Moreover, if* x_0 *is the unique solution to equation*

$$H_1(x) = xl\sqrt{\pi}/\left(c_0(T_b - T_m)\right), \quad x > 0,$$

and $x_0 < \eta_0^*$ *then* $\eta_0 \in \Omega$ *with* $\eta_0 < x_0$.

Proof.

$$\text{Eq. (21)} \iff H(x) = xl\sqrt{\pi}/\left(c_0(T_b - T_m)\right), \quad x > 0$$

and then, by Lemma 8, there exists at least one solution $\eta_0 > 0$ of Eq. (21). Due to the properties of $H_1(x)$ the equation

$$H_1(x) = xl\sqrt{\pi}/\left(c_0(T_b - T_m)\right), \quad x > 0, \tag{37}$$

has a unique solution x_0. Furthermore $\eta_0 < x_0$ and since β is an increasing function, then we have $\beta(x_0) < \beta(\eta_0^*) = 1$, and then we have $\beta(\eta_0) < 1$, that is $\eta_0 \in \Omega$. ■

Remark 2. The solution x_0 to Eq. (37) can be expressed as follows:

$$x_0 := M^{-1}(L_M\sqrt{N_M}c_0(T_b - T_m)/(\sqrt{\pi L_m}\, l)) \tag{38}$$

where

$$M(x) := x\operatorname{erf}(\sqrt{N_M/L_m}\, x)\exp(x^2 N_m/L_M)$$

is an increasing real function. Then we have

$$\beta(x_0) < 1 \iff \beta(M^{-1}(L_M\sqrt{N_M}c_0(T_b - T_m)/(\sqrt{\pi L_m}\, l))) < 1. \quad ■$$

And so we have the following theorem.

Theorem 10. (i) *If N and L verify the conditions (22)–(24) and (36) and $\beta(x_0) < 1$ where x_0 is defined by (38), then there exists at least one solution of the problem (1)–(5) where the free boundary $s(t)$ is given by (14) and the temperature is given by $T(x,t) = T_m + (T_b - T_m) f(\eta)$, with $\eta = x/2\sqrt{\alpha_0 t}$ where f is the unique solution to the integral equation (19) and η_0 is given by Theorem 9.*

(ii) *If N and L verify the conditions (22)–(24) and (36) then there exists at least one solution to the problem (1)–(5) for all latent heats of fusion $l > l_0$ for given other parameters where l_0 is given by*

$$l_0 := L_M \sqrt{N_M} c_0 (T_b - T_m) / (\sqrt{\pi} L_m M(\eta_0^*))$$

where $\eta_0^* > 0$ is characterized by the condition $\beta\left(\eta_0^*\right) = 1$. ■

Remark 3. The existence of a solution to the problem (1)–(5) is given for large latent heat of fusion ($\forall l > l_0$) if conditions (22)–(24) and (36) for the thermal coefficients are verified. This is equivalent to saying that there exists a solution for all small Stefan numbers Ste, i.e. $\forall \mathrm{Ste} < \mathrm{Ste}_0$ where

$$\mathrm{Ste} = c_0 (T_b - T_m)/l; \qquad \mathrm{Ste}_0 = c_0 (T_b - T_m)/l_0. \tag{39}$$

3. Solution to the free boundary problem with a heat flux condition on the fixed face

In this section we consider the problems (1)–(5), but condition (2) will be replaced by condition (6). If we define the following transformation:

$$\theta(x,t) = (T(x,t) - T_m)/T_m \qquad (T(x,t) = T_m + T_m \theta(x,t)) \tag{40}$$

then the problem to solve becomes

$$N(\theta)\theta_t = \alpha_0 \left(L(\theta)\theta_x\right)_x, \quad 0 < x < s(t) \tag{41}$$

$$k\left(T_m(\theta(0,t) + 1)\right)\theta_x(0,t) = -q_0/(T_m\sqrt{t}) \tag{42}$$

$$\theta(s(t),t) = 0 \tag{43}$$

$$k(T_m)\theta_x(s(t),t) = -\rho_0 l s'(t)/T_m \tag{44}$$

$$s(0) = 0. \tag{45}$$

Now we assume a similarity solution given by (13). Then the free boundary condition (10) implies that the free boundary $s(t)$ must be of the type (14) where η_0 is a positive parameter to be determined later.

Therefore, the conditions (41)–(45) become the following:

$$\left[L(f)f'(\eta)\right]' + 2\eta N(f) f'(\eta) = 0, \quad 0 < \eta < \eta_0 \tag{46}$$

$$k(T_m(f(0) + 1)) f'(0) = -2\sqrt{\alpha_0} q_0 / T_m \tag{47}$$

$$f(\eta_0) = 0 \tag{48}$$

$$f'(\eta_0) = -2\eta_0 \alpha_0 \rho_0 / (k(T_m) T_m) \tag{49}$$

$$s(0) = 0. \tag{50}$$

We have that the problem (46)–(48) is equivalent to the following nonlinear integral equation of Volterra type:

$$f(\eta) = l\eta_0\sqrt{\pi}\left(\Phi[\eta_0, L(f), N(f)] - \Phi[\eta, L(f), N(f)]\right) / \left(c_0 T_m E(\eta_0, f)\right) \tag{51}$$

where Φ and E were defined in (20).

The condition (47) becomes

$$E(\eta_0, f) = l\ \rho_0 \eta_0 \sqrt{\alpha_0}/q_0. \tag{52}$$

Theorem 11. *The solution to the free boundary problem (1) and (3)–(6) is given by (14) and $T(x,t) = T_m + T_m f(\eta)$, $\eta = x/(2\sqrt{\alpha_0 t})$ where the function $f = f(\eta)$ and the coefficient $\eta_0 > 0$ must satisfy the nonlinear integral equation (51) and the condition (52).* ■

Theorem 12. *Let η_0 be a given positive real number. We suppose that (22)–(24) hold. If η_0 satisfies the following inequality:*

$$\gamma(\eta_0) = \frac{4\eta_0^2 l}{c_0 T_m L_m^2} \exp^2\left(\frac{N_M}{L_m}\eta_0^2\right)\left\{2\exp\left(-\frac{N_m}{L_M}\eta_0^2\right)\left(\left(\widetilde{N} + \frac{N_M \widetilde{L}\eta_0^2}{L_m}\right)\frac{\eta_0^2}{3} + \widetilde{L}\right)\right.$$
$$\left. + \sqrt{\frac{L_M}{N_m}}\eta_0 \operatorname{erf}\left(\sqrt{\frac{N_m}{L_M}}\eta_0\right)\left(\widetilde{N} + \frac{N_M \widetilde{L}\eta_0^2}{L_m}\right)\right\} < 1 \tag{53}$$

then there exists a unique solution to the integral equation (51).

Proof. Let $R : C^0[0, \eta_0] \longrightarrow C^0[0, \eta_0]$ be the operator defined by

$$R(f)_{(\eta)} = l\eta_0\sqrt{\pi}\left(\Phi[\eta_0, f] - \Phi[\eta, f]\right)/\left(c_0 T_m E(\eta_0, f)\right), \quad f \in C^0[0, \eta_0], 0 < \eta < \eta_0. \tag{54}$$

The solution to the equation (51) is the fixed point of the operator R, that is

$$R(f(\eta)) = f(\eta), \quad 0 < \eta < \eta_0. \tag{55}$$

Let us have $f, f^* \in C^0[0, \eta_0]$; then we obtain

$$\begin{aligned}
&\left|R(f) - R(f^*)\right| \\
&\quad \le l\eta_0\sqrt{\pi}/\left(c_0 T_m E(\eta_0, f)E(\eta_0, f^*)\right) \\
&\quad \cdot\left|E(\eta_0, f)\left(\Phi\left[\eta_0, f^*\right] - \Phi\left[\eta, f^*\right]\right) + E(\eta, f)\left(\Phi[\eta, f] - \Phi[\eta_0, f]\right)\right| \\
&\quad \le l\eta_0\sqrt{\pi}\left(U_1 + U_2 + U_3\right)/\left(c_0 T_m E(\eta_0, f)E(\eta_0, f^*)\right)
\end{aligned}$$

where

$$\begin{aligned}
U_1 &= E(\eta_0, f)\left(\Phi\left[\eta_0, f^*\right] - \Phi[\eta_0, f]\right) \\
U_2 &= E(\eta_0, f^*)\left(\Phi\left[\eta_0, f^*\right] - \Phi[\eta_0, f]\right) \\
U_3 &= E(\eta_0, f)\Phi[\eta_0, f] - E(\eta_0, f^*)\Phi\left[\eta_0, f^*\right] - E(\eta_0, f)\Phi\left[\eta, f^*\right] + E(\eta_0, f^*)\Phi[\eta, f].
\end{aligned}$$

Taking into account Lemmas 2–5 we obtain

$$U_1 \le U_1^*\left\|f - f^*\right\|, \qquad U_2 \le U_2^*\left\|f - f^*\right\|, \qquad U_3 \le U_3^*\left\|f - f^*\right\|$$

where

$$\begin{aligned}
U_1^* &= \left(2\eta_0\Big/\left(L_m^2\sqrt{\pi}\right)\right)\exp\left(-N_m\eta_0^2/L_M\right)\left(\left(\widetilde{N} + N_M\widetilde{L}\eta_0^2/L_m\right)\eta_0^2/3 + \widetilde{L}\right) \\
U_2^* &= \left(2\eta_0\Big/\left(L_m^2\sqrt{\pi}\right)\right)\exp\left(-N_m\eta_0^2/L_M\right)\left(\left(\widetilde{N} + N_M\widetilde{L}\eta_0^2/L_m\right)\eta_0^2/3 + \widetilde{L}\right) \\
U_3^* &= 2U_1 + [4\eta_0^2\sqrt{L_M/N_m}/(L_m^2\sqrt{\pi})]\operatorname{erf}(\sqrt{N_m/L_M}\eta_0)(\widetilde{N} + N_M\widetilde{L}\eta_0^2/L_m).
\end{aligned}$$

Finally, we have

$$\left\|R(f) - R(f^*)\right\| \le \gamma(\eta_0)\left\|f^* - f\right\|.$$

Then, there exists a unique solution to the integral equation (51) if condition (53) is verified (i.e. R is a contraction operator). ■

Lemma 13. *Function $\gamma = \gamma(\eta)$, given by*

$$\gamma(\eta) = \frac{4\eta^2 l}{c_0 T_m L_m^2}\exp^2\left(\frac{N_M}{L_m}\eta^2\right)\left\{2\exp\left(-\frac{N_m}{L_M}\eta^2\right)\left(\left(\widetilde{N} + \frac{N_M\widetilde{L}}{L_m}\right)\frac{\eta^2}{3} + \widetilde{L}\right)\right.$$
$$\left. + \sqrt{\frac{L_M}{N_m}}\eta\operatorname{erf}\left(\sqrt{\frac{N_m}{L_M}}\eta\right)\left(\widetilde{N} + \frac{N_M\widetilde{L}}{L_m}\right)\right\}, \quad \eta > 0 \tag{56}$$

satisfies the following properties:
(i) $\gamma(0) = 0$, (ii) $\gamma(+\infty) = +\infty$,
(iii) γ *is an increasing function and* $\exists \widetilde{\eta} > 0 \;/\; \gamma(\eta) < 1 \;\forall \eta \in (0, \widetilde{\eta})$.

Theorem 14. *Eq. (52) has at least one solution* η_0. *Moreover, if* x_1 *is the unique solution to the equation*

$$I_1(x) = l\ \rho_0 x \sqrt{\alpha_0} \big/ q_0\ , \quad x > 0$$

and satisfies $\gamma(x_1) < 1$, *then* $\gamma(\eta_0) < 1$.

Proof. By Lemma 2 we have $I_2(x) \le E(x, f) \le I_1(x)$ where $I_1(x) = \exp\left(-N_m x^2 / L_M\right)$ and $I_2(x) = \exp\left(-N_M x^2 / L_m\right)$.

Since $I_1(0) = I_2(0) = +\infty$, $I_1(+\infty) = I_2(+\infty) = 0$ and they are decreasing functions, then there exists at least one solution η_0 to Eq. (52).

Moreover, $\eta_0 < x_1$, where x_1 is the unique solution to equation $I_1(x) = l\ \rho_0 x \sqrt{\alpha_0} \big/ q_0$, $x > 0$.

Furthermore, if $\gamma(x_1) < 1$ then $\gamma(\eta_0) < 1$. ∎

Remark 4. We note that $x_1 = M_*^{-1}\left(q_0 \big/ \left(l \rho_0 \eta_0 \sqrt{\alpha_0}\right)\right)$, where

$$M_*(x) := x \exp\left(N_m x^2 / L_M\right)$$

and then

$$\gamma(x_1) < 1 \Longleftrightarrow \gamma\left(M_*^{-1}\left(q_0 \big/ \left(l \rho_0 \eta_0 \sqrt{\alpha_0}\right)\right)\right) < 1. \quad \blacksquare \tag{57}$$

Theorem 15. (i) *If* N *and* L *satisfy* (22)–(24) *and* (57), *then there exists at least one solution of* (41)–(45) *given by* $s(t) = 2\eta_0 \sqrt{\alpha_0 t}$ *and*

$$T(x, t) = T_m\left(1 + f(\eta)\right), \quad \eta = x \big/ \left(2\sqrt{\alpha_0 t}\right)$$

where f *and* η_0 *satisfy* (51) *and* (52).

(ii) *If* N *and* L *verify the conditions* (22)–(24) *and* (57), *then there exists at least one solution to the problem* (41)–(45) *for all latent heats of fusion* $l > l_0^*$ *for given other parameters where* l_0^* *is given by*

$$l_0^* := q_0 \big/ \left(\rho_0 \eta_0 \sqrt{\alpha_0} M_*(\widetilde{\eta})\right)$$

where $\widetilde{\eta} > 0$ *is characterized by the condition* $\gamma(\widetilde{\eta}) = 1$. ∎

Remark 5. The existence of a solution to the problem (41)–(45) is given for large latent heat of fusion ($\forall l > l_0^*$) if conditions (22)–(24) and (57) for the thermal coefficients are verified.

Two examples with an explicit solution are well known.

Example 1. In the particular case $N = L = 1$, the solution of integral equation (19) is given by [16,17]

$$f(\eta) = 1 - \mathrm{erf}(\eta) \big/ \mathrm{erf}(\eta_0)\ , \quad 0 < \eta < \eta_0, \tag{58}$$

where $\eta_0 > 0$ is the unique solution to the equation

$$x\, \mathrm{erf}(x) \exp(x^2) = \mathrm{Ste}/\sqrt{\pi}, \quad x > 0 \tag{59}$$

where Ste is the Stefan number defined by (39).

Example 2. In [18,19], the case of $\rho(T) = \rho_0, c(T) = c_0$ and $k(T) = k_0\left[1 + \zeta(T - T_m) \big/ (T_b - T_m)\right] = k_0\left[1 + \zeta\theta\right]$ was considered, that is $N(T) = 1$ and $L(\theta) = k(T) \big/ k_0 = 1 + \zeta\theta$. In this case, the solution is given by

$$T(x, t) = T_b + (T_m - T_b)\, \Psi_\delta(\eta) \,/\, \Psi_\delta(\eta_0)\ , \quad 0 < \eta < \eta_0, \eta = x / \left(2\sqrt{\alpha_0 t}\right)$$
$$s(t) = 2\eta_0 \sqrt{\alpha_0 t}$$

where δ and η_0 must satisfy the following equations:

$$\zeta = \delta \Psi_\delta(\eta_0)$$
$$(1 + \delta \Psi_\delta(\eta_0)) \Psi'_\delta(\eta_0) / (\eta_0 \Psi_\delta(\eta_0)) = 2/\mathrm{Ste}$$

where $\Psi = \Psi_\delta$ is the error function defined as the unique solution to the ordinary differential problem

$$\left[(1 + \delta \Psi(\eta)) \Psi'(\eta)\right]' + 2\eta \Psi'(\eta) = 0$$
$$\Psi\left(0^+\right) = 0, \qquad \Psi(+\infty) = 1.$$

Note that if $\zeta = 0$ we obtain Example 1.

Other examples with nonlinear thermal coefficients and an explicit solution of a similarity type for the corresponding free boundary problem have been obtained in [20–28].

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