EXISTENCE AND UNIQUENESS OF DISTRIBUTED OPTIMAL CONTROL PROBLEMS GOVERNED BY PARABOLIC VARIATIONAL INEQUALITIES OF THE SECOND KIND

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Abstract: Let u_g be the unique solution of a parabolic variational inequality of second kind, with a second member g. Using a regularization method, we prove, for all g_1 and g_2, a monotony property between $\mu u_{g_1} + (1-\mu)u_{g_2}$ and $u_{\mu g_1+(1-\mu)g_2}$ for $\mu \in [0,1]$. This allowed us to prove the existence and uniqueness results to a family of distributed optimal control problems governed by parabolic variational inequalities of second kind over g for each heat transfer coefficient $h > 0$, associated to the Newton law, and of another distributed optimal control problem associated to a Dirichlet boundary condition.

Keywords: *Parabolic variational inequalities of the second kind, convex combination of solutions, monotony property, regularization method, strict convexity of cost functional, optimal control problems, free boundary problem*
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1 INTRODUCTION

Let consider the following problem governed by the parabolic variational inequality of the second kind

$$\langle \dot{u}(t)\,,\, v-u(t)\rangle + a(u(t)\,,\, v-u(t)) + \Phi(v) - \Phi(u(t)) \geq < g(t)\,,\, v-u(t) > \quad \forall v \in K, \tag{1}$$

a.e. $t \in]0,T[$, with the initial condition

$$u(0) = u_b, \tag{2}$$

where, a is a symmetric, continuous and coercive bilinear form (with λ its positive coerciveness constant) on the Hilbert space $V \times V$, Φ is a proper and convex function from V into $\mathbb{R}$ and is lower semi-continuous for the weak topology on V, $< \cdot, \cdot >$ denotes the duality brackets between V' and V, K is a closed convex non-empty subset of V, u_b is an initial value in another Hilbert space H with V being densely and continuously imbedded in H, and g is a given function in the space $L^2(0,T,V')$. It is well known [4, 5] that, there exists a unique solution

$$u \in \mathcal{C}(0,T,H) \cap L^2(0,T,V) \quad \text{with} \quad \dot{u} = \frac{\partial u}{\partial t} \in L^2(0,T,H)$$

to problem (1)-(2). So we can consider $g \mapsto u_g$ as a function from $L^2(0,T,H)$ to $\mathcal{C}(0,T,H) \cap L^2(0,T,V)$.

In Section 2, we establish in Theorem 1, the error estimate between $u_3(\mu)$ and $u_4(\mu)$, where

$$u_3(\mu) = \mu u_{g_1} + (1-\mu)u_{g_2}, \qquad u_4(\mu) = u_{g_3(\mu)}, \quad \text{with} \quad g_3(\mu) = \mu g_1 + (1-\mu)g_2. \tag{3}$$

This result generalizes our previous result obtained in [3] for the elliptic variational inequalities. We deduce in Corollary 1 a condition on the data to get $u_3(\mu) = u_4(\mu)$ for all $\mu \in [0,1]$.

Let Ω an open bounded set in $\mathbb{R}^N$ with its regular boundary $\partial\Omega = \Gamma_1 \cup \Gamma_2$. We suppose that $\Gamma_1 \cap \Gamma_2 = \emptyset$, and $mes(\Gamma_1) > 0$. Let given a time interval $[0,T]$ for some $T > 0$. Let consider the following free boundary problem

$$\frac{\partial u}{\partial t} - \Delta u = g, \quad in \quad \Omega\times]0,T[, \tag{4}$$

$$\left.\begin{array}{l} \left|\dfrac{\partial u}{\partial n}\right| < q \Longrightarrow u = 0, \\ \\ \left|\dfrac{\partial u}{\partial n}\right| = q \Longrightarrow \exists k > 0 : \quad u = -k\dfrac{\partial u}{\partial n} \end{array}\right\} \quad on \quad \Gamma_2\times]0,T[, \tag{5}$$

$$u = b \quad on \quad \Gamma_1\times]0,T[, \tag{6}$$

with the initial boundary condition (2) and the compatibility condition

$$u_b = b \quad on \quad \Gamma_1\times]0,T[, \tag{7}$$

where n is the unit outward normal to Γ_2, g is the external force, b is given on Γ_1, (5) is the Tresca type boundary condition (for more description see [1],[2],[5]) and q is the Tresca friction coefficient on Γ_2.

We consider a family of free boundary problems (4)-(5) with the intitial condition (2), where the Dirichlet boundary condition (6) is replaced by the following Robin condition which depends on a parameter $h > 0$

$$-\frac{\partial u}{\partial n} = h(u - b) \quad on \quad \Gamma_1\times]0,T[. \tag{8}$$

Let $H = L^2(\Omega)$, $V = H^1(\Omega)$. Let

$$V_0 = \{v \in V : \quad v_{|\Gamma_1} = 0\}, \quad and \quad K = \{v \in V : \quad v_{|\Gamma_1} = b\}.$$

So we consider the following variational problems:

Problem (P) Let given $b \in L^2(]0,T[\times\Gamma_1)$, $g \in L^2(0,T,H)$ and $q \in L^2(]0,T[\times\Gamma_2)$, $q > 0$. Find u in $\mathcal{C}([0,T],H) \cap L^2(0,T,K)$ solution of the parabolic variational inequality (1), where $< \cdot,\cdot >$ is the scalar product $(\cdot,\cdot)$ in H, with the initial condition (2), where $a(u,v) = \int_\Omega \nabla u \nabla v dx$ and $\Phi(v) = \int_{\Gamma_2} q|v|ds$.

Problem (P_h) Let given $b \in L^2(]0,T[\times\Gamma_1)$, $g \in L^2(0,T,H)$ and $q \in L^2(]0,T[\times\Gamma_2)$, $q > 0$. For all coefficient $h > 0$, find $u \in \mathcal{C}(0,T,H) \cap L^2(0,T,V)$ solution of the parabolic variational inequality

$$\begin{aligned} \langle \dot{u}(t)\,,\, v - u(t)\rangle + a_h(u(t)\,,\, v - u(t)) + \Phi(v) - \Phi(u(t)) \geq (g(t), v - u(t)) \\ + h\int_{\Gamma_1} b(t)(v - u(t))ds \quad \forall v \in V, \end{aligned} \tag{9}$$

and the initial condition (2), where $a_h(u,v) = a(u,v) + h\int_{\Gamma_1} uvds$.

We know that there exists $\lambda > 0$ such that $\lambda\|v\|_V^2 \leq a(v\,,\, v), \forall v \in V_0$. Moreover, it follows from [11] that there exists $\lambda_1 > 0$ such that $a_h(v,v) \geq \lambda_h\|v\|_V^2, \forall v \in V,$ with $\lambda_h = \lambda_1 \min\{1\,,\, h\}$ so a_h is a bilinear, continuous, symmetric and coercive form on V. Therefore, there exists an unique solution to each of the two problems (P) and (P_h). Then we can consider [7, 8] the cost functional J defined by

$$J(g) = \frac{1}{2}\|u_g\|^2_{L^2(0,T,H)} + \frac{M}{2}\|g\|^2_{L^2(0,T,H)}, \tag{10}$$

where M is a positive constant, and u_g is the unique solution to (1)-(2), corresponding to the control g. One of our main purposes is to prove the existence and uniqueness of the distributed optimal control problem:

$$\text{Find } g_{op} \in L^2(0,T,H) \quad \text{such that} \quad J(g_{op}) = \min_{g\in L^2(0,T,H)} J(g). \tag{11}$$

This can be reached if we prove the strictly convexity of the cost functional J, which follows from the following monotony property : for any two control g_1 and g_2 in $L^2(0,T,H)$,

$$u_4(\mu) \leq u_3(\mu) \qquad \forall\mu \in [0,1]. \tag{12}$$

In Section 3, by using a regularization method, we prove in Theorem 2 this monotony property (12), for the solutions of the two problems (P) and (P_h). This result with a new proof and simplified, generalizes that obtained by [10] for elliptic variational inequalities.

In Section 4, we also consider the family of distributed optimal control problems $(P_h)_{h>0}$:

$$\text{Find } g_{op_h} \in L^2(0,T,H) \quad \text{such that} \quad J(g_{op_h}) = \min_{g \in L^2(0,T,H)} J_h(g), \tag{13}$$

with the cost functional

$$J_h(g) = \frac{1}{2}\|u_{g_h}\|^2_{L^2(0,T,H)} + \frac{M}{2}\|g\|^2_{L^2(0,T,H)}, \tag{14}$$

where u_{g_h} is the unique solution of (9) and (2), corresponding to the control g for each $h > 0$.

We prove the strict convexity of the cost functional (10) and (14), associated to the distributed optimal control problems (11) and (13) respectively by using the crucial property of monotony (12) (see Theorem 2). Then, the corresponding existence and uniqueness of solutions to optimal control problems (11) and (13) follows from [8].

This paper generalizes the results obtained in [6, 10] for elliptic variational equalities, and in [9] for parabolic variational equalities, to the case of parabolic variational inequalities of second kind.

2 Error estimate for convex combinations of solutions

Theorem 1 *Let u_1 and u_2 be two solutions of the parabolic variational inequality* (1) *with the same initial condition, and corresponding to the two control g_1 and g_2 respectively. We have the following estimate*

$$\begin{aligned}&\frac{1}{2}\|u_4(\mu) - u_3(\mu)\|^2_{L^\infty(0,T,H)} + \lambda\|u_4(\mu) - u_3(\mu)\|^2_{L^2(0,T,V)} + \mu\mathcal{I}_{14}(\mu)(T) + (1-\mu)\mathcal{I}_{24}(\mu)(T)\\ &\qquad + \mu\Phi(u_1) + (1-\mu)\Phi(u_2) - \Phi(u_3(\mu)) \leq \mu(1-\mu)(\mathcal{A}(T,g_1) + \mathcal{B}(T,g_2)) \quad \forall\mu \in [0,1],\end{aligned}$$

where

$$\mathcal{I}_{j4}(\mu)(T) = \int_0^T I_{j4}(\mu)(t)dt \quad \text{for } j = 1,2, \quad \mathcal{A}(T,g_1) = \int_0^T \alpha(t)dt, \quad \mathcal{B}(T,g_2) = \int_0^T \beta(t)dt,$$

$$I_{j4}(\mu) = \langle \dot{u}_j \,,\, u_4(\mu) - u_j\rangle + a(u_j \,,\, u_4(\mu) - u_j) + \Phi(u_4(\mu)) - \Phi(u_j) - \langle g_j, u_4(\mu) - u_j\rangle \geq 0,$$

$$\alpha = \langle \dot{u}_1 \,,\, u_2 - u_1\rangle + a(u_1 \,,\, u_2 - u_1) + \Phi(u_2) - \Phi(u_1) - \langle g_1, u_2 - u_1\rangle \geq 0, \tag{15}$$

$$\beta = \langle \dot{u}_2 \,,\, u_1 - u_2\rangle + a(u_2 \,,\, u_1 - u_2) + \Phi(u_1) - \Phi(u_2) - \langle g_2, u_1 - u_2\rangle \geq 0. \tag{16}$$

Corollary 1 *From Theorem* 1 we get $a.e.\, t \in [0,T]$:

$$\mathcal{A}(T,g_1) = \mathcal{B}(T,g_2) = 0 \Rightarrow \begin{cases} u_3(\mu) = u_4(\mu) \qquad \forall\mu \in [0,1], \\ I_{14}(\mu) = I_{24}(\mu) = 0 \qquad \forall\mu \in [0,1], \\ \Phi(u_3(\mu)) = \mu\Phi(u_1) + (1-\mu)\Phi(u_2) \qquad \forall\mu \in [0,1]. \end{cases}$$

Lemma 1 *Let u_1 and u_2 be two solutions of the parabolic variational inequality of second kind* (1) *with respectively as second member g_1 and g_2, then we get*

$$\|u_1 - u_2\|^2_{L^\infty(0,T,H)} + \lambda\|u_1 - u_2\|^2_{L^2(0,T,V)} \leq \frac{1}{\lambda}\|g_1 - g_2\|^2_{L^2(0,T,V')}. \tag{17}$$

3 On the property of monotony

Theorem 2 *For any two control g_1 and g_2 in $L^2(0,T,H)$, it holds that*

$$u_4(\mu) \le u_3(\mu) \quad in \quad \Omega\times[0,T], \quad \forall\mu\in[0,1]. \tag{18}$$

Here $u_4(\mu) = u_{\mu g_1+(1-\mu)g_2}$, $u_3(\mu) = \mu u_{g_1} + (1-\mu)u_{g_2}$, $u_1 = u_{g_1}$ and $u_2 = u_{g_2}$ are the unique solutions of the variational problem P, with $g = g_1$ and $g = g_2$ respectively, and for the same q, and the same initial condition (2)*. Moreover, it holds also that*

$$u_{h4}(\mu) \le u_{h3}(\mu) \quad in \quad \Omega\times[0,T], \quad \forall\mu\in[0,1]. \tag{19}$$

Here $u_{4h}(\mu) = u_{\mu g_{1h}+(1-\mu)g_{2h}}$, $u_{3h}(\mu) = \mu u_{g_{1h}} + (1-\mu)u_{g_{2h}}$, $u_{1h} = u_{g_{1h}}$ and $u_{h2} = u_{g_{h2}}$ are the unique solutions of the variational problem P_h, with $g = g_1$ and $g = g_2$ respectively, and for the same q, h, b and the same initial condition (2)*.*

Proof. Because the functional Φ is not differentiable we use the regularization method by considering for $\varepsilon > 0$ the following approach $\Phi_\varepsilon(v) = \int_{\Gamma_2} q\sqrt{\varepsilon^2+|v|^2}ds, \forall v\in V$, which is Gateaux differentiable and then we consider the limit when $\varepsilon\to 0$ by using [5, 12].

□

4 Optimal Control problems

Lemma 2 *Assume that $g \ge 0$ in $\Omega\times]0,T[$, $b \ge 0$ on $\Gamma_1\times]0,T[$, $u_b \ge 0$ in Ω. Then as $q > 0$, we have $u_g \ge 0$. Assuming again that $h > 0$, then $u_{g_h} \ge 0$ in $\Omega\times]0,T[$.*

Theorem 3 *Assume the same hypotheses of Lemma 2. Then J and J_h, defined by (10) and (14) respectively, are strictly convex applications on $L^2(0,T,H)$, so there exist unique solutions g_{op} and g_{op_h} in $L^2(0,T,H)$ respectively to the distributed optimal control problems* (11) *and* (13) *for all $h > 0$.*

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