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Existence, uniqueness, and convergence of optimal control problems associated with parabolic variational inequalities of the second kind

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ABSTRACT

Let u_g be the unique solution of a parabolic variational inequality of second kind, with a given g. Using a regularization method, we prove, for all g_1 and g_2, a monotony property between $\mu u_{g_1} + (1-\mu)u_{g_2}$ and $u_{\mu g_1+(1-\mu)g_2}$ for $\mu \in [0, 1]$. This allowed us to prove the existence and uniqueness results to a family of optimal control problems over g for each heat transfer coefficient $h > 0$, associated with the Newton law, and of another optimal control problem associated with a Dirichlet boundary condition. We prove also, when $h \to +\infty$, the strong convergence of the optimal controls and states associated with this family of optimal control problems with the Newton law to that of the optimal control problem associated with a Dirichlet boundary condition.

1. Introduction

Consider the following problem governed by the parabolic variational inequality

$$\langle \dot{u}(t), v - u(t)\rangle + a(u(t), v - u(t)) + \Phi(v) - \Phi(u(t)) \geq \langle g(t), v - u(t)\rangle \quad \forall v \in K, \tag{1.1}$$

a.e. $t \in]0, T[$, with the initial condition

$$u(0) = u_b, \tag{1.2}$$

where, a is a symmetric continuous and coercive bilinear form on the Hilbert space $V \times V$, Φ is a proper and convex function from V into $\mathbb{R}$ and is lower semi-continuous for the weak topology on V, $\langle\cdot,\cdot\rangle$ denotes the duality brackets between V' and V, K is a closed convex non-empty subset of V, u_b is an initial value in another Hilbert space H with V being densely and continuously embedded in H, and g is a given function in the space $L^2(0, T, V')$. It is well known [1–4] that, there exists a unique solution

$$u \in \mathcal{C}(0, T, H) \cap L^2(0, T, V) \quad \text{with } \dot{u} = \frac{\partial u}{\partial t} \in L^2(0, T, H)$$

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to (1.1)–(1.2). So we can consider $g \mapsto u_g$ as a function from $L^2(0, T, H)$ to $\mathcal{C}(0, T, H) \cap L^2(0, T, V)$. Then we can consider [5–7] the cost functional J defined by

$$J(g) = \frac{1}{2}\|u_g\|^2_{L^2(0,T,H)} + \frac{M}{2}\|g\|^2_{L^2(0,T,H)}, \tag{1.3}$$

where M is a positive constant, and u_g is the unique solution to (1.1)–(1.2), corresponding to the control g. One of our main purposes is to prove the existence and uniqueness of the optimal control problem

$$\text{Find } g_{\text{op}} \in L^2(0, T, H) \quad \text{such that } J(g_{\text{op}}) = \min_{g \in L^2(0,T,H)} J(g). \tag{1.4}$$

This can be reached if we prove the strict convexity of the cost functional J, which follows (see Theorem 3.1) from the following monotony property: *for any two control g_1 and g_2 in $L^2(0, T, H)$*,

$$u_4(\mu) \le u_3(\mu) \quad \forall \mu \in [0, 1], \tag{1.5}$$

where

$$u_3(\mu) = \mu u_1 + (1-\mu)u_2, \qquad u_4(\mu) = u_{g_3(\mu)}, \quad \text{with } g_3(\mu) = \mu g_1 + (1-\mu)g_2. \tag{1.6}$$

In Section 2, first we establish in Theorem 2.2, the error estimate between $u_3(\mu)$ and $u_4(\mu)$. This result generalizes our previous result obtained in [8] for the elliptic variational inequalities. We deduce in Corollary 2.3 a condition on the data to get $u_3(\mu) = u_4(\mu)$ for all $\mu \in [0, 1]$. Then we assume that the convex K is a subset of $V = H^1(\Omega)$ and consider the parabolic variational problems (P) and (P$_h$). So, using a regularization method, we prove in Theorem 2.5 this monotony property (1.5), for the solutions of the two problems (P) and (P$_h$). This result with a new proof and simplified, generalizes that obtained by [9] for elliptic variational inequalities. In Section 2.1 we also obtain some properties of dependency solutions based on the data g and on a positive parameter h for the parabolic variational inequalities (1.1) and (2.1); see Propositions 2.6–2.8.

In Section 3, we consider the family of distributed optimal control problems $(P_h)_{h>0}$,

$$\text{Find } g_{\text{op}_h} \in L^2(0, T, H) \quad \text{such that } J(g_{\text{op}_h}) = \min_{g \in L^2(0,T,H)} J_h(g), \tag{1.7}$$

with the cost functional

$$J_h(g) = \frac{1}{2}\|u_{g_h}\|^2_{L^2(0,T,H)} + \frac{M}{2}\|g\|^2_{L^2(0,T,H)}, \tag{1.8}$$

where u_{g_h} is the unique solution of (2.1)–(1.2), corresponding to the control g for each $h > 0$. Using Theorem 2.5 with its crucial property of monotony (1.5), we prove the strict convexity of the cost functional (1.3) and also of the cost functional (1.8), associated with problems (1.4) and (1.7) respectively. Then, the existence and uniqueness of solutions to the optimal control problems (1.4) and (1.7) follow from [6].

In general, see for example [10] the relevant physical condition, to impose on the boundary, is the Newton law, or the Robin law, and not the Dirichlet. Therefore, the objective of this work is to approximate the optimal control problem (1.4), where the state is the solution to parabolic variational problems (1.1)–(1.2) associated with the Dirichlet condition (2.2), by a family indexed by a factor h of optimal control problem (2.1)–(1.2), where states are the solutions to parabolic variational problems, associated with the boundary condition of Newton (2.3). Moreover, from a numerical analysis point of view it maybe preferable to consider approximating Neumann problems in all space V (see (2.1)–(1.2)), with parameter h, rather than the Dirichlet problem in a subset of the space V (see (1.1)–(1.2)). So the asymptotic behavior can be considered very important in the optimal control.

In the last Section 3.1, which is also the goal of our paper, we prove that the optimal control g_{op_h} (unique solution of the optimization problem (1.7)) and its corresponding state $u_{g_{\text{op}_h}h}$ (the unique solution of the parabolic variational problem (2.1)–(1.2)) for each $h > 1$, are strongly convergent to g_{op} (the unique solution of the optimization problem (1.4)), and $u_{g_{\text{op}}}$ (the unique solution of the parabolic variational problem (1.1)–(1.2)) in $L^2([0, T] \times \Omega)$ and $L^2(0, T, H^1(\Omega))$ respectively when $h \to +\infty$.

This paper generalizes the results obtained in [11], for elliptic variational equalities, and in [12] for parabolic variational equalities, to the case of parabolic variational inequalities of second kind. Various problems with distributed optimal control, associated with elliptic variational inequalities are given; see for example [13–19,9,20–22] and for the parabolic case see for example [23,14,24–30].

2. On the property of monotony

As we cannot prove the property of monotony (1.5) for any convex set K. Let Ω a bounded open set in $\mathbb{R}^N$ with smooth boundary $\partial\Omega = \Gamma_1 \cup \Gamma_2$. We assume that $\Gamma_1 \cap \Gamma_2 = \emptyset$, and meas$(\Gamma_1) > 0$. Let $H = L^2(\Omega)$, $V = H^1(\Omega)$. We can prove the property of monotony (1.5) for any convex subset of V. Let

$$K = \{v \in V : v_{|\Gamma_1} = 0\}, \quad \text{and} \quad u_b \in K_b = \{v \in V : v_{|\Gamma_1} = b\}.$$

So we consider the following variational problems with such a convex subset.

Problem (P). Let given $b \in L^2(]0,T[\times\Gamma_1)$, $g \in L^2(0,T,H)$ and $q \in L^2(]0,T[\times\Gamma_2)$, $q > 0$. Find u in $\mathcal{C}([0,T],H)\cap L^2(0,T,K_b)$ solution of the parabolic problem (1.1), where $\langle\cdot,\cdot\rangle$ is only the scalar product $(\cdot,\cdot)$ in H, with the initial condition (1.2), and $\Phi(v) = \int_{\Gamma_2} q|v|\mathrm{d}s$.

Problem (P_h)**.** Let given $b \in L^2(]0,T[\times\Gamma_1)$, $g \in L^2(0,T,H)$ and $q \in L^2(]0,T[\times\Gamma_2)$, $q > 0$. For all coefficient $h > 0$, find $u \in \mathcal{C}(0,T,H)\cap L^2(0,T,V)$ solution of the parabolic variational inequality

$$\langle \dot{u}(t), v-u(t)\rangle + a_h(u(t), v-u(t)) + \Phi(v) - \Phi(u(t)) \geq (g(t), v-u(t)) + h\int_{\Gamma_1} b(t)(v-u(t))\mathrm{d}s \quad \forall v \in V, \tag{2.1}$$

and the initial condition (1.2), where $a_h(u,v) = a(u,v) + h\int_{\Gamma_1} uv\mathrm{d}s$.

It is easy to see that Problem (P) is with the Dirichlet condition

$$u = b \quad \text{on } \Gamma_1\times]0,T[, \tag{2.2}$$

and Problem (P_h) is with the following Newton–Robin type condition

$$-\frac{\partial u}{\partial n} = h(u-b) \quad \text{on } \Gamma_1\times]0,T[\tag{2.3}$$

where n is the exterior unit vector normal to the boundary. The integral on Γ_2 in the expression of Φ comes from the Tresca boundary condition (see [31–33,4]) with q is the Tresca friction coefficient on Γ_2. Note that only for the proof of Theorem 2.5 we need to specify an expression of the functional Φ.

By the assumption there exists $\lambda > 0$ such that $\lambda\|v\|_V^2 \leq a(v,v)\ \forall v \in V$. Moreover, it follows from [34,35] that there exists $\lambda_1 > 0$ such that

$$a_h(v,v) \geq \lambda_h\|v\|_V^2 \quad \forall v \in V,\ \text{with } \lambda_h = \lambda_1\min\{1,h\}$$

so a_h is a bilinear, continuous, symmetric and coercive form on V. So there exists a unique solution to each of the two problems (P) and (P_h).

We recall that u_g is the unique solution of the parabolic variational problem (P), corresponding to the control $g \in L^2(0,T,H)$, and also that u_{g_h} is the unique solution of the parabolic variational problem (P_h), corresponding to the control $g \in L^2(0,T,H)$.

Proposition 2.1. *Assume that $g \geq 0$ in $\Omega\times]0,T[$, $b \geq 0$ on $\Gamma_1\times]0,T[$, $u_b \geq 0$ in Ω. Then as $q > 0$, we have $u_g \geq 0$. Assuming again that $h > 0$, then $u_{g_h} \geq 0$ in $\Omega\times]0,T[$.*

Proof. For $u = u_{g_h}$, it is enough to take $v = u^+$ in (2.1), to get

$$\begin{aligned}&\|u^-(T)\|^2_{L^2(\Omega)} + \lambda\int_0^T \|u^-(t)\|_V^2\mathrm{d}t + h\int_0^T\int_{\Gamma_1}(u^-(t))^2\mathrm{d}s\mathrm{d}t\\ &\quad\leq -\int_0^T (g(t),u^-(t))\mathrm{d}t - \int_0^T\int_{\Gamma_2} q(|u(t)| - |u^+(t)|)\mathrm{d}s\mathrm{d}t - h\int_0^T\int_{\Gamma_1} b(t)u^-(t)\mathrm{d}s\mathrm{d}t + \|u^-(0)\|^2_{L^2(\Omega)}\end{aligned} \tag{2.4}$$

so the result follows. □

Theorem 2.2. *Let u_1 and u_2 be two solutions of the parabolic variational inequality (1.1) with the same initial condition, and corresponding to the two control g_1 and g_2 respectively. We have the following estimate*

$$\begin{aligned}&\frac{1}{2}\|u_4(\mu) - u_3(\mu)\|^2_{L^\infty(0,T,H)} + \lambda\|u_4(\mu) - u_3(\mu)\|^2_{L^2(0,T,V)} + \mu\mathcal{I}_{14}(\mu)(T) + (1-\mu)\mathcal{I}_{24}(\mu)(T)\\ &\quad + \mu\Phi(u_1) + (1-\mu)\Phi(u_2) - \Phi(u_3(\mu)) \leq \mu(1-\mu)(\mathcal{A}(T,g_1) + \mathcal{B}(T,g_2)) \quad \forall\mu \in [0,1],\end{aligned}$$

where

$$\mathcal{I}_{j4}(\mu)(T) = \int_0^T I_{j4}(\mu)(t)\mathrm{d}t \quad \textit{for } j = 1,2, \qquad \mathcal{A}(T,g_1) = \int_0^T \alpha(t)\mathrm{d}t, \qquad \mathcal{B}(T,g_2) = \int_0^T \beta(t)\mathrm{d}t,$$

$$I_{j4}(\mu) = \langle \dot{u}_j, u_4(\mu) - u_j\rangle + a(u_j, u_4(\mu) - u_j) + \Phi(u_4(\mu)) - \Phi(u_j) - \langle g_j, u_4(\mu) - u_j\rangle \geq 0,$$

$$\alpha = \langle \dot{u}_1, u_2 - u_1\rangle + a(u_1, u_2 - u_1) + \Phi(u_2) - \Phi(u_1) - \langle g_1, u_2 - u_1\rangle \geq 0, \tag{2.5}$$

$$\beta = \langle \dot{u}_2, u_1 - u_2\rangle + a(u_2, u_1 - u_2) + \Phi(u_1) - \Phi(u_2) - \langle g_2, u_1 - u_2\rangle \geq 0. \tag{2.6}$$

Proof. As $u_3(\mu)(t) \in K$ so with $v = u_3(\mu)(t)$, in the variational inequality (1.1) where $u = u_4(\mu)$ and $g = g_3(\mu)$, we obtain

$$\begin{aligned}&\langle \dot{u}_4(\mu), u_3(\mu) - u_4(\mu)\rangle + a(u_4(\mu), u_3(\mu) - u_4(\mu)) + \Phi(u_3(\mu)) - \Phi(u_4(\mu))\\ &\quad \geq \langle g_3(\mu), u_3(\mu) - u_4(\mu)\rangle \quad \text{a.e. } t \in]0, T[,\end{aligned}$$

then

$$\begin{aligned}&\langle \dot{u}_4(\mu) - \dot{u}_3(\mu), u_4(\mu) - u_3(\mu)\rangle + a(u_4(\mu) - u_3(\mu), u_4(\mu) - u_3(\mu))\\ &\quad \leq \langle \dot{u}_3(\mu), u_3(\mu) - u_4(\mu)\rangle + a(u_3(\mu), u_3(\mu) - u_4(\mu)) + \Phi(u_3(\mu)) - \Phi(u_4(\mu)(t))\\ &\qquad - \langle g_3(\mu), u_3(\mu) - u_4(\mu)\rangle \quad \text{a.e. } t \in]0, T[,\end{aligned}$$

thus

$$\begin{aligned}\frac{1}{2}\frac{\partial}{\partial t}\left(\|u_4(\mu) - u_3(\mu)\|_H^2\right) + \lambda\|u_4(\mu) - u_3(\mu)\|_V^2 \leq{}& \langle \dot{u}_3(\mu), u_3(\mu) - u_4(\mu)\rangle\\ &+ a(u_3(\mu), u_3(\mu) - u_4(\mu)) + \Phi(u_3(\mu)) - \Phi(u_4(\mu))\\ &- \langle g_3(\mu), u_3(\mu) - u_4(\mu)\rangle, \quad \text{a.e. } t \in]0, T[,\end{aligned}$$

using that $u_3(\mu) = \mu(u_1 - u_2) + u_2$, $g_3(\mu) = \mu(g_1 - g_2) + g_2$ we get

$$\begin{aligned}&\frac{1}{2}\frac{\partial}{\partial t}\left(\|u_4(\mu) - u_3(\mu)\|_H^2\right) + \lambda\|u_4(\mu) - u_3(\mu)\|_V^2 + \mu\Phi(u_1) + (1-\mu)\Phi(u_2) - \Phi(u_3(\mu))\\ &\quad \leq \mu(1-\mu)(\alpha+\beta) - \mu I_{14}(\mu) - (1-\mu)I_{24}(\mu) \quad \text{a.e. } t \in]0, T[,\end{aligned}$$

so by integration between $t = 0$ and $t = T$, we deduce the required result. □

Corollary 2.3. *From Theorem 2.2 we get a.e. $t \in [0, T]$*

$$\mathcal{A}(T, g_1) = \mathcal{B}(T, g_2) = 0 \Rightarrow \begin{cases} u_3(\mu) = u_4(\mu) \quad \forall \mu \in [0, 1],\\ I_{14}(\mu) = I_{24}(\mu) = 0 \quad \forall \mu \in [0, 1],\\ \Phi(u_3(\mu)) = \mu\Phi(u_1) + (1-\mu)\Phi(u_2) \quad \forall \mu \in [0, 1].\end{cases}$$

Lemma 2.4. *Let u_1 and u_2 be two solutions of the parabolic variational inequality of second kind (1.1) with respectively as second members g_1 and g_2, then we get*

$$\|u_1 - u_2\|^2_{L^\infty(0,T,H)} + \lambda\|u_1 - u_2\|^2_{L^2(0,T,V)} \leq \frac{1}{\lambda}\|g_1 - g_2\|^2_{L^2(0,T,V')} \tag{2.7}$$

where λ is the coerciveness constant of the bilinear form a.

Proof. Taking $v = u_2$ in (1.1) where $u = u_1$ and $g = g_1$; then $v = u_1$ in (1.1) where $u = u_2$ and $g = g_2$, so by addition (2.7) holds. □

We generalize now in our case the result on a monotony property, obtained by [9] for the elliptic variational inequality. This theorem is the cornerstone to prove the strict convexity of the cost functional J defined in problem (1.4) and the cost functional J_h defined in problem (1.7). Remark first that with the duality brackets $\langle\cdot,\cdot\rangle$ defined by

$$\langle g(t), \varphi\rangle = (g(t), \varphi) + h\int_{\Gamma_1} b(t)\varphi \mathrm{d}s$$

(2.1) leads to (1.1). We prove the following theorem for Φ such that $\Phi(v) = \int_{\Gamma_2} q|v|\mathrm{d}s$.

Theorem 2.5. *For any two control g_1 and g_2 in $L^2(0, T, H)$, it holds that*

$$u_4(\mu) \leq u_3(\mu) \quad \text{in } \Omega \times [0, T],\ \forall \mu \in [0, 1]. \tag{2.8}$$

Here $u_4(\mu) = u_{\mu g_1 + (1-\mu)g_2}$, $u_3(\mu) = \mu u_{g_1} + (1-\mu)u_{g_2}$, $u_1 = u_{g_1}$ and $u_2 = u_{g_2}$ are the unique solutions of the variational problem (P), with $g = g_1$ and $g = g_2$ respectively, and for the same q, and the same initial condition (1.2). Moreover, it holds also that

$$u_{h4}(\mu) \leq u_{h3}(\mu) \quad \text{in } \Omega \times [0, T],\ \forall \mu \in [0, 1]. \tag{2.9}$$

Here $u_{4h}(\mu) = u_{\mu g_{1h} + (1-\mu)g_{2h}}$, $u_{3h}(\mu) = \mu u_{g_{1h}} + (1-\mu)u_{g_{2h}}$, $u_{1h} = u_{g_{1h}}$ and $u_{h2} = u_{g_{h2}}$ are the unique solutions of the variational problem (P_h), with $g = g_1$ and $g = g_2$ respectively, and for the same q, h, b and the same initial condition (1.2).

Proof. The main difficulty, to prove this result comes from the fact that the functional Φ is not differentiable. To overcome this difficulty, we use the regularization method and consider for $\varepsilon > 0$ the following approach of Φ

$$\Phi_\varepsilon(v) = \int_{\Gamma_2} q\sqrt{\varepsilon^2 + |v|^2}\mathrm{d}s, \quad \forall v \in V,$$

which is Gateaux differentiable, with

$$\langle \Phi'_\varepsilon(w), v\rangle = \int_{\Gamma_2} \frac{qwv}{\sqrt{\varepsilon^2 + |w|^2}}\mathrm{d}s \quad \forall (w, v) \in V^2.$$

Let u^ε be the unique solution of the variational inequality

$$\langle \dot{u}^\varepsilon, v - u^\varepsilon\rangle + a(u^\varepsilon, v - u^\varepsilon) + \langle \Phi'_\varepsilon(u^\varepsilon), v - u^\varepsilon\rangle \geq \langle g, v - u^\varepsilon\rangle \quad \text{a.e. } t \in [0, T]\ \forall v \in K, \text{ and } u^\varepsilon(0) = u_b. \tag{2.10}$$

Let us show first that for all $\mu \in [0, 1] u_4^\varepsilon(\mu) \leq u_3^\varepsilon(\mu)$, then that $u_3^\varepsilon(\mu) \to u_3(\mu)$ and $u_4^\varepsilon(\mu) \to u_4(\mu)$ strongly in $L^2(0, T; H)$ when $\varepsilon \to 0$. Indeed for all $\mu \in [0, 1]$, let consider $U_\varepsilon(\mu) = u_4^\varepsilon(\mu) - u_3^\varepsilon(\mu)$ thus $u_4^\varepsilon(\mu)(t) - U_\varepsilon^+(\mu)(t)$ is in K. So we can take $v = u_4^\varepsilon(\mu)(t) - U_\varepsilon^+(\mu)(t)$ in (2.10) where $u^\varepsilon = u_4^\varepsilon(\mu)$ and $g = g_3(\mu) = \mu(g_1 - g_2) + g_2$. We also can take $v = u_1^\varepsilon(t) + U_\varepsilon^+(\mu)(t)$ in (2.10) where $u^\varepsilon = u_1^\varepsilon$ and $g = g_1$, and we multiply the two sides of the obtained inequality by μ then we take $v = u_2^\varepsilon + U_\varepsilon^+(\mu)$ in (2.10) where $u^\varepsilon = u_2^\varepsilon$ and $g = g_2$ and we multiply the two sides of the obtained inequality by $(1 - \mu)$. By adding the three obtained inequalities we get a.e. $t \in]0, T[$,

$$\frac{1}{2}\frac{\partial}{\partial t}(\|U_\varepsilon^+(\mu)\|_H^2) + \lambda\|U_\varepsilon^+(\mu)\|_V^2 \leq \langle \mu\Phi'_\varepsilon(u_1^\varepsilon) + (1 - \mu)\Phi'_\varepsilon(u_2^\varepsilon) - \Phi'_\varepsilon(u_4^\varepsilon(\mu)), U_\varepsilon^+(\mu)\rangle,$$

hence as $U_\varepsilon^+(\mu)(0) = 0$, by integration from $t = 0$ to $t = T$ we obtain a.e. $t \in]0, T[$

$$\frac{1}{2}\|U_\varepsilon^+(\mu)(T)\|_H^2 + \lambda \int_0^T \|U_\varepsilon^+(\mu)(t)\|_V^2\mathrm{d}t \leq \int_0^T \langle \mu\Phi'_\varepsilon(u_1^\varepsilon(t)) + (1 - \mu)\Phi'_\varepsilon(u_2^\varepsilon(t)) - \Phi'_\varepsilon(u_4^\varepsilon(\mu)(t)), U_\varepsilon^+(\mu)(t)\rangle\mathrm{d}t.$$

As

$$\begin{aligned}\langle \mu\Phi'_\varepsilon(u_1^\varepsilon) + (1 - \mu)\Phi'_\varepsilon(u_2^\varepsilon) - \Phi'_\varepsilon(u_4^\varepsilon(\mu)), U_\varepsilon^+(\mu)\rangle &= \int_{\Gamma'_2} \frac{q\mu u_1^\varepsilon U_\varepsilon^+(\mu)}{\sqrt{\varepsilon^2 + |u_1^\varepsilon|^2}}\mathrm{d}s + \int_{\Gamma'_2} \frac{q(1 - \mu)u_2^\varepsilon U_\varepsilon^+(\mu)}{\sqrt{\varepsilon^2 + |u_2^\varepsilon|^2}}\mathrm{d}s \\ &\quad - \int_{\Gamma'_2} \frac{qu_4^\varepsilon(\mu)U_\varepsilon^+(\mu)}{\sqrt{\varepsilon^2 + |u_4^\varepsilon|^2}}\mathrm{d}s\end{aligned}$$

where $\Gamma'_2 = \Gamma_2 \cap \{u_4^\varepsilon(\mu) > u_3^\varepsilon(\mu)\}$. The function $x \mapsto \psi(x) = \frac{x}{\sqrt{\varepsilon^2+x^2}}$ for $x \in \mathbb{R}$ is increasing $\left(\psi'(x) = \varepsilon^2(\varepsilon^2 + x^2)^{\frac{-3}{2}} > 0\right)$ so

$$\begin{aligned}&\int_{\Gamma'_2} \frac{q\mu u_1^\varepsilon U_\varepsilon^+(\mu)}{\sqrt{\varepsilon^2 + \|u_1^\varepsilon\|_{\mathbb{R}^N}^2}}\mathrm{d}s + \int_{\Gamma'_2} \frac{q(1 - \mu)u_2^\varepsilon U_\varepsilon^+(\mu)}{\sqrt{\varepsilon^2 + |u_2^\varepsilon|^2}}\mathrm{d}s - \int_{\Gamma'_2} \frac{qu_4^\varepsilon(\mu)U_\varepsilon^+(\mu)}{\sqrt{\varepsilon^2 + |u_4^\varepsilon|^2}}\mathrm{d}s \\ &\quad \leq \int_{\Gamma'_2} \frac{q\mu u_1^\varepsilon U_\varepsilon^+(\mu)}{\sqrt{\varepsilon^2 + |u_1^\varepsilon|^2}}\mathrm{d}s + \int_{\Gamma'_2} \frac{q(1 - \mu)u_2^\varepsilon U_\varepsilon^+(\mu)}{\sqrt{\varepsilon^2 + |u_2^\varepsilon|^2}}\mathrm{d}s - \int_{\Gamma'_2} \frac{qu_3^\varepsilon(\mu)U_\varepsilon^+(\mu)}{\sqrt{\varepsilon^2 + |u_3^\varepsilon|^2}}\mathrm{d}s.\end{aligned}$$

Moreover, the function ψ is concave on $\mathbb{R}^+ \setminus \{0\}$ $\left(\psi''(x) = -3\varepsilon^2 x(\varepsilon^2 + x^2)^{\frac{-5}{2}} < 0\right)$ thus

$$\frac{1}{2}\|U^+(\mu)(T)\|_H^2 + \lambda \int_0^T \|U^+(\mu)(t)\|_V^2\mathrm{d}t \leq 0. \tag{2.11}$$

As $U_\varepsilon^+(\mu) = 0$ on $\{\Gamma_2 \times [0, T]\} \cap \{u_4^\varepsilon(\mu) \leq u_3^\varepsilon(\mu)\}$ so

$$u_4^\varepsilon(\mu) \leq u_3^\varepsilon(\mu) \quad \forall \mu \in [0, 1]. \tag{2.12}$$

Now we must prove that $u_3^\varepsilon(\mu) \to u_3(\mu)$ and $u_4^\varepsilon(\mu) \to u_4(\mu)$ strongly in $L^2(0, T; H)$ when $\varepsilon \to 0$. Taking $v = u_b \in K$ with $u^\varepsilon = u_i^\varepsilon$ $(i = 1, 2)$ in (2.10), we deduce that

$$\langle \dot{u}_i^\varepsilon, u_i^\varepsilon - u_b\rangle + a(u_i^\varepsilon - u_b, u_i^\varepsilon - u_b) + \langle \Phi'_\varepsilon(u_i^\varepsilon), u_i^\varepsilon\rangle \leq a(u_b, u_b - u_i^\varepsilon) + \langle \Phi'_\varepsilon(u_i^\varepsilon), u_b\rangle - \langle g_i, u_b - u_i^\varepsilon\rangle.$$

As

$$\langle \Phi'_\varepsilon(u_i^\varepsilon), u_i^\varepsilon\rangle \geq 0 \quad \text{and} \quad |\langle \Phi'_\varepsilon(u_i^\varepsilon), u_b\rangle| \leq \int_{\Gamma_2} q|u_b|\mathrm{d}s$$

we deduce, using the Cauchy–Schwarz inequality, that $\|u_i^\varepsilon\|_{L^2(0,T;V)}$ so also $\|u_3^\varepsilon(\mu)\|_{L^2(0,T;V)}$ are bounded independently from ε. By Theorem 2.2 we get

$$\frac{1}{2}\|u_3^\varepsilon(\mu)-u_4^\varepsilon(\mu)\|_{L^\infty(0,T;H)}+\lambda\|u_3^\varepsilon(\mu)-u_4^\varepsilon(\mu)\|_{L^2(0,T;V)}\le\mu(1-\mu)(\mathcal{A}^\varepsilon(T,g_1)+\mathcal{B}^\varepsilon(T,g_2))$$
$$\le\mu(1-\mu)\frac{1}{2}\left(\|g_1-g_2\|^2_{L^2(0,T;H)}+\|u_1^\varepsilon-u_2^\varepsilon\|^2_{L^2(0,T;H)}\right)\quad\forall\mu\in[0,1],$$

thus $\|u_4^\varepsilon(\mu)\|_{L^2(0,T;V)}$ is also bounded independently from ε. So there exists $l_i\in V$, for $i=1,\dots,4$, such that

$$u_i^\varepsilon\rightharpoonup l_i\quad\text{in } L^2(0,T;V)\text{ weak, and in } L^\infty(0,T;H)\text{ weak star.}\tag{2.13}$$

Now we check that $l_i=u_i$. Indeed for $i=1,2$ or 4 and as Φ is convex functional, we have

$$\begin{aligned}\langle\dot u_i^\varepsilon,v-u_i^\varepsilon\rangle+a(u_i^\varepsilon,v-u_i^\varepsilon)+\Phi_\varepsilon(v)-\Phi_\varepsilon(u_i^\varepsilon)&\ge\langle\dot u_i^\varepsilon,v-u_i^\varepsilon\rangle+a(u_i^\varepsilon,v-u_i^\varepsilon)+\langle\Phi_\varepsilon'(u_i^\varepsilon),v-u_i^\varepsilon\rangle\\&\ge\langle g_i,v-u_i^\varepsilon\rangle,\quad\text{a.e. } t\in]0,T[,\end{aligned}$$

thus

$$\langle\dot u_i^\varepsilon,v-u_i^\varepsilon\rangle+a(u_i^\varepsilon,v-u_i^\varepsilon)+\Phi_\varepsilon(v)-\Phi_\varepsilon(u_i^\varepsilon)\ge\langle g_i,v-u_i^\varepsilon\rangle,\quad\text{a.e. } t\in]0,T[.\tag{2.14}$$

Taking $v=u_i^\varepsilon\pm\varphi$, in (2.14) we have

$$\langle\dot u_i^\varepsilon,\varphi\rangle=-a(u_i^\varepsilon,\varphi)+\langle g_i,\varphi\rangle,\quad\forall\varphi\in L^2(0,T,H_0^1(\Omega)).\tag{2.15}$$

As $H_0^1(\Omega)\subset V$ with continuous inclusion but not dense, so V' (the topological dual of the space V) is not identifiable with a subset of $H^{-1}(\Omega)$. However, following [12] we can use the Hahn–Banach Theorem in order to extend any element in $H^{-1}(\Omega)$ to an element of V' preserving its norm. So from (2.13) and (2.15) we conclude that

$$\left.\begin{array}{l}u_i^\varepsilon\rightharpoonup l_i\quad\text{in } L^2(0,T,V)\text{ weak, in } L^\infty(0,T,H)\text{ weak star,}\\ \text{and}\quad\dot u_i^\varepsilon\rightharpoonup\dot l_i\text{ in } L^2(0,T,V')\text{ weak}\end{array}\right\}.\tag{2.16}$$

Then from (2.14), and following [4,36] we can write

$$\begin{aligned}\int_0^T\left\{\langle\dot u_i^\varepsilon,v\rangle+a(u_i^\varepsilon,v)+\Phi_\varepsilon(v)-\langle g_i,v-u_i^\varepsilon\rangle\right\}dt&\ge\int_0^T\left\{\langle\dot u_i^\varepsilon,u_i^\varepsilon\rangle+a(u_i^\varepsilon,u_i^\varepsilon)+\Phi_\varepsilon(u_i^\varepsilon)\right\}dt\\&=\frac{1}{2}\|u_i^\varepsilon(T)\|_H^2-\frac{1}{2}\|u_b(T)\|_H^2+\int_0^T\left\{a(u_i^\varepsilon,u_i^\varepsilon)+\Phi_\varepsilon(u_i^\varepsilon)\right\}dt.\end{aligned}$$

Using the property of Φ_ε we have $\liminf_{\varepsilon\to0}\Phi_\varepsilon(u_i^\varepsilon)\ge\Phi(l_i)$, and (2.16) we obtain

$$\int_0^T\left\{\langle\dot l_i,v\rangle+a(l_i,v)+\Phi(v)-\langle g_i,v-l_i\rangle\right\}dt\ge\int_0^T\left\{\langle\dot l_i,l_i\rangle+a(l_i,l_i)+\Phi(l_i)\right\}dt.\tag{2.17}$$

Let $w\in K$ and any $t_0\in]0,T[$ then we consider the open interval $\mathcal{O}_j=]t_0-\frac{1}{j},t_0+\frac{1}{j}[\subset]0,T[$ for $j\in\mathbb{N}^\star$ sufficiently large we take $v=\begin{cases}w\text{ if } t\in\mathcal{O}_j,\\ l_i(t)\text{ if } t\in]0,T[\setminus\mathcal{O}_j\end{cases}$ in (2.17) to get

$$\int_{\mathcal{O}_j}\left\{\langle\dot l_i,w-l_i\rangle+a(l_i,w-l_i)+\Phi(w)-\Phi(l_i)\right\}dt\ge\int_{\mathcal{O}_j}\langle g_i,w-l_i\rangle dt.\tag{2.18}$$

Now we use the Lebesgue Theorem to obtain, when $j\to+\infty$

$$\langle\dot l_i,w-l_i\rangle+a(l_i,w-l_i)+\Phi(w)-\Phi(l_i)\ge\langle g_i,w-l_i\rangle,\quad\text{a.e. } t\in]0,T[.\tag{2.19}$$

So by the uniqueness of the solution of the parabolic variational inequality of second kind (1.1), we deduce that $l_i=u_i$.

To finish the proof we check the strong convergence of u_i^ε to u_i. Indeed for $i=1,2$ or 4 taking $v=u_i(t)$ in (1.1) where $u=u_i^\varepsilon$ then $v=u_i^\varepsilon(t)$ in (1.1) where $u=u_i$, then by addition, and integration over the time interval $[0,T]$ we obtain

$$\begin{aligned}&\frac{1}{2}\|u_i(T)-u_i^\varepsilon(T)\|_H^2+\int_0^T a(u_i(t)-u_i^\varepsilon(t),u_i(t)-u_i^\varepsilon(t))dt\\&\quad\le\int_0^T\Phi_\varepsilon(u_i(t))-\Phi(u_i(t))+\Phi(u_i^\varepsilon(t))-\Phi_\varepsilon(u_i^\varepsilon(t))dt\end{aligned}\tag{2.20}$$

as

$$\Phi_\varepsilon(v) - \Phi(v) = \int_{\Gamma_2} q(\sqrt{\varepsilon^2 + |v|^2} - |v|)\mathrm{d}s \le \varepsilon\sqrt{|\Gamma_2|}\|q\|_{L^2(\Gamma_2)},$$

so from (2.20)

$$\frac{1}{2}\|u_i - u_i^\varepsilon\|^2_{L^\infty(0,T,H)} + \int_0^T a(u_i(t) - u_i^\varepsilon(t), u_i(t) - u_i^\varepsilon(t))\mathrm{d}t \le 2T\varepsilon\sqrt{|\Gamma_2|}\|q\|_{L^2(\Gamma_2)}$$

thus

$$u_i^\varepsilon \to u_i \quad \text{strongly in } L^2(0, T; V) \cap L^\infty(0, T; H) \text{ for } i = 1, 2, 4 \tag{2.21}$$

then also

$$u_3^\varepsilon(\mu) = \mu u_1^\varepsilon + (1-\mu)u_2^\varepsilon \to u_3 \quad \text{strongly in } L^2(0, T; V) \cap L^\infty(0, T; H) \tag{2.22}$$

from (2.12), (2.21) and (2.22) we get (2.8). As the proof is given for any two control $g = g_1$ and $g = g_2$ in $L^2(0, T, H)$, but for the same q, h, b and the same initial condition (1.2), so we get also (2.9). □

2.1. Dependency of the solutions on the data

Note that this subsection is not needed in the last section. We just would like to establish three propositions which allow us to deduce some additional and interesting properties on the solutions of the variational problems (P) and (P_h).

Proposition 2.6. *Let u_{g_n}, u_g be two solutions of Problem (P), with $g = g_n$ and $g = g$ respectively. Assume that $g_n \rightharpoonup g$ in $L^2(0, T, H)$ (weak), we get*

$$u_{g_n} \to u_g \quad \text{in } L^2(0, T, V) \cap L^\infty(0, T, H) \text{ (strong)} \tag{2.23}$$

$$\dot{u}_{g_n} \to \dot{u}_g \quad \text{in } L^2(0, T, V') \text{ (strong)}. \tag{2.24}$$

Moreover,

$$g_1 \ge g_2 \quad \text{in } \Omega \times [0, T] \text{ then } u_{g_1} \ge u_{g_2} \text{ in } \Omega \times [0, T]. \tag{2.25}$$

$$u_{\min(g_1,g_2)} \le u_4(\mu) \le u_{\max(g_1,g_2)}, \quad \forall\mu \in [0, 1]. \tag{2.26}$$

Let u_{g_1h}, u_{g_2h} be two solutions of (P_h), with $g = g_1$ and $g = g_2$ respectively for all $h > 0$, we get

$$g_1 \ge g_2 \quad \text{in } \Omega \times [0, T] \text{ then } u_{g_1h} \ge u_{g_2h} \text{ in } \Omega \times [0, T]. \tag{2.27}$$

$$u_{\min(g_1,g_2)h} \le u_{h4}(\mu) \le u_{\max(g_1,g_2)h} \quad \forall\mu \in [0, 1]. \tag{2.28}$$

Proof. Let $g_n \rightharpoonup g$ in $L^2(0, T, H)$, u_{g_n} and u_g be in $L^2(0, T, K)$ such that

$$\langle \dot{u}_{g_n}, v - u_{g_n}\rangle + a(u_{g_n}, v - u_{g_n}) + \Phi(v) - \Phi(u_{g_n}) \ge (g_n, v - u_{g_n}) \quad \forall v \in K,\ \text{a.e. } t \in]0, T[. \tag{2.29}$$

Remark also that $V_2 = \{v \in V : v_{|\Gamma_2} = 0\} \subset V$ with continuous inclusion but not dense, so V' is not identifiable with a subset of V_2'. However, following again [12] we can use the Hahn–Banach Theorem in order to extend any element in V_2' to an element of V' preserving its norm. So with the same arguments as in (2.14)–(2.19), we conclude that there exists η such that (eventually for a subsequence)

$$\left.\begin{array}{l} u_{g_n} \rightharpoonup \eta \quad \text{in } L^2(0, T, V) \text{ weak, in } L^\infty(0, T, H) \text{ weak star,} \\ \text{and} \quad \dot{u}_{g_n} \rightharpoonup \dot{\eta} \text{ in } L^2(0, T, V') \text{ weak} \end{array}\right\}. \tag{2.30}$$

Using (2.30) and taking $n \to +\infty$ in (2.29), we get

$$\langle \dot{\eta}, v - \eta\rangle + a(\eta, v - \eta) + \Phi(v) - \Phi(u_\eta) \ge (g, v - \eta), \quad \forall v \in K,\ \text{a.e. } t \in]0, T[, \tag{2.31}$$

by the uniqueness of the solution of (1.1) we obtain that $\eta = u_g$. Taking now $v = u_g(t)$ in (2.29) and $v = u_{g_n}(t)$ in (2.31), we get by addition and integration over $[0, T]$ we obtain

$$\frac{1}{2}\|u_{g_n}(T) - u_g(T)\|_H^2 + \lambda\|u_{g_n} - u_g\|^2_{L^2(0,T,V)} \le \int_0^T (g_n(t) - g(t), u_{g_n}(t) - u_g(t))\mathrm{d}t,$$

so from the above inequality and (2.30) we deduce (2.23). To prove (2.25) we take first $v = u_1(t) + (u_1(t) - u_2(t))^-$ (which is in K) in (1.1) where $u = u_1$ and $g = g_1$, then taking $v = u_2(t) - (u_1(t) - u_2(t))^-$ (which also is in K) in (1.1) where $u = u_2$ and $g = g_2$, we get

$$\frac{1}{2}\|(u_1(T) - u_2(T))^-\|_H^2 + \lambda\|(u_1 - u_2)^-\|^2_{L^2(0,T,V)} \leq \int_0^T (g_2(t) - g_1(t), (u_1(t) - u_2(t))^-)\mathrm{d}t$$

as

$$\Phi(u_1) - \Phi(u_1 + (u_1 - u_2)^-) + \Phi(u_2) - \Phi(u_2 - (u_1 - u_2)^-) = 0.$$

So if $g_2 - g_1 \leq 0$ in $\Omega \times [0, T]$ then $\|(u_1 - u_2)^-\|_{L^2(0,T,V)} = 0$, and as $(u_1 - u_2)^- = 0$ on $\Gamma_1 \times]0, T[$ we have by the Poincaré inequality that $u_1 - u_2 \geq 0$ in $\Omega \times [0, T]$. Then (2.26) follows from (2.25) because

$$\min\{g_1, g_2\} \leq \mu g_1 + (1 - \mu)g_2 \leq \max\{g_1, g_2\} \quad \forall \mu \in [0, T].$$

Similarly taking $v = u_{g_1h}(t) + (u_{g_1h}(t) - u_{g_2h}(t))^-$ (which is in V) in (2.1) where $u = u_{g_1h}$ and $g = g_1h$, then taking $v = u_{g_2h}(t) - (u_{g_1h}(t) - u_{g_2h}(t))^-$ (which also is in V) in (2.1) where $u = u_{g_2h}$ and $g = g_2h$, we get

$$\frac{1}{2}\|(u_{g_1h}(T) - u_{g_2h}(T))^-\|_H^2 + \lambda\|(u_{g_1h} - u_{g_2h})^-\|^2_{L^2(0,T,V)} + h\|(u_{g_1h} - u_{g_2h})^-\|^2_{L^2(0,T,L^2(\Gamma_1))}$$
$$\leq \int_0^T (g_2(t) - g_1(t), (u_1(t) - u_2(t))^-)\mathrm{d}t$$

so we get also (2.27), then (2.28) follows. □

The following Propositions 2.7 and 2.8 are to give, with some assumptions, a first information that the sequence $(u_{g_h})_{h>0}$ is increasing and bounded, therefore it is convergent in some sense. Remark from (2.4) that $u_{g_h} \geq 0$ although $g < 0$, provided to take the parameter h sufficiently large.

Proposition 2.7. *Assume that $h > 0$ and is sufficiently large, b is a positive constant, $q \geq 0$ on $\Gamma_2 \times [0, T]$, then we have*

$$g \leq 0 \quad \text{in } \Omega \times [0, T] \Longrightarrow 0 \leq u_{g_h} \leq b \text{ in } \Omega \cup \Gamma_1 \times [0, T]. \tag{2.32}$$

Proof. Taking in (2.1) $u = u_{g_h}(t)$ and $v = u_{g_h}(t) - (u_{g_h}(t) - b)^+$, we get

$$\langle \dot{u}_{g_h}, (u_{g_h} - b)^+\rangle + a_h(u_{g_h}, (u_{g_h} - b)^+) - \Phi(u_{g_h} - (u_{g_h} - b)^+) + \Phi(u_{g_h})$$
$$\leq (g, (u_{g_h} - b)^+) + h\int_{\Gamma_1} b(u_{g_h} - b)^+\mathrm{d}s, \quad \text{a.e. } t \in]0, T[$$

as b is constant we have $a(b, (u_{g_h}(t) - b)^+) = 0$ so a.e. $t \in]0, T[$

$$\frac{1}{2}\frac{\partial}{\partial t}\left(\|(u_{g_h}(t) - b)^+\|_H^2\right) + a((u_{g_h} - b)^+, (u_{g_h} - b)^+) + h\int_{\Gamma_1} u_{g_h}(u_{g_h} - b)^+\mathrm{d}s$$
$$\leq (g, (u_{g_h} - b)^+) + h\int_{\Gamma_1} b(u_{g_h} - b)^+\mathrm{d}s + \Phi(u_{g_h} - (u_{g_h} - b)^+) - \Phi(u_{g_h}),$$

as $u_{g_h}(0) = b$ and

$$\Phi(u_{g_h} - (u_{g_h} - b)^+) - \Phi(u_{g_h}) = \int_{\Gamma_2} q(|u_{g_h} - (u_{g_h} - b)^+| - |u_{g_h}|)\mathrm{d}s \leq 0,$$

so

$$\frac{1}{2}\|(u_{g_h}(T) - b)^+\|_H^2 + \int_0^T a_h((u_{g_h}(t) - b)^+, (u_{g_h}(t) - b)^+)\mathrm{d}t \leq \int_0^T (g(t), (u_{g_h}(t) - b)^+)\mathrm{d}t \leq 0,$$

thus (2.32) holds. □

Proposition 2.8. *Assume that $h > 0$ and is sufficiently large. Let g, g_1, g_2 in $L^2(0, T, H)$, $q \in L^2(0, T, L^2(\Gamma_2))$ and b is a positive constant, we have*

$$g_2 \leq g_1 \leq 0 \quad \text{in } \Omega \times [0, T] \text{ and } h_2 \leq h_1 \Longrightarrow 0 \leq u_{g_2h_2} \leq u_{g_1h_1} \text{ in } \Omega \times [0, T], \tag{2.33}$$

$$g \leq 0 \quad \text{in } \Omega \times [0, T] \Longrightarrow 0 \leq u_{g_h} \leq u_g \text{ in } \Omega \times [0, T], \ \forall h > 0. \tag{2.34}$$

$$h_2 \leq h_1 \Longrightarrow \|u_{g_{h_2}} - u_{g_{h_1}}\|_{L^2(0,T,V)} \leq \frac{\|\gamma_0\|}{\lambda_1 \min(1, h_2)}\|b - u_{g_{h_1}}\|_{L^2(0,T,\mathbf{L}^2(\Gamma_1))}(h_1 - h_2), \tag{2.35}$$

where γ_0 is the trace embedding from V to $L^2(\Gamma_1)$.

Proof. To check (2.33) we take first $v = u_{g_1h_1}(t) + (u_{g_2h_2}(t) - u_{g_1h_1}(t))^+$, for $t \in [0, T]$, in (2.1) where $u = u_{g_1h_1}$, $g = g_1h_1$ and $h = h_1$, then taking $v = u_{g_2h_2}(t) - (u_{g_2h_2}(t) - u_{g_1h_1}(t))^+$ in (2.1) where $u = u_{g_2h_2}$, $g = g_2h_2$ and $h = h_2$,

adding the two obtained inequalities, as

$$\Phi(u_{g_1h_1} + (u_{g_2h_2} - u_{g_1h_1})^+) - \Phi(u_{g_1h_1}) + \Phi(u_{g_2h_2} - (u_{g_2h_2} - u_{g_1h_1})^+) - \Phi(u_{g_2h_2}) = 0$$

we get

$$-\frac{1}{2}\frac{\partial}{\partial t}\left(\|(u_{g_2h_2} - u_{g_1h_1})^+\|_H^2\right) - a(u_{g_2h_2} - u_{g_1h_1}, (u_{g_2h_2} - u_{g_1h_1})^+) + \int_{\Gamma_1} (h_1u_{g_1h_1} - h_2u_{g_2h_2})(u_{g_2h_2} - u_{g_1h_1})^+ ds$$
$$\geq (g_1 - g_2, (u_{g_2h_2} - u_{g_1h_1})^+) + (h_1 - h_2)\int_{\Gamma_1} b(u_{g_2h_2} - u_{g_1h_1})^+ ds, \quad \text{a.e. } t \in]0, T[,$$

so by integration on $]0, T[$, we deduce

$$\frac{1}{2}\|(u_{g_2h_2}(T) - u_{g_1h_1}(T))^+\|_H^2 + \int_0^T a_{h_2}((u_{g_2h_2} - u_{g_1h_1})^+, (u_{g_2h_2} - u_{g_1h_1}(t))^+)dt$$
$$\leq \int_0^T (g_2 - g_1, (u_{g_2h_2}(t) - u_{g_1h_1})^+)dt + (h_1 - h_2)\int_0^T\int_{\Gamma_1} (u_{g_1h_1} - b)(u_{g_2h_2} - u_{g_1h_1})^+ ds dt,$$

and from (2.32) we get (2.33). To check (2.34), let $W = u_{g_h}(t) - u_g(t)$, and choose, in (2.1), $v = u_{g_h}(t) - W^+(t)$, so a.e. $t \in]0, T[$

$$\langle \dot{u}_{g_h}, W^+\rangle + a_h(u_{g_h}, W^+) \leq +\Phi(u_{g_h} - W^+) - \Phi(u_{g_h}) + (g, W^+) + h\int_{\Gamma_1} bW^+ ds,$$

as $u_g = b$ on $\Gamma_1 \times [0, T]$ we obtain a.e. $t \in]0, T[$

$$\langle \dot{u}_{g_h}, W^+\rangle + a(u_{g_h}, W^+) + h\int_{\Gamma_1} |W^+|^2 ds \leq (g, W^+) + \Phi(u_{g_h} - W^+) - \Phi(u_{g_h}). \tag{2.36}$$

Then we choose, in (1.1), $v = u_g(t) + W^+(t)$, which is in K because from (2.32) we have $W^+ = 0$ on $\Gamma_1 \times [0, T]$, so

$$\langle \dot{u}_g, W^+(t)\rangle + a(u_g, W^+) \geq (g, W^+) - \Phi(u_g + W^+) + \Phi(u_g), \quad \text{a.e. } t \in]0, T[. \tag{2.37}$$

So from (2.36) and (2.37) we deduce that

$$\frac{1}{2}\|W^+(T)\|_H^2 + \int_0^T a(W^+, W^+)dt + h\int_{\Gamma_1} |W^+|^2 ds \leq \Phi(u_{g_h} - W^+) - \Phi(u_{g_h}) + \Phi(u_g + W^+) - \Phi(u_g) = 0.$$

Then (2.34) holds. To finish the proof we must check (2.35). We choose $v = u_{g_{h_1}}(t)$ in (2.1) where $u = u_{g_{h_2}}(t)$, then choosing $v = u_{g_{h_2}}(t)$ in (2.1) where $u = u_{g_{h_1}}(t)$, we get

$$-\langle \dot{u}_{g_{h_2}} - \dot{u}_{g_{h_1}}, u_{g_{h_2}} - u_{g_{h_1}}\rangle - a(u_{g_{h_2}} - u_{g_{h_1}}, u_{g_{h_2}} - u_{g_{h_1}}) - h_2\int_{\Gamma_1} u_{g_{h_2}}(u_{g_{h_2}} - u_{g_{h_1}})ds + h_1\int_{\Gamma_1} u_{g_{h_1}}(u_{g_{h_2}} - u_{g_{h_1}})ds$$
$$\geq -(h_2 - h_1)\int_{\Gamma_1} b(u_{g_{h_2}} - u_{g_{h_1}})ds, \quad \text{a.e. } t \in]0, T[,$$

then

$$\frac{1}{2}\|u_{g_{h_2}}(T) - u_{g_{h_1}}(T)\|_H^2 + \int_0^T a_{h_2}(u_{g_{h_2}} - u_{g_{h_1}}, u_{g_{h_2}} - u_{g_{h_1}})dt \leq (h_1 - h_2)\int_0^T\int_{\Gamma_1} (u_{g_{h_1}} - b)(u_{g_{h_2}} - u_{g_{h_1}})ds dt.$$

So

$$\frac{1}{2}\|u_{g_{h_2}} - u_{g_{h_1}}\|_{L^\infty(0,T,H)}^2 + \lambda_1 \min\{1, h_2\}\|u_{g_{h_2}} - u_{g_{h_1}}\|_{L^2(0,T,V)}^2$$
$$\leq \|\gamma_0\|(h_1 - h_2)\|b - u_{g_{h_1}}\|_{L^2(0,T,\mathbf{L}^2(\Gamma_1))}\|u_{g_{h_2}} - u_{g_{h_1}}\|_{L^2(0,T,V)}$$

Thus (2.35) holds. $\square$

3. Optimal control problems and convergence for $h \to +\infty$

In this section, b is not constant but a given function in $L^2(]0, T[\times \Gamma_1)$. First, we prove the existence and uniqueness of the solution for the optimal control problem associated with the parabolic variational inequalities of second kind (1.1), and for the optimal control problem associated also with (2.1), then in Section 3.1 we prove (see Lemma 3.2 and Theorem 3.3) the convergence of the state $u_{g_{op_h}h}$ and the optimal control g_{op_h}, when the coefficient h on Γ_1, goes to infinity.

The existence and uniqueness of the solution to the parabolic variational inequalities of second kind (1.1) and (2.1), with the initial condition (1.2), allow us to consider $g \mapsto u_g$ and $g \mapsto u_{g_h}$ as functions from $L^2(0, T, H)$ to $L^2(0, T, V)$, for all $h > 0$.

Using the monotony property (2.8) and (2.9), established in Theorem 2.5, we prove in the following that J and J_h, defined by (1.3) and (1.8), are strictly convex applications on $L^2(0, T, H)$, so [6] there exists a unique solution g_{op} in $L^2(0, T, H)$ of problem (1.4), and there exists also a unique solution g_{op_h} in $L^2(0, T, H)$ of problem (1.7) for all $h > 0$.

Theorem 3.1. *Assume the same hypotheses of Proposition 2.1. Then J and J_h, defined by (1.3) and (1.8) respectively, are strictly convex applications on $L^2(0, T, H)$, so there exist unique solutions g_{op} and g_{op_h} in $L^2(0, T, H)$ respectively of problems (1.4) and (1.7).*

Proof. Let $u = u_{g_i}$ and u_{g_ih} be respectively the solution of the variational inequalities (1.1) and (2.1) with $g = g_i$ for $i = 1, 2$. We have

$$\|u_3(\mu)\|^2_{L^2(0,T,H)} = \mu^2\|u_{g_1}\|^2_{L^2(0,T,H)} + (1-\mu)^2\|u_{g_2}\|^2_{L^2(0,T,H)} + 2\mu(1-\mu)(u_{g_1}, u_{g_2})$$

then the following equalities hold

$$\|u_3(\mu)\|^2_{L^2(0,T,H)} = \mu\|u_{g_1}\|^2_{L^2(0,T,H)} + (1-\mu)\|u_{g_2}\|^2_{L^2(0,T,H)} - \mu(1-\mu)\|u_{g_2} - u_{g_1}\|^2_{L^2(0,T,H)}, \tag{3.1}$$

$$\|u_{3h}(\mu)\|^2_{L^2(0,T,H)} = \mu\|u_{g_1h}\|^2_{L^2(0,T,H)} + (1-\mu)\|u_{g_2h}\|^2_{L^2(0,T,H)} - \mu(1-\mu)\|u_{g_2h} - u_{g_1h}\|^2_{L^2(0,T,H)}. \tag{3.2}$$

Now let $\mu \in [0, 1]$ and $g_1, g_2 \in L^2(0, T, H)$ so

$$\begin{aligned}\mu J(g_1) + (1-\mu)J(g_2) - J(g_3(\mu)) = {} & \frac{\mu}{2}\|u_{g_1}\|^2_{L^2(0,T,H)} + \frac{(1-\mu)}{2}\|u_{g_2}\|^2_{L^2(0,T,H)} \\ & - \frac{1}{2}\|u_4(\mu)\|^2_{L^2(0,T,H)} + \frac{M}{2}\Big\{\mu\|g_1\|^2_{L^2(0,T,H)} + (1-\mu)\|g_2\|^2_{L^2(0,T,H)} \\ & - \|g_3(\mu)\|^2_{L^2(0,T,H)}\Big\}\end{aligned}$$

using (3.1) and $g_3(\mu) = \mu g_1 + (1-\mu)g_2$ we obtain

$$\begin{aligned}\mu J(g_1) + (1-\mu)J(g_2) - J(g_3(\mu)) = {} & \frac{1}{2}\left(\|u_3(\mu)\|^2_{L^2(0,T,H)} - \|u_4(\mu)\|^2_{L^2(0,T,H)}\right) + \frac{1}{2}\mu(1-\mu)\|u_1 - u_2\|^2_{L^2(0,T,H)} \\ & + \frac{M}{2}\mu(1-\mu)\|g_1 - g_2\|^2_{L^2(0,T,H)},\end{aligned} \tag{3.3}$$

for all $\mu \in]0, 1[$ and for all g_1, g_2 in $L^2(0, T, H)$. From Proposition 2.1 we have $u_4(\mu) \geq 0$ in $\Omega \times [0, T]$ for all $\mu \in [0, 1]$, so using the monotony property (2.8) (Theorem 2.5) and we deduce

$$\|u_4(\mu)\|^2_{L^2(0,T,H)} \leq \|u_3(\mu)\|^2_{L^2(0,T,H)}. \tag{3.4}$$

Finally from (3.3) the cost functional J is strictly convex, thus [6] the uniqueness of the optimal control of problem (1.4) holds.

The uniqueness of the optimal control of problem (1.7) follows using the analogous inequalities (3.3)–(3.4) for any $h > 0$. $\square$

3.1. Convergence when $h \to +\infty$

In this last subsection we study the convergence of the state $u_{g_{op_h}h}$ and the optimal control g_{op_h}, when the coefficient h on Γ_1, goes to infinity. For a given g in $L^2(0, T, H)$, first we have the following estimate which generalizes [34,35].

Lemma 3.2. *Let u_{g_h} be the unique solution of the parabolic variational inequality (2.1) and u_g the unique solution of the parabolic variational inequality (1.1), then*

$$u_{g_h} \to u_g \in L^2(0, T, V) \quad \text{strongly as } h \to +\infty, \ \forall g \in L^2(0, T, H).$$

Proof. We take $v = u_g(t)$ in (2.1) where $u = u_{g_h}$, and recalling that $u_g(t) = b$ on $\Gamma_1 \times]0, T[$, taking $u_{g_h}(t) - u_g(t) = \phi_h(t)$ we obtain for $h > 1$, a.e. $t \in]0, T[$

$$\langle \dot{\phi}_h, \phi_h \rangle + a_1(\phi_h, \phi_h) + (h-1) \int_{\Gamma_1} |\phi_h|^2 ds \le -\langle \dot{u}_g, \phi_h \rangle - a(u_g, \phi_h) + (g, \phi_h) + \Phi(\phi_h),$$

so we deduce that

$$\frac{1}{2} \|\phi_h\|^2_{L^\infty(0,T,H)} + \|\phi_h\|^2_{L^2(0,T,V)} + (h-1)\|\phi_h\|^2_{L^2(0,T,L^2(\Gamma_1))}$$

is bounded for all $h > 1$, then $\|u_{g_h}\|_{L^2(0,T,V)} \le \|\phi_h\|_{L^2(0,T,V)} + \|u_g\|_{L^2(0,T,V)}$ is also bounded for all $h > 1$. So there exists $\eta \in L^2(0, T, V)$ such that $u_{g_h} \rightharpoonup \eta$ weakly in $L^2(0, T, V)$ and $u_{g_h} \to b$ strongly on Γ_1 when $h \to +\infty$ so $\eta(0) = u_b$.

Let $\varphi \in L^2(0, T, H^1_0(\Omega))$ and taking $u = u_{g_h}$, $v = u_{g_h}(t) \pm \varphi(t)$ in (2.1), we obtain

$$\langle \dot{u}_{g_h}, \varphi \rangle = -a(u_{g_h}, \varphi) + (g, \varphi) \quad \text{a.e. } t \in]0, T[.$$

As $\|u_{g_h}\|_{L^2(0,T,V)}$ is bounded for all $h > 1$, we deduce that $\|\dot{u}_{g_h}\|_{L^2(0,T,V'_2)}$ is also bounded for all $h > 1$. Following the proof of Theorem 2.5, we conclude that

$$\left.\begin{array}{l} u_{g_h} \rightharpoonup \eta \quad \text{in } L^2(0,T,V) \text{ weak, and in } L^\infty(0,T,H) \text{ weak star,} \\ \text{and } \dot{u}_{g_n} \rightharpoonup \dot{\eta} \text{ in } L^2(0,T,V') \text{ weak} \end{array}\right\}. \tag{3.5}$$

From (2.1) and taking $v \in K_b$ so $v = b$ on Γ_1, we obtain

$$\langle \dot{u}_{g_h}, v - u_{g_h} \rangle + a(u_{g_h}, v - u_{g_h}) - h \int_{\Gamma_1} |u_{g_h} - b|^2 ds \ge \Phi(u_{g_h}) - \Phi(v) + (g, v - u_{g_h}) \quad \forall v \in K_b, \text{ a.e. } t \in]0, T[,$$

then

$$\langle \dot{u}_{g_h}, v - u_{g_h} \rangle + a(u_{g_h}, v - u_{g_h}) \ge \Phi(u_{g_h}) - \Phi(v) + (g, v - u_{g_h}) \quad \forall v \in K_b, \text{ a.e. }, t \in]0, T[. \tag{3.6}$$

So with (3.5) and the same arguments as in (2.14)–(2.19), we obtain

$$\langle \dot{\eta}, v - \eta \rangle + a(\eta, v - \eta) + \Phi(v) - \Phi(\eta) \ge (g, v - \eta) \quad \forall v \in K_b, \text{ a.e. } t \in]0, T[$$

and $\eta(0) = u_b$. Using the uniqueness of the solution of (1.1)–(1.2) we get that $\eta = u_g$.

To prove the strong convergence, we take $v = u_g(t)$ in (2.1)

$$\langle \dot{u}_{g_h}, u_g - u_{g_h} \rangle + a_h(u_{g_h}, u_g - u_{g_h}) + \Phi(u_g) - \Phi(u_{g_h}) \ge (g, u_g - u_{g_h}) + h \int_{\Gamma_1} b(u_g - u_{g_h}) ds, \quad \text{a.e. } t \in]0, T[$$

thus as $u_g = b$ on $\Gamma_1 \times]0, T[$, we put $u_{g_h} - u_g = \phi_h$, so a.e. $t \in]0, T[$

$$\langle \dot{\phi}_h, \phi_h \rangle + a(\phi_h, \phi_h) + h \int_{\Gamma_1} |\phi_h|^2 ds + \Phi(u_{g_h}) - \Phi(u_g) \le \langle \dot{u}_g, \phi_h \rangle + a(u_g, \phi_h) + (g, \phi_h),$$

so

$$\begin{aligned} &\frac{1}{2} \|\phi_h\|^2_{L^\infty(0,T,H)} + \lambda_h \|\phi_h\|^2_{L^2(0,T,V)} + \Phi(u_{g_h}) - \Phi(u_g) \\ &\le -\int_0^T \langle \dot{u}_g(t), \phi_h(t) \rangle dt - \int_0^T a(u_g(t), \phi_h(t)) dt + \int_0^T (g(t), \phi_h(t)) dt, \end{aligned}$$

using the weak semi-continuity of Φ and the weak convergence (2.30) the right side of the just above inequality tends to zero when $h \to +\infty$, then we deduce the strong convergence of $\phi_h = u_{g_h} - u_g$ to 0 in $L^2(0, T, V) \cap L^\infty(0, T, H)$, for all $g \in L^2(0, T, H)$. This ends the proof. □

Now we give, without need to use the notion of adjoint states [6], the convergence result which generalizes the result obtained in [12] for a parabolic variational equations (see also [37,38,11,39]).

Theorem 3.3. *Let $u_{g_{op_h}h}$, g_{op_h} and $u_{g_{op}}$, g_{op} be respectively the states and the optimal control defined in problems (1.4) and (1.7). Then*

$$\begin{aligned} \lim_{h\to+\infty} \|u_{g_{op_h}h} - u_{g_{op}}\|_{L^2(0,T,V)} &= \lim_{h\to+\infty} \|u_{g_{op_h}h} - u_{g_{op}}\|_{L^\infty(0,T,H)}, \\ &= \lim_{h\to+\infty} \|u_{g_{op_h}h} - u_{g_{op}}\|_{L^2(0,T,L^2(\Gamma_1))} = 0, \end{aligned} \tag{3.7}$$

$$\lim_{h\to+\infty} \|g_{op_h} - g_{op}\|_{L^2(0,T,H)} = 0. \tag{3.8}$$

Proof. First, we have

$$J_h(g_{\mathrm{op}_h}) = \frac{1}{2}\|u_{g_{\mathrm{op}_h}h}\|^2_{L^2(0,T,H)} + \frac{M}{2}\|g_{\mathrm{op}_h}\|^2_{L^2(0,T,H)} \le \frac{1}{2}\|u_{g_h}\|^2_{L^2(0,T,H)} + \frac{M}{2}\|g\|^2_{L^2(0,T,H)},$$

for all $g \in L^2(0, T, H)$, then for $g = 0 \in L^2(0, T, H)$ we obtain that

$$J_h(g_{\mathrm{op}_h}) = \frac{1}{2}\|u_{g_{\mathrm{op}_h}h}\|^2_{L^2(0,T,H)} + \frac{M}{2}\|g_{\mathrm{op}_h}\|^2_{L^2(0,T,H)} \le \frac{1}{2}\|u_{0_h}\|^2_{L^2(0,T,H)} \tag{3.9}$$

where $u_{0_h} \in L^2(0, T, V)$ is the solution of the following parabolic variational inequality

$$\langle \dot{u}_{0_h}, v - u_{0_h}\rangle + a_h(u_{0_h}, v - u_{0_h}) + \Phi(v) - \Phi(u_{0_h}) \ge h\int_{\Gamma_1} b(v - u_{0_h})\mathrm{d}s, \quad \text{a.e. } t \in]0, T[$$

for all $v \in V$ and $u_{0_h}(0) = u_b$. Taking $v = u_b \in K_b$ we get that $\|u_{0_h} - u_b\|_{L^2(0,T,V)}$ is bounded independently of h, then $\|u_{0_h}\|_{L^2(0,T,H)}$ is bounded independently of h. So we deduce with (3.9) that $\|u_{g_{\mathrm{op}_h}h}\|_{L^2(0,T,H)}$ and $\|g_{\mathrm{op}_h}\|_{L^2(0,T,H)}$ are also bounded independently of h. So there exist f and η in $L^2(0, T, H)$ such that

$$g_{\mathrm{op}_h} \rightharpoonup f \quad \text{in } L^2(0, T, H) \text{ (weak) and } u_{g_{\mathrm{op}_h}h} \rightharpoonup \eta \text{ in } L^2(0, T, H) \text{ (weak).} \tag{3.10}$$

Taking now $v = u_{g_{\mathrm{op}}}(t) \in K_b$ in (2.1), for $t \in]0, T[$, with $u = u_{g_{\mathrm{op}_h}h}$ and $g = g_{\mathrm{op}_h}$, we obtain

$$\langle \dot{u}_{g_{\mathrm{op}_h}h}, u_{g_{\mathrm{op}}} - u_{g_{\mathrm{op}_h}h}\rangle + a_1(u_{g_{\mathrm{op}_h}h}, u_{g_{\mathrm{op}}} - u_{g_{\mathrm{op}_h}h}) + (h-1)\int_{\Gamma_1} u_{g_{\mathrm{op}_h}h}(u_{g_{\mathrm{op}}} - u_{g_{\mathrm{op}_h}h})\mathrm{d}s + \Phi(u_{g_{\mathrm{op}}}) - \Phi(ug_{\mathrm{op}_h}h)$$
$$\ge (g_{\mathrm{op}_h}, u_{g_{\mathrm{op}}} - u_{g_{\mathrm{op}_h}h}) + h\int_{\Gamma_1} b(u_{g_{\mathrm{op}}} - u_{g_{\mathrm{op}_h}h})\mathrm{d}s, \quad \text{a.e. } t \in]0, T[$$

as $u_{g_{\mathrm{op}}} = b$ on $\Gamma_1 \times [0, T]$, taking $u_{g_{\mathrm{op}}} - u_{g_{\mathrm{op}_h}h} = \phi_h$ we obtain

$$\langle \dot{\phi}_h, \phi_h\rangle + a_1(\phi_h, \phi_h) + (h-1)\int_{\Gamma_1} |\phi_h|^2\mathrm{d}s \le -(g_{\mathrm{op}_h}, \phi_h) + \int_{\Gamma_2} q|\phi_h|\mathrm{d}s + \langle \dot{u}_{g_{\mathrm{op}}}, \phi_h\rangle + a(u_{g_{\mathrm{op}}}, \phi_h),$$
$$\text{a.e. } t \in]0, T[$$

then

$$\frac{1}{2}\|\phi_h\|^2_{L^\infty(0,T,H)} + \lambda_1\|\phi_h\|^2_{L^2(0,T,V)} + (h-1)\int_0^T\int_{\Gamma_1} |\phi_h(t)|^2\mathrm{d}s\mathrm{d}t$$
$$\le -\int_0^T (g_{\mathrm{op}_h}(t), \phi_h(t))\mathrm{d}t + \int_0^T\int_{\Gamma_2} q|\phi_h(t)|\mathrm{d}s\mathrm{d}t + \int_0^T \langle \dot{u}_{g_{\mathrm{op}}}(t), \phi_h(t)\rangle\mathrm{d}t + \int_0^T a(u_{g_{\mathrm{op}_h}h}(t), \phi_h(t))\mathrm{d}t.$$

There exists a constant $C >$ which does not depend on h such that

$$\|\phi_h\|_{L^2(0,T,V)} = \|u_{g_{\mathrm{op}_h}h} - u_{g_{\mathrm{op}}}\|_{L^2(0,T,V)} \le C, \quad \|\phi_h\|_{L^\infty(0,T,H)} \le C$$
$$\text{and} \quad (h-1)\int_0^T\int_{\Gamma_1} |u_{g_{\mathrm{op}_h}h} - b|^2\mathrm{d}s\mathrm{d}t \le C,$$

then $\eta \in L^2(0, T, V)$ and

$$u_{g_{\mathrm{op}_h}h} \rightharpoonup \eta \quad \text{in } L^2(0, T, V) \text{ weak and in } L^\infty(0, T, H) \text{ weak star} \tag{3.11}$$

$$u_{g_{\mathrm{op}_h}h} \to b \quad \text{in } L^2(0, T, L^2(\Gamma_1)) \text{ strong,} \tag{3.12}$$

so $\eta(t) \in K_b$ for all $t \in [0, T]$. Now taking $v \in K_b$ in (2.1) where $u = u_{g_{\mathrm{op}_h}h}$ and $g = g_{\mathrm{op}_h}$ so

$$\langle \dot{u}_{g_{\mathrm{op}_h}h}, v - u_{g_{\mathrm{op}_h}h}\rangle + a_h(u_{g_{\mathrm{op}_h}h}, v - u_{g_{\mathrm{op}_h}h}) + \Phi(v) - \Phi(u_{g_{\mathrm{op}_h}h})$$
$$\ge (g_{\mathrm{op}_h}, v - u_{g_{\mathrm{op}_h}h}) + h\int_{\Gamma_1} b(v - u_{g_{\mathrm{op}_h}h})\mathrm{d}s, \quad \text{a.e. } t \in]0, T[$$

as $v \in K_b$ so $v = b$ on Γ_1, thus we have

$$\langle \dot{u}_{g_{\mathrm{op}_h}h}, u_{g_{\mathrm{op}_h}h} - v\rangle + a(u_{g_{\mathrm{op}_h}h}, u_{g_{\mathrm{op}_h}h} - v) + h\int_{\Gamma_1} |u_{g_{\mathrm{op}_h}h} - b|^2\mathrm{d}s + \Phi(u_{g_{\mathrm{op}_h}h}) - \Phi(v)$$
$$\le -(g_{\mathrm{op}_h}, v - u_{g_{\mathrm{op}_h}h}) \quad \text{a.e. } t \in]0, T[.$$

Thus

$$\langle \dot{u}_{g_{op_h}h}, u_{g_{op_h}h} - v\rangle + a(u_{g_{op_h}h}, u_{g_{op_h}h} - v) + \Phi(u_{g_{op_h}h}) - \Phi(v) \leq -(g_{op_h}, v - u_{g_{op_h}h}) \quad \text{a.e. } t \in]0, T[.$$

Using (3.10) and (3.11) and the same arguments as in (2.14)–(2.19), we deduce that

$$\langle \dot{\eta}, v - \eta\rangle + a(\eta, v - \eta) + \Phi(v) - \Phi(\eta) \geq (f, v - \eta), \quad \forall v \in K_b, \text{ a.e. } t \in]0, T[,$$

so also by the uniqueness of the solution of (1.1) we obtain that

$$u_f = \eta. \tag{3.13}$$

We prove that $f = g_{op}$. Indeed we have

$$\begin{aligned} J(f) &= \frac{1}{2}\|\eta\|^2_{L^2(0,T;H)} + \frac{M}{2}\|f\|^2_{L^2(0,T;H)} \\ &\leq \liminf_{h\to+\infty}\left\{\frac{1}{2}\|u_{g_{op_h}h}\|^2_{L^2(0,T;H)} + \frac{M}{2}\|g_{op_h}\|^2_{L^2(0,T;H)}\right\} = \liminf_{h\to+\infty} J_h(g_{op_h}) \\ &\leq \liminf_{h\to+\infty} J_h(g) = \liminf_{h\to+\infty}\left\{\frac{1}{2}\|u_{g_h}\|^2_{L^2(0,T;H)} + \frac{M}{2}\|g\|^2_{L^2(0,T;H)}\right\}. \end{aligned}$$

Using now the strong convergence $u_{g_h} \to u_g$ as $h \to +\infty$, $\forall g \in H$ (see Lemma 3.2), we obtain that

$$J(f) \leq \liminf_{h\to+\infty} J_h(g_{op_h}) \leq \frac{1}{2}\|u_g\|^2_{L^2(0,T;H)} + \frac{M}{2}\|g\|^2_{L^2(0,T;H)} = J(g), \quad \forall g \in L^2(0, T; H) \tag{3.14}$$

then by the uniqueness of the optimal control problem (1.4) we get

$$f = g_{op}. \tag{3.15}$$

Now we prove the strong convergence of $u_{g_{op_h}h}$ to $\eta = u_f$ in $L^2(0, T, V) \cap L^\infty(0, T, H) \cap L^2(0, T, L^2(\Gamma_1))$, indeed taking $v = \eta$ in (2.1) where $u = u_{g_{op_h}h}$ and $g = g_{op_h}$, as $\eta(t) \in K_b$ for $t \in [0, T]$, so $\eta = b$ on Γ_1, we obtain

$$\begin{aligned} &\langle \dot{u}_{g_{op_h}h} - \dot{\eta}, u_{g_{op_h}h} - \eta\rangle + a_1(u_{g_{op_h}h} - \eta, u_{g_{op_h}h} - \eta) + (h-1)\int_{\Gamma_1} |u_{g_{op_h}h} - \eta|^2 ds \\ &\quad + \Phi(u_{g_{op_h}h}) - \Phi(\eta) \leq (g_{op_h}, u_{g_{op_h}h} - \eta) + \langle \dot{\eta}, u_{g_{op_h}h} - \eta\rangle + a(\eta, u_{g_{op_h}h} - \eta) \end{aligned}$$

thus

$$\begin{aligned} &\frac{1}{2}\|u_{g_{op_h}h} - \eta\|^2_{L^\infty(0,T;H)} + \lambda_1\|u_{g_{op_h}h} - \eta\|^2_{L^2(0,T,V)} + \int_0^T \{\Phi(u_{g_{op_h}h}) - \Phi(\eta)\}dt + (h-1)\|u_{g_{op_h}h} - \eta\|^2_{L^2(0,T,L^2(\Gamma_1))} \\ &\quad \leq \int_0^T (g_{op_h}(t), u_{g_{op_h}h}(t) - \eta(t))dt + \int_0^T \langle \dot{\eta}, u_{g_{op_h}h} - \eta\rangle dt + \int_0^T a(\eta(t), \eta(t) - u_{g_{op_h}h}(t))dt. \end{aligned}$$

Using (3.11) and the weak semi-continuity of Φ we deduce that

$$\begin{aligned} \lim_{h\to+\infty}\|u_{g_{op_h}h} - \eta\|_{L^\infty(0,T;H)} &= \lim_{h\to+\infty}\|u_{g_{op_h}h} - \eta\|_{L^2(0,T,V)} \\ &= \|u_{g_{op_h}h} - \eta\|_{L^2(0,T,L^2(\Gamma_1))} = 0, \end{aligned}$$

and with (3.13) and (3.15) we deduce (3.7). As $f \in L^2(0, T, H)$, then from (3.14) with $g = f$ and (3.15) we can write

$$\begin{aligned} J(f) = J(g_{op}) &= \frac{1}{2}\|u_{g_{op}}\|^2_{L^2(0,T,H)} + \frac{M}{2}\|g_{op}\|^2_{L^2(0,T,H)} \\ &\leq \liminf_{h\to+\infty} J_h(g_{op_h}) = \liminf_{h\to+\infty}\left\{\frac{1}{2}\|u_{g_{op_h}h}\|^2_{L^2(0,T,H)} + \frac{M}{2}\|g_{op_h}\|^2_{L^2(0,T,H)}\right\} \\ &\leq \lim_{h\to+\infty} J_h(g_{op}) = J(g_{op}) \end{aligned} \tag{3.16}$$

and using the strong convergence (3.7), we get

$$\lim_{h\to+\infty}\|g_{op_h}\|_{L^2(0,T,H)} = \|g_{op}\|_{L^2(0,T,H)}. \tag{3.17}$$

Finally, as

$$\|g_{op_h} - g_{op}\|^2_{L^2(0,T;H)} = \|g_{op_h}\|^2_{L^2(0,T;H)} + \|g_{op}\|^2_{L^2(0,T;H)} - 2(g_{op_h}, g_{op}) \tag{3.18}$$

and by the first part of (3.10) we have

$$\lim_{h\to+\infty}\left(g_{\mathrm{op}_h}, g_{\mathrm{op}}\right) = \|g_{\mathrm{op}}\|^2_{L^2(0,T,H)},$$

so from (3.17) and (3.18) we get (3.8). This ends the proof. □

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