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Explicit solutions for a two-phase unidimensional Lamé–Clapeyron–Stefan problem with source terms in both phases

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Abstract

A two-phase Stefan problem with heat source terms of a general similarity type in both liquid and solid phases for a semi-infinite phase-change material is studied. We assume the initial temperature is a negative constant and we consider two different boundary conditions at the fixed face $x = 0$, a constant temperature or a heat flux of the form $-q_0/\sqrt{t}$ ($q_0 > 0$). The internal heat source functions are given by $g_j(x,t) = \frac{\rho l}{t}\beta_j(\frac{x}{2a_j\sqrt{t}})$ ($j = 1$ solid phase; $j = 2$ liquid phase) where $\beta_j = \beta_j(\eta)$ are functions with appropriate regularity properties, ρ is the mass density, l is the fusion latent heat by unit of mass, a_j^2 is the diffusion coefficient, x is the spatial variable and t is the temporal variable. We obtain for both problems explicit solutions with a restriction for data only for the second boundary conditions on $x = 0$. Moreover, the equivalence of the two free boundary problems is also proved. We generalize the solution obtained in [J.L. Menaldi, D.A. Tarzia, Generalized Lamé–Clapeyron solution for a one-phase source Stefan problem, Comput. Appl. Math. 12 (2) (1993) 123–142] for the one-phase Stefan problem. Finally, a particular case where β_j ($j = 1, 2$) are of exponential type given by $\beta_j(x) = \exp(-(x + d_j)^2)$ with x and $d_j \in \mathbb{R}$ is also studied in details for both boundary temperature conditions at $x = 0$. This type of heat source terms is important through the use of microwave energy following [E.P. Scott, An analytical solution and sensitivity study of sublimation–dehydration within a porous medium with volumetric heating, J. Heat Transfer 116 (1994) 686–693]. We obtain a unique solution of the similarity type for any data when a temperature

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boundary condition at the fixed face $x = 0$ is considered; a similar result is obtained for a heat flux condition imposed on $x = 0$ if an inequality for parameter q_0 is satisfied.

Keywords: Stefan problem; Free boundary problem; Lamé–Clapeyron solution; Neumann solution; Phase-change process; Fusion; Sublimation–dehydration process; Heat source; Similarity solution

1. Introduction

Following Scott [20], sublimation–dehydration or freeze–drying, is used as a method for removing moisture from biological materials, such as food. Some of the advantages of sublimation–dehydration over evaporative drying are that the structural integrity of the material is maintained and product degradation is minimized (Ang et al. [1], Rosenberg, Bögl [19]). The major disadvantage of the freeze–drying process is that it is generally slow, and consequently, the process is economically unfeasible for certain materials. One of the means of alleviating this problem is through the use of microwave energy.

Several mathematical models have been proposed to describe the freeze–drying process without microwave heating (Fey, Boles [10], Lin [13]). Only a few studies have also included a microwave heat source in the model (Ang et al. [1]). Phase-change problems appear frequently in industrial processes; a large bibliography on the subject was given recently in Tarzia [22].

In Menaldi, Tarzia [14] the one-phase Lamé–Clapeyron–Stefan problem [12] with internal heat sources of general similarity type was studied and a generalized Lamé–Clapeyron explicit solution was obtained. Moreover, necessary and sufficient conditions were given in order to characterize the source term which provides a unique solution.

In Bouillet, Tarzia [5], the self-similar solutions $\theta(x,t) = \theta(\eta) = \theta(x/\sqrt{t})$ of the problem

$$
\begin{aligned}
&E(\theta)_t - A(\theta)_{xx} = \frac{1}{t}B(\eta), \quad \eta > 0,\\
&\theta(x,t) = C > 0, \quad t > 0,\\
&E\big(\theta(x,0)\big) = 0, \quad x > 0,
\end{aligned}
$$

were studied where E and A are monotone increasing functions, A being continuous, with $E(0) = A(0) = 0$ and $\lambda = E(0^+) > 0$. This equation can describe the conservation of thermal energy in a heat conduction process for a semi-infinite material with a "self-similar" source or sink term of the type $B(x/\sqrt{t})/t$. Moreover, $E(\theta)$ represents an energy per unit volume at level (temperature) θ, $A'(\theta) \geqslant 0$ is the thermal conductivity and $B(\eta)/t$ represents a singular source or sink depending of the sign of the function B. It was obtained for the inverse function $\eta = \eta(\theta)$ an integral equation equivalent to the above problem and it was proven that for certain hypotheses over data there exists at least a solution of the corresponding integral equation following Bouillet [4].

Several applied papers give us the significance of the source terms in the interior of the material which can undergo a change of phase, e.g. Bhattacharya et al. [3], Carslaw, Jaeger [6], Feng [9], Grigor'ev et al. [11], Mercado et al. [15], Ratanadecho et al. [17], Ward [23]. In Scott [20] there is a mathematical model for sublimation–dehydration with volumetric heating of a particular exponential type from which analytical solutions for dimensionless temperature, vapor concentration, and pressure were obtained for two different temperature boundary conditions. It was considered a semi-infinite frozen porous medium with constant thermal properties subject to a sublimation–dehydration process involving a volumetric heat source of the type

$$g(x,t)=\frac{\text{const.}}{t}\exp\bigl(-(x+d)^2\bigr).$$

A sensitivity study was also conducted in which the effects of the material properties inherent in these solutions were analyzed. The mathematical analysis of the analytical solutions is only given from the numerical computation point of view. In one phase is taken d equals to 0 and in the other one d is proportional to the constant λ which characterizes the interface position; this last choice is, for us, a nonadequate choice of a parameter because it depends on the solution itself.

Analytical solutions can provide important insights into the importance of different material properties on the solution, which can aid in the development of improved mathematical models for this process. These solutions provide an important means of evaluating numerical schemes which can later be used with less restrictive assumptions, if necessary, to simulate actual processes. Moreover, it can be used to obtain super and sub solutions for general conditions by using the maximum principle.

In this paper a semi-infinite homogeneous phase-change material initially in solid phase at the uniform temperature $-C<0$, with a volumetric heat source, is considered. A mathematical description for the temperature within the material is given by

$$\frac{\partial T_2}{\partial t}(x,t)=a_2^2\frac{\partial^2 T_2}{\partial x^2}(x,t)+\frac{1}{\rho c_2}g_2(x,t),\quad 0<x<s(t),\ t>0; \tag{1}$$

$$\frac{\partial T_1}{\partial t}(x,t)=a_1^2\frac{\partial^2 T_1}{\partial x^2}(x,t)+\frac{1}{\rho c_1}g_1(x,t),\quad x>s(t),\ t>0; \tag{2}$$

for two given internal source functions (Bouillet, Tarzia [5], Menaldi, Tarzia [14], Scott [20]) given by

$$g_j=g_j(x,t)=\frac{\rho l}{t}\beta_j\left(\frac{x}{2a_j\sqrt{t}}\right),\quad j=1,2, \tag{3}$$

where $\beta_j=\beta_j(\eta)$ are integrable functions in $(0,\epsilon)$ $\forall\epsilon>0$ and $\beta_j(\eta)\exp(\eta^2)$ are integrable functions in $(M,+\infty)$ $\forall M>0$. We assume that $\beta_1(\eta)\geqslant 0$, $\beta_2(\eta)\leqslant 0$ and ρ is the mass density, l is the fusion latent heat per unit of mass, a_j^2 is the diffusion coefficient, c_j is the specified heat per unit of mass and k_j is the thermal conductivity, for $j=1,2$.

The initial temperature and the temperature as $x\to\infty$ are assumed to be constant

$$T_1(x,0)=T_1(+\infty,t)=-C<0,\quad x>0,\ t>0. \tag{4}$$

At $x=0$, two different temperature boundary conditions are considered, the first is a constant temperature condition

$$T_2(0,t)=B>0,\quad t>0, \tag{5}$$

which is studied in Section 2.1, and the second is an assumed heat flux of the form

$$k_2\frac{\partial T_2}{\partial x}(0,t)=\frac{-q_0}{\sqrt{t}},\quad t>0, \tag{6}$$

which is studied in Section 3.

We remark that $-q_0/\sqrt{t}$ denotes the prescribed heat flux on the boundary $x=0$ which is of the type imposed in Tarzia [21] where it was proven that the heat flux condition (6) on the fixed face $x = 0$ is equivalent to the constant temperature boundary condition (5) for the two phase Stefan problem for a semi-infinite material with constant thermal coefficient in both phases without source terms. This kind of heat flux condition was also considered in several papers, e.g. Barber [2], Coelho Pinheiro [7], Polyanin, Dil'man [16], Rogers [18].

The phase-change interface condition is determined from an energy balance at the free boundary $x = s(t)$:

$$k_1 \frac{\partial T_1}{\partial x}\big(s(t), t\big) - k_2 \frac{\partial T_2}{\partial x}\big(s(t), t\big) = \rho l \dot{s}(t), \quad t > 0, \tag{7}$$

where the temperature conditions at the interface are assumed to be constant:

$$T_1\big(s(t), t\big) = T_2\big(s(t), t\big) = 0, \quad t > 0. \tag{8}$$

Moreover, the initial position of the free boundary is

$$s(0) = 0. \tag{9}$$

In Section 2.1 we obtain an explicit solution for the problem (1)–(5), (7)–(9), when the general type of sources given by (3) verifies appropriate properties, and in Section 2.2 we give monotonicity properties of the solution. Both results are obtained for any data and thermal coefficients (particularly for all β's source terms). We remark that when we consider the particular case $C = 0$ and $\beta_1 = 0$ we obtain the solutions given in Menaldi, Tarzia [14] for the one-phase case.

In Section 3 we solve the same free boundary problem but with the heat flux condition of the type $-\frac{q_0}{\sqrt{t}}$ ($q_0 > 0$) prescribed on the fixed face $x = 0$, and we obtain an explicit solution to this problem if the coefficient q_0 satisfies an appropriate particular inequality given by (46). This result is new for the analytical solution. Furthermore, if we take $\beta_1 = \beta_2 = 0$ we get the inequality (46) which was given in Tarzia [21] for the classical two-phase Stefan problem.

In Section 4 we prove the equivalence of the two free boundary problems: the first one with the Dirichlet constant boundary condition (5) considered in Section 2, and the second one with the Neumann boundary condition (6) considered in Section 3.

In Section 5 we will consider the volumetric heat sources of the type given by expressions (56) proposed by Scott [20] in thermal processes. In this particular case we can explicitly obtain conditions (45) and (46) which guarantees the existence of a unique solution, as a function of the parameters of the two problems, in order to have the corresponding exact similarity solution in both phases. If we take $d_1 = d_2 = 0$ in β's expressions (56) our solution (63) coincides with Scott's solution taking a null vapor mass flow rate.

2. Free boundary problem with temperature boundary condition

2.1. *Solution of the free boundary problem with temperature boundary condition at $x = 0$*

Applying the immobilization domain method (see Crank [8]), we are looking for solutions of the type

$$T_j(x, t) = \theta_j(y), \quad j = 1, 2, \tag{10}$$

where the new independent spatial variable y is defined by

$$y = \frac{x}{s(t)}. \tag{11}$$

Then, the condition (7) is transformed into

$$k_1\theta_1'(1) - k_2\theta_2'(1) = \rho l s(t)\dot{s}(t), \tag{12}$$

and we must have necessarily that $s(t)\dot{s}(t) = \text{const.}$ i.e.,

$$s(t) = 2a_2\lambda\sqrt{t}, \tag{13}$$

where the dimensionless parameter $\lambda > 0$ is unknown.

Next, we define

$$R_j(\eta) = \theta_j\left(\frac{\eta}{\lambda}\right), \quad j = 1, 2,\ \eta = \lambda y, \tag{14}$$

then the problem (1)–(5), (7)–(9) is equivalent to the following one:

$$R_2''(\eta) + 2\eta R_2'(\eta) = -\frac{4l}{c_2}\beta_2(\eta), \quad 0 < \eta < \lambda; \tag{15}$$

$$R_1''(\eta) + 2\frac{a_2^2}{a_1^2}\eta R_1'(\eta) = -\frac{4a_2^2 l}{a_1^2 c_1}\beta_1\left(\frac{a_2}{a_1}\eta\right), \quad \eta > \lambda; \tag{16}$$

$$R_1(\lambda) = R_2(\lambda) = 0; \tag{17}$$

$$k_1 R_1'(\lambda) - k_2 R_2'(\lambda) = 2\rho l\lambda a_2^2; \tag{18}$$

$$R_1(+\infty) = -C; \tag{19}$$

$$R_2(0) = B. \tag{20}$$

After some elementary computations, from (15), (17) and (20) we obtain

$$R_2(\eta) = B - \big(B + \varphi_2(\lambda)\big)\frac{\operatorname{erf}(\eta)}{\operatorname{erf}(\lambda)} + \varphi_2(\eta), \quad 0 < \eta < \lambda,$$

$$\varphi_2(\eta) = \frac{2l\sqrt{\pi}}{c_2}\int_0^{\eta} \beta_2(u)\exp\big(u^2\big)\big(\operatorname{erf}(u) - \operatorname{erf}(\eta)\big)\,du \tag{21}$$

and, from (16), (17) and (19), we have

$$R_1(\eta) = -\frac{(C + \varphi_1(+\infty))}{\operatorname{erf}c(\frac{a_2}{a_1}\lambda)}\frac{2}{\sqrt{\pi}}\int_{\frac{a_2}{a_1}\lambda}^{\frac{a_2}{a_1}\eta} \exp\big(-u^2\big)\,du + \varphi_1(\eta), \quad \eta > \lambda,$$

$$\varphi_1(\eta) = \frac{2l\sqrt{\pi}}{c_1}\int_{\frac{a_2}{a_1}\lambda}^{\frac{a_2}{a_1}\eta} \beta_1(u)\exp\big(u^2\big)\left[\operatorname{erf}(u) - \operatorname{erf}\left(\frac{a_2}{a_1}\eta\right)\right]du \tag{22}$$

where λ is the unknown coefficient which must verify the condition (18).

Furthermore, Eq. (18) for λ is equivalent to the following equation

$$f_1(x, \beta_1) = f_2(x, \beta_2), \quad x > 0, \tag{23}$$

where

$$f_1(x,\beta_1) = F_0(x)\, h_1(x,\beta_1), \tag{24}$$

$$f_2(x,\beta_2) = Q\left(\frac{a_2}{a_1}x\right) h_2(x,\beta_2) \tag{25}$$

with

$$Q(x) = \sqrt{\pi}\, x \exp\left(x^2\right)\left(1 - \operatorname{erf}(x)\right), \quad x > 0, \tag{26}$$

$$F_0(x) = x \operatorname{erf}(x) \exp\left(x^2\right), \quad x > 0, \tag{27}$$

$$h_1(x,\beta_1) = \mathrm{Ste}_1 - 2\sqrt{\pi} \int_{\frac{a_2}{a_1}x}^{+\infty} \operatorname{erf} c(u) \beta_1(u) \exp\left(u^2\right) du, \tag{28}$$

$$h_2(x,\beta_2) = \frac{\mathrm{Ste}_2}{\sqrt{\pi}} - F(x,\beta_2), \quad x > 0, \tag{29}$$

with

$$F(x,\beta_2) = F_0(x) - 2\int_0^x \operatorname{erf}(u)\beta_2(u) \exp\left(u^2\right) du, \quad x > 0, \tag{30}$$

and

$$\mathrm{Ste}_1 = \frac{Cc_1}{l}, \qquad \mathrm{Ste}_2 = \frac{Bc_2}{l} \tag{31}$$

are the Stefan numbers for phases $j = 1$ and $j = 2$, respectively.

Theorem 1. *Equation* (23) *has a unique solution* $\lambda > 0$. *Moreover, the free boundary problem with heat source terms* (1)–(5), (7)–(9) *has an explicit solution given by*

$$T_1(x,t) = \frac{-(C + \varphi_1(+\infty))}{\operatorname{erf} c(\frac{a_2}{a_1}\lambda)} \left[\operatorname{erf}\left(\frac{x}{2a_1\sqrt{t}}\right) - \operatorname{erf}\left(\frac{a_2}{a_1}\lambda\right)\right] + \varphi_1\left(\frac{x}{2a_2\sqrt{t}}\right),$$

$$\text{for } x > s(t),\ t > 0;$$

$$T_2(x,t) = \frac{2l\sqrt{\pi}}{c_2} \int_0^{\frac{x}{2a_2\sqrt{t}}} \beta_2(u) \exp\left(u^2\right)\left(\operatorname{erf}(u) - \operatorname{erf}\left(\frac{x}{2a_2\sqrt{t}}\right)\right) du$$

$$+ B - \left(B + \varphi_2(\lambda)\right)\frac{\operatorname{erf}(\frac{x}{2a_2\sqrt{t}})}{\operatorname{erf}(\lambda)} \quad \text{for } 0 < x < s(t),\ t > 0, \tag{32}$$

where $\varphi_1(\eta)$ *and* $\varphi_2(\eta)$ *are defined in* (22), (21) *respectively and the free boundary* $s(t)$ *is given by* (13) *where the coefficient* λ *is the unique solution of Eq.* (23).

Proof. Taking into account Appendix A (Lemma A.1) we can prove that Eq. (23) has a unique solution $\lambda > 0$. We invert relations (14), (10) and (11) in order to obtain an explicit solution of problem (1)–(5), (7)–(9) with the source terms g_j defined by (3). □

Remark 1. If the initial temperature $C = 0$ and the solid phase source $\beta_1 = 0$ then we have the one-phase Stefan problem with a constant temperature B at the fixed face $x = 0$ which is the problem considered in Menaldi, Tarzia [14]. The solution is given by

$$\begin{cases} T(x,t) = T_2(x,t) = B - \big(B + \varphi_2(\lambda)\big)\dfrac{\operatorname{erf}(\frac{x}{2a_2\sqrt{t}})}{\operatorname{erf}(\lambda)} \\ \qquad + \dfrac{2l\sqrt{\pi}}{c_2}\displaystyle\int_0^{\frac{x}{2a_2\sqrt{t}}} \beta_2(u)\exp\big(u^2\big)\left(\operatorname{erf}(u) - \operatorname{erf}\left(\frac{x}{2a_2\sqrt{t}}\right)\right) du, \\ \quad 0 < x < s(t),\ t > 0; \\ s(t) = 2\lambda a_2\sqrt{t}, \end{cases} \tag{33}$$

where λ is the unique solution of equation $F(x, \beta_2) = \frac{\mathrm{Ste}_2}{\sqrt{\pi}}$, $x > 0$.

Remark 2. In the particular case $\beta_1 = \beta_2 = 0$ we have the classic Neumann solution (see Carslaw, Jaeger [6]).

2.2. *Monotonicity properties*

We denote by $T_{\beta_1\beta_2,1}(x,t)$, $T_{\beta_1\beta_2,2}(x,t)$ and $s_{\beta_1\beta_2}(t)$ (i.e., $\lambda_{\beta_1\beta_2}$) the solution to problem (1)–(5), (7)–(9) for data β_1 and β_2. We will compare this solution with that corresponding to the case $\beta_1 = 0$ and $\beta_1 = \beta_2 = 0$.

We obtain a monotonicity property for the corresponding free-boundaries in Lemma 2 and for temperatures in Theorem 3.

Lemma 2. *If $\beta_1 \geqslant 0$ and $\beta_2 \leqslant 0$ then we have the following monotonicity properties*:

$$\text{(i)}\quad s_{0\beta_2}(t) \leqslant s_{\beta_1\beta_2}(t) \leqslant s_{\beta_1 0}(t), \quad t > 0,$$
$$\text{(ii)}\quad s_{0\beta_2}(t) \leqslant s_{00}(t) \leqslant s_{\beta_1 0}(t), \quad t > 0. \tag{34}$$

Proof. In order to prove (34) it is sufficient to show the same inequality for the coefficient λ, that is,

$$\begin{aligned} &\text{(i)}\quad \lambda_{0\beta_2} \leqslant \lambda_{\beta_1\beta_2} \leqslant \lambda_{\beta_1 0}, \\ &\text{(ii)}\quad \lambda_{0\beta_2} \leqslant \lambda_{00} \leqslant \lambda_{\beta_1 0}. \end{aligned} \tag{35}$$

We can rewrite Eq. (23) for λ by the following

$$G_1(x, \beta_1) = G_2(x, \beta_2) \tag{36}$$

where the real functions G_1 and G_2 are defined by

$$G_1(x, \beta_1) = F_0(x)\left[\mathrm{Ste}_1 + Q\left(\frac{a_2}{a_1}x\right) - 2\sqrt{\pi}\int_{\frac{a_2}{a_1}x}^{+\infty} \operatorname{erf}c(u)\beta_1(u)\exp\big(u^2\big)\,du\right], \tag{37}$$

$$G_2(x, \beta_2) = Q\left(\frac{a_2}{a_1}x\right)\left[\frac{\mathrm{Ste}_2}{\sqrt{\pi}} + 2\int_0^x \operatorname{erf}(u)\beta_2(u)\exp\big(u^2\big)\,du\right]. \tag{38}$$

Taking into account $\beta_1 \geqslant 0$ and $\beta_2 \leqslant 0$ and by comparison of functions G_1 and G_2 we obtain (35)(i), (ii). See Appendix A (Lemma A.2). □

Theorem 3. *The solution to problem* (1)–(5), (7)–(9) *for data* $\beta_1 \geqslant 0$ *and* $\beta_2 \leqslant 0$ *satisfies the following monotonicity properties*:

$$\begin{aligned}
&\text{(i)} && T_{\beta_1\beta_2,2}(x,t) \leqslant T_{\beta_1 0,2}(x,t), \quad 0 \leqslant x \leqslant s_{\beta_1\beta_2}(t),\ t>0,\\
&\text{(ii)} && T_{00,2}(x,t) \leqslant T_{\beta_1 0,2}(x,t), \quad 0 \leqslant x \leqslant s_{00}(t),\ t>0,\\
&\text{(iii)} && T_{0\beta_2,2}(x,t) \leqslant T_{00,2}(x,t), \quad 0 \leqslant x \leqslant s_{0\beta_2}(t),\ t>0,\\
&\text{(iv)} && T_{0\beta_2,1}(x,t) \leqslant T_{00,1}(x,t), \quad x > s_{00}(t),\ t>0,\\
&\text{(v)} && T_{0\beta_2,2}(x,t) \leqslant T_{\beta_1\beta_2,2}(x,t), \quad 0 \leqslant x \leqslant s_{0\beta_2}(t),\ t>0,\\
&\text{(vi)} && T_{0\beta_2,1}(x,t) \leqslant T_{\beta_1\beta_2,1}(x,t), \quad x > s_{\beta_1\beta_2}(t),\ t>0,\\
&\text{(vii)} && T_{00,1}(x,t) \leqslant T_{\beta_1 0,1}(x,t), \quad x > s_{\beta_1 0}(t),\ t>0,\\
&\text{(viii)} && T_{\beta_1\beta_2,1}(x,t) \leqslant T_{\beta_1 0,1}(x,t), \quad x > s_{\beta_1 0}(t),\ t>0.
\end{aligned} \tag{39}$$

Proof. From maximum principle we obtain (39). We will only give the proof of the property (vii).

Let $u(x,t) = T_{\beta_1 0,1}(x,t) - T_{00,1}(x,t)$. Function u satisfies the following conditions:

$$\begin{aligned}
&u_t - a_1^2 u_{xx} = \frac{l}{c_1 t}\beta_1\left(\frac{x}{2a_1\sqrt{t}}\right) \geqslant 0, \quad x > s_{\beta_1 0}(t),\ t>0,\\
&u\big(s_{\beta_1 0}(t),t\big) = -T_{00,1}\big(s_{\beta_1 0}(t),t\big) \geqslant 0, \quad t>0,\\
&u(x,0) = T_{\beta_1 0,1}(x,0) - T_{00,1}(x,0) = -C-(-C) = 0, \quad x > s_{\beta_1 0}(t).
\end{aligned}$$

Then we have $u(x,t) \geqslant 0$ for $x > s_{\beta_1 0}(t)$, $t>0$. □

These monotonicity properties can be interpreted by physical considerations and can be used in order to obtain super and sub explicit solutions for general conditions by using the maximum principle.

3. Solution of the free boundary problem with a heat flux condition on the fixed face $x = 0$

In this section we consider problem (1)–(5), (7)–(9), but condition (5) will be replaced by condition (6) (see Rogers [18], Tarzia [21]). We can define the same transformations (10), (11) and (14) as were done for the previous problem, and we obtain (15)–(19) and

$$R_2'(0) = \frac{-2q_0}{\rho c_2 a_2}. \tag{40}$$

It easy to see that the free boundary must be of the type $s(t) = 2a_2\mu\sqrt{t}$ where μ is a dimensionless constant to be determined. The solution to problem (15)–(19) and (40) is given by

$$R_1(\eta) = -\frac{(C+\varphi_1(+\infty))}{\operatorname{erf}c(\frac{a_2}{a_1}\mu)}\left[\operatorname{erf}\left(\frac{a_2}{a_1}\eta\right) - \operatorname{erf}\left(\frac{a_2}{a_1}\mu\right)\right] + \varphi_3(\eta), \quad \eta > \mu,$$

$$\varphi_3(\eta) = \frac{2l\sqrt{\pi}}{c_1} \int_{\frac{a_2}{a_1}\mu}^{\frac{a_2}{a_1}\eta} \beta_1(u) \exp(u^2) \left[\operatorname{erf}(u) - \operatorname{erf}\left(\frac{a_2}{a_1}\eta \right) \right] du \tag{41}$$

and

$$R_2(\eta) = \frac{q_0\sqrt{\pi}}{\rho c_2 a_2} \big(\operatorname{erf}(\mu) - \operatorname{erf}(\eta) \big) + \varphi_2(\eta) - \varphi_2(\mu), \quad 0 < \eta < \mu, \tag{42}$$

where φ_2 was defined in (21) and the unknown μ must satisfy the following equation

$$W(x, \beta_1) = V(x, \beta_2), \quad x > 0, \tag{43}$$

where

$$W(x, \beta_1) = \frac{x \exp(x^2)}{Q(\frac{a_2}{a_1}x)} \left[\mathrm{Ste}_1 - 2\sqrt{\pi} \int_{\frac{a_2}{a_1}x}^{+\infty} \operatorname{erf}c(u)\beta_1(u) \exp(u^2)\, du \right]$$

and

$$V(x, \beta_2) = \frac{q_0}{\rho l a_2} - x \exp(x^2) + 2 \int_0^x \beta_2(u) \exp(u^2)\, du. \tag{44}$$

Theorem 4.

(a) *If condition*

$$\int_0^{+\infty} \operatorname{erf}c(u)\beta_1(u) \exp(u^2)\, du \leqslant \frac{\mathrm{Ste}_1}{2\sqrt{\pi}} \tag{45}$$

holds then Eq. (43) *has a unique solution* $\mu > 0$ *if and only if* q_0 *satisfies the following inequality*:

$$q_0 \geqslant 2a_1\rho l \left[\frac{\mathrm{Ste}_1}{2\sqrt{\pi}} - \int_0^{+\infty} \operatorname{erf}c(u)\beta_1(u) \exp(u^2)\, du \right]. \tag{46}$$

(b) *If*

$$\int_0^{+\infty} \operatorname{erf}c(u)\beta_1(u) \exp(u^2)\, du > \frac{\mathrm{Ste}_1}{2\sqrt{\pi}} \tag{47}$$

holds, then Eq. (43) *has at least a solution* $\mu > 0\ \forall q_0 > 0$.

(c) *Under the hypothesis assumed for* β_i $(i = 1, 2)$ *given in the Introduction, the free boundary problem with sources term* (1)–(4), (6)–(9) *has an explicit solution given by*

$$T_1(x, t) = \frac{-(C + \varphi_3(+\infty))}{\operatorname{erf}c(\frac{a_2}{a_1}\mu)} \left[\operatorname{erf}\left(\frac{x}{2a_1\sqrt{t}} \right) - \operatorname{erf}\left(\frac{a_2}{a_1}\mu \right) \right] + \varphi_3\left(\frac{x}{2a_2\sqrt{t}} \right)$$
$$\text{for } x > s(t),\ t > 0, \tag{48}$$

$$T_2(x,t) = \frac{q_0\sqrt{\pi}}{\rho c_2 a_2}\left[\operatorname{erf}(\mu) - \operatorname{erf}\left(\frac{x}{2a_2\sqrt{t}}\right)\right] + \varphi_2\left(\frac{x}{2a_2\sqrt{t}}\right) - \varphi_2(\mu)$$
$$\text{for } 0 < x < s(t),\ t > 0, \tag{49}$$

where φ_3 and φ_2 are defined in (41) *and* (21) *respectively, the free boundary is given by*

$$s(t) = 2a_2\mu\sqrt{t},$$

and μ is the unique solution given in (a) *or* (b).

Proof. To prove (a) and (b) we use the definitions of the functions W and V, and Lemma A.2 (see Appendix A).

We invert relations (14), (10) and (11) in order to obtain (48)–(49). □

Remark 3. In the particular case $\beta_1 \equiv 0$ and $\beta_2 \leqslant 0$ we have that

$$\exists!\mu > 0 \quad \text{solution of Eq. (43)} \quad \Longleftrightarrow \quad q_0 > \frac{Ck_1}{a_1\sqrt{\pi}}$$

which was the result obtained by Tarzia [21].

Remark 4. Taking into account Lemma A.2 (Appendix A) we can prove the same monotonicity properties given in Section 2.2.

4. Equivalence of the two free boundary problems

We consider the solution $T_2(x,t)$ of problem (1)–(4), (6)–(9) given by (49). We compute $T_2(0,t)$ and we have

$$\begin{aligned} T_2(0,t) &= \frac{q_0\sqrt{\pi}}{\rho c_2 a_2}\operatorname{erf}(\mu) - \varphi_2(\mu) \\ &= \frac{q_0\sqrt{\pi}}{\rho c_2 a_2}\operatorname{erf}(\mu) - \frac{2l\sqrt{\pi}}{c_2}\int_0^{\mu} \beta_2(z)\exp(z^2)\big(\operatorname{erf}(z) - \operatorname{erf}(\mu)\big)\,dz \\ &= B_0(\mu) \end{aligned} \tag{50}$$

which is constant in time.

If we replace B by $B_0(\mu)$ in condition (5) and we solve problem (1)–(5), (7)–(9) we obtain the similarity solutions

$$T_1^*(x,t) = \frac{-(C+\varphi_1(+\infty))}{\operatorname{erf}c(\frac{a_2}{a_1}\lambda)}\left[\operatorname{erf}\left(\frac{x}{2a_1\sqrt{t}}\right) - \operatorname{erf}\left(\frac{a_2}{a_1}\lambda\right)\right] + \varphi_1\left(\frac{x}{2a_2\sqrt{t}}\right),$$
$$\text{for } x > s(t),\ t > 0,$$

$$T_2^*(x,t) = B_0(\mu) - \big(B_0(\mu) + \varphi_2(\lambda)\big)\frac{\operatorname{erf}(\frac{x}{2a_2\sqrt{t}})}{\operatorname{erf}(\lambda)}$$
$$+ \frac{2l\sqrt{\pi}}{c_2}\int_0^{\frac{x}{2a_2\sqrt{t}}} \beta_2(u)\exp(u^2)\left(\operatorname{erf}(u) - \operatorname{erf}\left(\frac{x}{2a_2\sqrt{t}}\right)\right)du,$$
$$\text{for } 0 < x < s(t),\ t > 0,$$

where $\varphi_1(\eta)$ and $\varphi_2(\eta)$ are defined in (22), (21) respectively and $s(t) = 2\lambda a_2\sqrt{t}$ is the free boundary. The coefficient λ must be the solution of the following equation:

$$f_1(x,\beta_1) = Q\left(\frac{a_2}{a_1}x\right)\left[\frac{\mathrm{Ste}_2^*}{\sqrt{\pi}} - F(x,\beta_2)\right], \quad x>0,\ \mathrm{Ste}_2^* = \frac{B_0(\mu)c_2}{l}. \tag{51}$$

We remark that Eq. (51) is Eq. (23) where Ste_2 has been replaced by Ste_2^*.

Theorem 5. *Under the hypotheses* (45) *and* (46) *the solution* μ *of Eq.* (43) *is also solution of Eq.* (51)*, i.e.,* $\mu = \lambda$.

Proof. We have:

$$\begin{aligned}
&\mu \text{ is a solution of Eq. (51)}\\
\iff\quad & f_1(\mu,\beta_1) = Q\left(\frac{a_2}{a_1}\mu\right)\left[\frac{B_0(\mu)c_2}{l\sqrt{\pi}} - F(\mu,\beta_2)\right]\\
\iff\quad & F_0(\mu)\left(\mathrm{Ste}_1 - 2\sqrt{\pi}\int_{\frac{a_2}{a_1}\mu}^{+\infty} \operatorname{erf}c(z)\beta_1(z)\exp(z^2)\,dz\right)\\
&\qquad = Q\left(\frac{a_2}{a_1}\mu\right)\operatorname{erf}(\mu)\left(\frac{q_0}{\rho l a_2} + 2\int_0^{\mu}\beta_2(z)\exp(z^2)\,dz - \mu\exp(\mu^2)\right)\\
\iff\quad & W(\mu,\beta_1) = V(\mu,\beta_2)\\
\iff\quad & \mu \text{ is a solution of Eq. (43), i.e., } \mu=\lambda. \qquad \square
\end{aligned}$$

Corollary 6. *The coefficient* λ *a solution of Eq.* (23) *satisfies the following inequality*:

$$\frac{B+\varphi_2(\lambda)}{\operatorname{erf}(\lambda)} \geqslant \frac{l a_1}{c_2 a_2}\left[\mathrm{Ste}_1 - 2\sqrt{\pi}\int_0^{+\infty}\operatorname{erf}c(z)\beta_1(z)\exp(z^2)\,dz\right]. \tag{52}$$

Inequality (52) is a generalization of the inequality for the coefficient which characterizes the free boundary $s(t)$ of the Neumann solution for the particular case $\beta_1 = \beta_2 = 0$ obtained in Tarzia [21], given by

$$\operatorname{erf}(\lambda) < \frac{B a_2 c_2}{C a_1 c_1} = \frac{B}{C}\sqrt{\frac{c_2 k_2}{c_1 k_1}}. \tag{53}$$

5. Study of a particular case

We study the important particular case which has been considered in Scott [20] for sublimation–dehydration with volumetric heating since it is of interest in microwave energy. Taking into account the g's internal source functions given in [20] and definition (3) we can choose in our computation the following expressions for β_i's function:

$$\beta_1(x/2a_1\sqrt{t}) = \exp\left(-(x/2a_1\sqrt{t}+d_1)^2\right), \tag{54}$$

$$\beta_2(x/2a_2\sqrt{t}) = -\exp\left(-(x/2a_2\sqrt{t}+d_2)^2\right), \quad d_1,d_2\in\mathbb{R}. \tag{55}$$

From (11) and (14) we can take from now on

$$\beta_1(\eta)=\exp\bigl(-(\eta+d_1)^2\bigr),\qquad \beta_2(\eta)=-\exp\bigl(-(\eta+d_2)^2\bigr),\quad d_1,d_2\in\mathbb{R}. \tag{56}$$

The functions φ_1, φ_2 and φ_3 defined by (22), (21) and (41) respectively, are given by

$$\begin{aligned}\varphi_1(\eta)={}&\frac{l\sqrt{\pi}}{c_1d_1}\exp\bigl(-d_1^2\bigr)\biggl[\exp\biggl(-2\frac{a_2}{a_1}\lambda d_1\biggr)\biggl(\operatorname{erf}\biggl(\frac{a_2}{a_1}\lambda\biggr)-\operatorname{erf}\biggl(\frac{a_2}{a_1}\eta\biggr)\biggr)\\&+\exp\bigl(d_1^2\bigr)\biggl(\operatorname{erf}\biggl(\frac{a_2}{a_1}\eta+d_1\biggr)-\operatorname{erf}\biggl(\frac{a_2}{a_1}\lambda+d_1\biggr)\biggr)\biggr],\quad\text{if } d_1\neq 0,\end{aligned} \tag{57}$$

$$\begin{aligned}\varphi_1(\eta)={}&\frac{2l\sqrt{\pi}}{c_1}\biggl[\frac{a_2}{a_1}\lambda\biggl(\operatorname{erf}\biggl(\frac{a_2}{a_1}\eta\biggr)-\operatorname{erf}\biggl(\frac{a_2}{a_1}\lambda\biggr)\biggr)\\&+\frac{1}{\sqrt{\pi}}\biggl(\exp\biggl(-\biggl(\frac{a_2}{a_1}\eta\biggr)^2\biggr)-\exp\biggl(-\biggl(\frac{a_2}{a_1}\lambda\biggr)^2\biggr)\biggr)\biggr],\quad\text{if } d_1=0,\end{aligned} \tag{58}$$

$$\varphi_2(\eta)=\frac{-l\sqrt{\pi}}{c_2d_2}\bigl[\operatorname{erf}(\eta+d_2)-\operatorname{erf}(d_2)-\operatorname{erf}(\eta)\exp\bigl(-d_2^2\bigr)\bigr],\quad\text{if } d_2\neq 0, \tag{59}$$

$$\varphi_2(\eta)=\frac{2l}{c_2}\bigl[1-\exp\bigl(-\eta^2\bigr)\bigr],\quad\text{if } d_2=0, \tag{60}$$

$$\begin{aligned}\varphi_3(\eta)={}&\frac{l\sqrt{\pi}}{c_1d_1}\exp\bigl(-d_1^2\bigr)\biggl[\exp\biggl(-2\frac{a_2}{a_1}\mu d_1\biggr)\biggl(\operatorname{erf}\biggl(\frac{a_2}{a_1}\mu\biggr)-\operatorname{erf}\biggl(\frac{a_2}{a_1}\eta\biggr)\biggr)\\&+\exp\bigl(d_1^2\bigr)\biggl(\operatorname{erf}\biggl(\frac{a_2}{a_1}\eta+d_1\biggr)-\operatorname{erf}\biggl(\frac{a_2}{a_1}\mu+d_1\biggr)\biggr)\biggr],\quad\text{if } d_1\neq 0,\end{aligned} \tag{61}$$

and

$$\begin{aligned}\varphi_3(\eta)={}&\frac{2l\sqrt{\pi}}{c_1}\biggl[\frac{a_2}{a_1}\mu\biggl(\operatorname{erf}\biggl(\frac{a_2}{a_1}\eta\biggr)-\operatorname{erf}\biggl(\frac{a_2}{a_1}\mu\biggr)\biggr)\\&+\frac{1}{\sqrt{\pi}}\biggl(\exp\biggl(-\biggl(\frac{a_2}{a_1}\eta\biggr)^2\biggr)-\exp\biggl(-\biggl(\frac{a_2}{a_1}\mu\biggr)^2\biggr)\biggr)\biggr],\quad\text{if } d_1=0.\end{aligned} \tag{62}$$

Theorem 7. *The explicit solution to the free boundary problem with sources term* (1)–(5)*,* (7)–(9) *is given by*

$$\begin{aligned}&T_1(x,t)=\frac{-(C+\varphi_1(+\infty))}{\operatorname{erf}c(\frac{a_2}{a_1}\lambda)}\biggl[\operatorname{erf}\biggl(\frac{x}{2a_1\sqrt{t}}\biggr)-\operatorname{erf}\biggl(\frac{a_2}{a_1}\lambda\biggr)\biggr]+\varphi_1\biggl(\frac{x}{2a_2\sqrt{t}}\biggr),\\&\quad\textit{for } x>s(t),\ t>0;\\&T_2(x,t)=\varphi_2\biggl(\frac{x}{2a_2\sqrt{t}}\biggr)+B-\bigl(B+\varphi_2(\lambda)\bigr)\frac{\operatorname{erf}(\frac{x}{2a_2\sqrt{t}})}{\operatorname{erf}(\lambda)},\\&\quad\textit{for } 0<x<s(t),\ t>0,\end{aligned} \tag{63}$$

where φ_1 *and* φ_2 *are given by* (57)–(60)*, and*

$$s(t)=2\lambda a_2\sqrt{t} \tag{64}$$

is the free boundary with λ *the unique solution of Eq.* (23)*.*

Proof. Taking into account expressions (57)–(60) we obtain the explicit expressions (63) for the temperatures T_1 and T_2. □

Theorem 8.

(a) *Inequality* (45) *is equivalent to*

$$\mathrm{Ste}_1 \geqslant 2, \quad \textit{for } d_1 \geqslant 0, \qquad \mathrm{Ste}_1 \geqslant 2\sqrt{\pi}\,P(d_1), \quad \textit{for } d_1 < 0, \tag{65}$$

where

$$P(x) = \frac{\exp(-x^2) - \operatorname{erf} c(x)}{2x}. \tag{66}$$

(b) *Inequality* (46) *is equivalent to*

$$q_0 \geqslant a_1 \rho l \left[\frac{\mathrm{Ste}_1}{\sqrt{\pi}} - \frac{1}{d_1}\bigl(\exp(-d_1^2) - \operatorname{erf} c(d_1)\bigr) \right] \quad \textit{if } d_1 \neq 0, \tag{67}$$

$$q_0 \geqslant \frac{a_1 \rho l}{\sqrt{\pi}}[\mathrm{Ste}_1 - 2] \quad \textit{if } d_1 = 0. \tag{68}$$

(c) *Inequality* (52) *is equivalent to*

$$\frac{B - \frac{l\sqrt{\pi}}{c_2 d_2}(\operatorname{erf}(\lambda + d_2) - \operatorname{erf}(d_2) - \operatorname{erf}(\lambda)\exp(-d_2^2))}{\operatorname{erf}(\lambda)} \geqslant \frac{l a_1}{c_2 a_2}\left[\mathrm{Ste}_1 - \frac{\sqrt{\pi}}{d_1}\bigl(\exp(-d_1^2) - \operatorname{erf} c(d_1)\bigr) \right] \quad \textit{if } d_1 \neq 0, \tag{69}$$

and

$$\frac{B - \frac{2l}{c_2}[1 - \exp(-\lambda^2)]}{\operatorname{erf}(\lambda)} \geqslant \frac{l a_1}{c_2 a_2}[\mathrm{Ste}_1 - 2] \quad \textit{if } d_1 = 0. \tag{70}$$

(d) *The free boundary problem with sources term* (1)–(4), (6)–(9) *has an explicit solution given by*

$$T_1(x,t) = \frac{-(C + \varphi_3(+\infty))}{\operatorname{erf} c(\frac{a_2}{a_1}\mu)}\left[\operatorname{erf}\left(\frac{x}{2a_1\sqrt{t}}\right) - \operatorname{erf}\left(\frac{a_2}{a_1}\mu\right) \right] + \varphi_3\left(\frac{x}{2a_2\sqrt{t}}\right) \quad \textit{for } x > s(t),\ t > 0; \tag{71}$$

$$T_2(x,t) = \frac{q_0\sqrt{\pi}}{\rho c_2 a_2}\left[\operatorname{erf}(\mu) - \operatorname{erf}\left(\frac{x}{2a_2\sqrt{t}}\right) \right] + \varphi_2\left(\frac{x}{2a_2\sqrt{t}}\right) - \varphi_2(\mu) \quad \textit{for } 0 < x < s(t),\ t > 0, \tag{72}$$

where φ_3 *and* φ_2 *are defined in* (61)–(62) *and* (59)–(60) *respectively, the free boundary is given by*

$$s(t) = 2a_2\mu\sqrt{t}, \tag{73}$$

and μ *is the unique solution of Eq.* (43).

Proof. (a) We have

$$\int_0^{+\infty} \operatorname{erf} c(u)\beta_1(u)\exp(u^2)\,du = \begin{cases} P(d_1) = \frac{\exp(-d_1^2) - \operatorname{erf} c(d_1)}{2d_1}, & \text{if } d_1 \neq 0, \\ \frac{1}{\sqrt{\pi}}, & \text{if } d_1 = 0, \end{cases} \tag{74}$$

where the function $P(x)$ satisfies the following properties:

$$P(0)=\frac{1}{\sqrt{\pi}}, \qquad P(+\infty)=0, \qquad P(-\infty)=0, \qquad P(x)>0 \quad \forall x.$$

Then we obtain that condition (45) is equivalent to

$$2\leqslant \mathrm{Ste}_1, \quad \text{if } d_1=0 \quad \text{or} \quad 2\sqrt{\pi}P(d_1)\leqslant \mathrm{Ste}_1, \quad \text{if } d_1\neq 0.$$

(b) To obtain (67) we replace expression (74) in (46).

(c) If we replace $\varphi_2(\lambda)$ for expressions (59) or (60) in (52) we obtain (69) or (70) respectively.

(d) Taking into account expressions (59)–(62) we obtain explicit expressions (71) and (72) for the temperatures T_1 and T_2. □

Remark 5. If we take $d_1=d_2=0$ in (56) solution (63) was given by Scott [20] by taking

$$T_d(x,t)=\frac{T_s-T_v}{B}T_2(x,t)+T_v \quad \text{and} \quad T_f(x,t)=\frac{T_v-T_i}{C}T_1(x,t)+T_v$$

where T_s, T_v and T_d were defined in Scott [20].

6. Conclusions

As regards the two-phase Stefan problem with general source terms of a similarity type in both liquid and solid phases for a semi-infinite phase-change material we have arrived at the following conclusions:

(1) An explicit solution for a constant temperature condition $B>0$ at the fixed face $x=0$ for any data has been obtained.
(2) An explicit solution for an assumed heat flux of the form $-\frac{q_0}{\sqrt{t}}$ ($q_0>0$) has been obtained for data verifying restrictions (45) and (46).
(3) The equivalence of the two previous free boundary problems has also been proved and an inequality (52) for the coefficient λ which characterizes the phase change position is obtained.
(4) An explicit solution for the particular case (56) where functions β_j ($j=1,2$) are of an exponential type which are of interest in microwave energy is obtained for any temperature boundary condition $B>0$.
(5) An explicit solution for the particular case (56) is obtained when a heat flux condition of the type (6) is imposed on $x=0$; this kind of solution there exists when the parameter q_0 satisfies the inequalities (67) and (68); this is new with respect to Scott [20].

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Appendix A. Mathematical properties of some useful functions

Lemma A.1.

(A) *Functions $Q(x)$, $F_0(x)$ and $F(x,\beta_2)$ satisfy the following properties*:

(i) $Q(0)=0, \quad Q(+\infty)=1, \quad Q'(x)>0,\ \forall x>0, \quad Q'(0)=\sqrt{\pi}.$

(ii) $F_0(0)=0, \quad F_0(+\infty)=+\infty, \quad F_0'(x)>0,\ \forall x>0.$

(iii) $F(0,\beta_2)=0, \quad F(+\infty,\beta_2)=+\infty, \quad \dfrac{\partial F}{\partial x}(x,\beta_2)>0,\ \forall x>0.$ (A.1)

(B) *Functions* $h_j(x,\beta_j)$ $(j=1,2)$ *satisfy the following properties*:

(i) $h_1(0^+,\beta_1)=\mathrm{Ste}_1-2\sqrt{\pi}\displaystyle\int_0^{+\infty}\operatorname{erf}c(u)\beta_1(u)\exp(u^2)\,du;$

(ii) $h_1(+\infty,\beta_1)=\mathrm{Ste}_1;$

(iii) $\dfrac{\partial h_1}{\partial x}(x,\beta_1)=2\sqrt{\pi}\dfrac{a_2}{a_1}\operatorname{erf}c\left(\dfrac{a_2}{a_1}x\right)\exp\left(\dfrac{a_2}{a_1}x\right)^2\beta_1\left(\dfrac{a_2}{a_1}x\right)>0, \quad \forall x>0;$

(iv) *if*

$$\int_0^{+\infty}\operatorname{erf}c(u)\beta_1(u)\exp(u^2)\,du\leqslant\frac{\mathrm{Ste}_1}{2\sqrt{\pi}} \tag{A.2}$$

then $h_1(x,\beta_1)>0,\ \forall x>0;$

(v) *if*

$$\int_0^{+\infty}\operatorname{erf}c(u)\beta_1(u)\exp(u^2)\,du>\frac{\mathrm{Ste}_1}{2\sqrt{\pi}} \tag{A.3}$$

then there exists a unique $\xi_1>0$*, such that* $h_1(\xi_1,\beta_1)=0$ *and* $h_1(x,\beta_1)$ *is negative in* $(0,\xi_1)$*, is positive in* $(\xi_1,+\infty)$*;*

(vi) $h_2(0^+,\beta_2)=\dfrac{\mathrm{Ste}_2}{\sqrt{\pi}};$

(vii) $h_2(+\infty,\beta_2)=-\infty;$

(viii) $\dfrac{\partial h_2}{\partial x}(x,\beta_2)=-\left\{\dfrac{2x}{\sqrt{\pi}}+\exp(x^2)\operatorname{erf}(x)\left[1+2x^2-2\beta_2(x)\right]\right\}<0;$

(ix) *there exist a unique* $\xi_2>0$ *such that* $h_2(\xi_2,\beta_2)=0$.

(C) (a) *Function* $f_1(x,\beta_1)$*, satisfies the following properties*:

(i) $f_1(0^+,\beta_1)=0;$

(ii) $f_1(+\infty,\beta_1)=+\infty;$

(iii) *if condition* (A.2) *is verified then* $f_1(x,\beta_1)>0\ \forall x>0,$

$$\frac{\partial f_1}{\partial x}(x,\beta_1)>0 \quad \textit{and} \quad \frac{\partial f_1}{\partial x}(0^+,\beta_1)=0^+;$$

(iv) *if condition* (A.3) *is verified then* $f_1(\xi_1,\beta_1)=0$ *and* $f_1(x,\beta_1)$ *is negative in* $(0,\xi_1)$*, and is positive in* $(\xi_1,+\infty)$*; then there exists* $x_1\in(0,\xi_1)$ *such that* $\frac{\partial f_1}{\partial x}(x_1,\beta_1)=0$*. Moreover we have* $\frac{\partial f_1}{\partial x}(x,\beta_1)>0\ \forall x>\xi_1$.

(b) *Function* $f_2(x,\beta_2)$ *satisfies the following properties*:

(i) $f_2(0^+,\beta_2)=0;$

(ii) $f_2(+\infty,\beta_2)=-\infty;$

(iii) $f_2(\xi_2, \beta_2) = 0$;

(iv) $\dfrac{\partial f_2}{\partial x}(x, \beta_2) = \dfrac{a_2}{a_1} Q'\left(\dfrac{a_2}{a_1}x\right) h_2(x, \beta_2) + Q\left(\dfrac{a_2}{a_1}x\right)\dfrac{\partial h_2}{\partial x}(x, \beta_2)$;

(v) $\dfrac{\partial f_2}{\partial x}(0^+, \beta_2) = \dfrac{a_2}{a_1}\mathrm{Ste}_2 > 0$;

(vi) *there exists* $x_2 \in (0, \xi_2)$ *such that* $\frac{\partial f_2}{\partial x}(x_2, \beta_2) = 0$;

(vii) $\frac{\partial f_2}{\partial x}(x, \beta_2) < 0,\ \forall x > \xi_2$.

Proof. (A) The properties for F_0 and Q are easy to check and the function F appears for the one-phase case which was considered in Menaldi, Tarzia [14].

(B) It easily follows from (A) and definitions (28)–(29).

(C) We use the definitions of the corresponding real functions and (A) and (B). We remark that in (a)(iv) we have $f_1(x, \beta_1) < 0\ \forall x \in (0, \xi_1)$ and in (b)(vi) we have $f_2(x, \beta_2) > 0$ in $(0, \xi_2)$. □

Lemma A.2. *Function* G_1 *has the following properties*:

(i) $G_1(0, \beta_1) = 0$,

(ii) $G_1(+\infty, \beta_1) = +\infty$,

(iii) *if condition* (A.2) *is verified then* $G_1(x, \beta_1) > 0,\ \forall x > 0$,

(iv) *if condition* (A.3) *is verified then there exists a unique* $\xi > 0$ *such that* $G_1(\xi, \beta_1) = 0$ *and* $G_1(x, \beta_1)$ *is* negative *in* $(0, \xi)$, G_1 *is* positive *in* $(\xi, +\infty)$,

(v) $G_1(0, 0) = 0$,

(vi) $G_1(+\infty, 0) = +\infty$,

(vii) $\dfrac{\partial G_1}{\partial x}(x, 0) > 0, \quad \forall x > 0, \quad$ *and* $\quad \dfrac{\partial G_1}{\partial x}(0, 0) = 0$.

Function G_2 *has the following properties*:

(i) $G_2(0, \beta_2) = 0$,

(ii) $G_2(0, 0) = 0$,

(iii) $G_2(+\infty, 0) = \frac{\mathrm{Ste}_2}{\sqrt{\pi}}$,

(iv) $G_2(+\infty, \beta_2) = \dfrac{\mathrm{Ste}_2}{\sqrt{\pi}} + 2\displaystyle\int_0^{+\infty} \mathrm{erf}(u)\beta_2(u)\exp\left(u^2\right) du$,

(v) $\frac{\partial G_2}{\partial x}(x, 0) > 0\ \forall x > 0$,

(vi) $G_2(x, \beta_2) \leqslant G_2(x, 0)\ \forall x \geqslant 0$.

Lemma A.3.

(a) *Function* $W(x, \beta_1)$ *satisfies the following properties*:

(i) $W(0, \beta_1) = \dfrac{a_1}{a_2\sqrt{\pi}}\left[\mathrm{Ste}_1 - 2\sqrt{\pi}\displaystyle\int_0^{+\infty} \mathrm{erf}\,c(u)\beta_1(u)\exp\left(u^2\right) du\right]$,

(ii) *$W(+\infty, \beta_1) = +\infty$,*
(iii) *$W(x, \beta_1) \leqslant W(x, 0)$, $\forall x > 0$, $\beta_1 > 0$,*
(iv) *if condition* (A.2) *is verified then $W(0, \beta_1) \geqslant 0$ and*

$$\frac{\partial W}{\partial x}(x, \beta_1) > 0, \quad \forall x > 0,$$

(v) *if condition* (A.3) *is verified then $W(0, \beta_1) < 0$.*

(b) *Function $V(x, \beta_2)$ satisfies the following properties*:
(i) *$V(0, \beta_2) = \frac{q_0}{\rho l a_2}$,*
(ii) *$V(+\infty, \beta_2) = -\infty$,*
(iii) *$\frac{\partial V}{\partial x}(x, \beta_2) < 0$, $\forall x > 0$,*
(iv) *$V(x, \beta_2) \leqslant V(x, 0)$, $\forall x > 0$, $\beta_2 < 0$.*

Proof. In order to prove (a)(iii) we use that $Q'(x)$ is given by $Q'(x) = \frac{Q(x)(1+2x^2)-2x^2}{x}$.
We demonstrate the other properties by elementary computations. □

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