INSTITUTO ARGENTINO DE MATEMATICA - CONICET
Y FACULTAD DE CIENCIAS EXACTAS Y NATURALES
DEPARTAMENTO DE MATEMATICA - UBA

TRABAJOS DE MATEMATICA

PREPRINT 190

Encuentro de ecuaciones diferenciales II

J. E. Bouillet, editor

Julio 1992

PUBLICACIONES PREVIAS

ISSN 0325-6677

Encuentros de Ecuaciones Diferenciales II
Trabajos de Matemática 190
Instituto Argentino de Matemática – CONICET
Buenos Aires, Julio 1992.

An Integral Equation for a Stefan Problem with Many Phases and a Singular Source

Julio E. Bouillet[1] – Domingo A. Tarzia[2]

[1] Departamento de Matemática, FCEyN
Universidad de Buenos Aires, and
IAM (CONICET)
Viamonte 1636 1er c. 1er p.
(1055) Buenos Aires – Argentina,

[2] Departamento de Matemática
Universidad Austral,
Moreno 1056,
2000 Rosario – Argentina, and
PROMAR (CONICET – UNR)
Avda. Pellegrini 250
2000 Rosario – Argentina.

(Received June, 1991)

Employing (3), (4) and $[A(D)' + \frac{1}{2}\eta E(D)]' = \frac{1}{2}E(D)$ we obtain by substraction

$$\left\{(A\circ\theta)' + \frac{1}{2}\eta[E(\theta(\eta)) - E(D)]\right\}' = \frac{1}{2}[E(\theta(\eta)) - E(D)] - B(\eta)$$

integrating in (η_1, η_2) gives

$$0 \geq (A\circ\theta)'(\eta_2^-) - (A\circ\theta)'(\eta_1^+) \geq \frac{1}{2}\int_{\eta_1}^{\eta_2} [E(\theta(\eta)) - E(D)]d\eta > 0\,,$$

a contradiction.

Case (ii). Again by contradiction, if $\theta(\eta)$ were not monotone there would be a $D > 0$, $D \leq C$, and $0 \leq \eta_1 < \eta_2$, $\theta(\eta_1) = \theta(\eta_2) = D$, $\theta(\eta) < D$ in (η_1, η_2) and thus, with an argument similar to Case (i) we would have

$$0 \leq (A\circ\theta)'(\eta_2^-) - (A\circ\theta)'(\eta_1^+) = \frac{1}{2}\int_{\eta_1}^{\eta_2} [E(\theta(\eta)) - E(D)]d\eta -$$
$$- \int_{\eta_1}^{\eta_2} B(\eta)d\eta < 0, \text{ a contradiction.}$$

(N.B. A situation like $\theta(\eta) > C$ in $(0, \eta_2)$ is excluded.)

In both cases we conclude that $\theta(\eta)$ is a decreasing function of η due to the fact that $\lim_{\eta\to\infty}\theta(\eta) = 0$. ∎

When the thesis of the Lemma 1 is verified we can consider the inverse function $\eta = \eta(\theta)$ for $0 < \theta < C$, which satisfies the following property

THEOREM 2. *Assume that* $\int_1^{+\infty} \frac{|B(s)|}{s}ds < +\infty$. *For the inverse function* $\eta = \eta(\theta)$ *we have the integral equation equivalent to* (1)-(4):

$$\eta(\theta) = T(\eta)(\theta)\,, \quad \theta \in (0, C)\,, \tag{6}$$

where the operator is defined by

$$(T(\eta(\theta)))^2 = 2\int_\theta^C \left\{\frac{1}{2}E(\psi) - \int_{\eta(\psi)}^{+\infty} \frac{B(s)}{s}ds + \int_0^\psi \frac{dA(r)}{\eta^2(r)}\right\}^{-1} dA(\psi) \tag{7}$$

Proof. We have

$$h'(\eta) = \frac{1}{2}E(\theta) - B(\eta) = \frac{h(\eta)}{\eta} - \frac{(A\circ\theta)'(\eta)}{\eta} - B(\eta)\,,$$

hence

$$\eta\left(\frac{h(\eta)}{\eta}\right)' = -\frac{(A\circ\theta)'(\eta)}{\eta} - B(\eta)\,,$$

$$\left(\frac{h(\eta)}{\eta}\right)' = -\frac{(A\circ\theta)'(\eta)}{\eta^2} - \frac{B(\eta)}{\eta}\,.$$

Assuming for the moment that $\dfrac{h(\eta)}{\eta} \to 0,\ \eta \to \infty$:

$$-\frac{h(\eta)}{\eta} = -\int_\eta^\infty \frac{(A\circ\theta)'(s)ds}{s^2} - \int_\eta^\infty \frac{B(s)}{s}ds\,,$$

and hence

$$\frac{(A\circ\theta)'(\eta)}{\eta} + \frac{1}{2}E(\theta) = \int_\eta^\infty \frac{(A\circ\theta)'(s)ds}{s^2} + \int_\eta^\infty \frac{B(s)}{s}ds$$

$$\frac{d(A\circ\theta)}{\eta} = \frac{(A\circ\theta)'(\eta)d\eta}{\eta} = -\left\{\frac{1}{2}E(\theta) - \int_{\eta(\theta)}^{+\infty} \frac{B(s)}{s}ds + \int_0^\theta \frac{dA(\psi)}{\eta^2(\psi)}\right\}d\eta$$

whence

$$\eta^2(\theta) = 2\int_\theta^C \left(\frac{1}{2}E(\psi) - \int_{\eta(\psi)}^{+\infty} \frac{B(s)}{s}ds + \int_0^\psi \frac{dA(r)}{\eta^2(r)}\right)^{-1} dA(\psi)\,,$$

i.e. (6), (7).

We now prove that $h(\eta)/\eta \to 0,\ \eta \to \infty$ (under the hypothesis $\displaystyle\int_0^{+\infty} B(s)ds < +\infty$ for $B \geq 0$). From (4) it follows

$$h(\eta) = h(0^+) + \frac{1}{2}\int_0^\eta E(\theta(s))ds - \int_0^\eta B ds\,,$$

i.e. $(A\circ\theta)'(\eta)+\frac{1}{2}\eta\, E(\theta(\eta))=(A\circ\theta)'(0^+)+\frac{1}{2}\int_0^\eta E(\theta(s))ds-\int_0^\eta B(s)ds$.

Hence

$$0=(A\circ\theta)'(\eta)=(A\circ\theta)'(0^+)+\frac{1}{2}\int_0^\eta [E(\theta(s)-E(\theta(\eta))]ds-\int_0^\eta Bds$$

Therefore, if $B\le 0$ or if $B\ge 0$ and $\int_0^\infty B(s)ds<\infty$, we will have $0\ge (A\circ\theta)'(\eta)\ge$ constant. The claim now follows due to $\lim_{\eta\to\infty} E(\theta(\eta))=0$. More precisely if $\lim_{\eta\to\infty}\frac{1}{\eta}\int_0^\eta B(s)ds=0$,

$$\frac{h(\eta)}{\eta}=\frac{h(0^+)}{\eta}+\frac{1}{2}\frac{1}{\eta}\int_0^\eta E(\theta(s))ds-\frac{1}{\eta}\int_0^\eta B(s)ds$$

and therefore $\lim_{\eta\to\infty}\frac{h(\eta)}{\eta}=0$, due again to the fact that $E(\theta(+\infty))=0$.

Theorem 3. (i) *If* $B(\eta)\le 0$, $\forall\eta\ge 0$ *then*

$$T(\eta(\theta))\le 2\sqrt{\frac{A(c)}{\lambda}}\,, \tag{9}$$

and T *is a monotone operator in the following sense*

$$\eta_1(\theta)\le\eta_2(\theta)\Longrightarrow T(\eta_1(\theta))\le T(\eta_2(\theta))\,. \tag{10}$$

(ii) *If* $B(\eta)\le 0$, $\forall\eta\ge 0$, *and*

$$-\int_0^{+\infty}\frac{B(s)}{s}ds=K<+\infty\,, \tag{11}$$

then, there exists a function $\eta_0=\eta_0(\theta)$ *which verifies the condition*

$$\eta_0\le T(\eta_0)\,. \tag{12}$$

Moreover, η_0 *is given by*

$$\eta_0(\theta)=\mu[A(C)-A(\theta)]>0 \quad \text{in } (0,C)\,,$$

where $\mu > 0$ *is a parameter to be chosen so that*

$$0 < \mu^2 \le \frac{1}{A(C)\left[K + \frac{1}{2}E(C)\right]} \,. \tag{14}$$

(iii) *Under the previous hypotheses there exists at least a solution* $\eta = \eta(\theta)$ *of the integral equation* (6)-(7).

Proof. (i) It is clear that

$$(T\eta(\theta))^2 \le 2\int_\theta^C \frac{dA(\psi)}{\frac{1}{2}E(\psi)} \le 4\int_\theta^C \frac{dA(\psi)}{E(\psi)} \le 4\frac{A(C)}{E(0^+)} \,.$$

The monotonicity is obvious.

(ii) By (11) and (13)

$$(T\eta_0(\theta))^2 \ge 2\int_\theta^C \left(\frac{1}{2}E(\psi) + K + \frac{1}{\mu^2}\Big(\frac{1}{A(C) - A(\psi)} - \frac{1}{A(C)}\Big)\right)^{-1} dA(\psi) \,.$$

Select μ so that $\frac{1}{2}E(\psi) + K \le \frac{1}{\mu^2 A(C)}$, (i.e. (14)). We find

$$(T\eta_0(\theta))^2 \ge 2\mu^2 \int_\theta^C (A(C) - A(\psi))dA(\psi) = (\eta_0(\theta))^2 \,.$$

Therefore $\eta_0 \le T\eta_0 \le T^2\eta_0 \le \cdots \le 2\Big(\int_\theta^C \frac{dA(\psi)}{E(\psi)}\Big)^{1/2} \le 2\Big(\frac{A(C)}{E(0^+)}\Big)^{1/2}$. It follows that there exists the pointwise limit $\eta(\theta) = \lim_{\eta\to\infty} (T^n\eta_0)(\theta)$. Repeated application of the monotone convergence theorem to the integrals in the definition of T gives $\eta(\theta) = T\eta(\theta)$, $0 < \theta \le C$.

Theorem 4. *There is at most one solution to* (1)-(4) *that is monotone decreasing in* $(0, \infty)$.

Proof. Assume $\theta_1(\eta)$, $\theta_2(\eta)$ are two such solutions. Two cases are possible:

(i) There are $0 \le \eta_1 \le \eta_2$ such that $\theta_1(\eta_1^+) = \theta_2(\eta_1^+)$

$$\theta_1(\eta_2^-) = \theta_2(\eta_2^-) \,, \quad \theta_1(\eta) < \theta_2(\eta) \text{ in } (\eta_1, \eta_2) \,;$$

(ii) for $0 \leq \eta_1 < \eta$, $\theta_1(\eta_1^+) = \theta_2(\eta_1^+)$ and $\theta_1(\eta) < \theta_2(\eta)$.

Case (i). By (4),

$$h_i(\eta_2^-) - h_i(\eta_1^+) = \frac{1}{2}\int_{\eta_1}^{\eta_2} E(\theta_i(\eta))d\eta - \int_{\eta_1}^{\eta_2} B(\eta)d\eta\,,$$

$i = 1, 2$. Substracting we get

$$\begin{aligned}0 \geq& (A\circ\theta_2)'(\eta_2^-) - (A\circ\theta_1)'(\eta_2^-) - \big((A\circ\theta_2)'(\eta_1^+) - (A\circ\theta_1)'(\eta_1^+)\big) + \\ &+ \frac{1}{2}\eta_2\big(E(\theta_2(\eta_2^-)) - E(\theta_1(\eta_2^-))\big) + \frac{1}{2}\eta_1\big(E(\theta_2(\eta_1^+)) - E(\theta_1(\eta_1^+))\big) \\ =& \frac{1}{2}\int_{\eta_1}^{\eta_2}\big[E(\theta_2(\eta)) - E(\theta_1(\eta))\big]d\eta > 0\end{aligned}$$

(the second line above being zero by the strict monotonicity of θ_1, θ_2).

Case (ii). With analogous considerations $(\eta_2 = +\infty)$

$$\begin{aligned}&(A\circ\theta_2)'(\eta_1^+) - (A\circ\theta_1)'(\eta_1^+) + \frac{1}{2}\eta_1\big(E(\theta_2(\eta_1^+)) - E(\theta_1(\eta_1^+))\big) = \\ =& \frac{1}{2}\int_{\eta_1}^{\infty}\big[E(\theta_2(\eta)) - E(\theta_1(\eta))\big]d\eta > 0\,.\end{aligned}$$

Therefore, the assumptions $\theta_1 \neq \theta_2$ leads to contradiction.

References

[1] J.E. Bouillet, "Self-similar solutions, having jumps and intervals of constancy, of a diffusion-heat conduction equation", IMA Preprints N° 230, U. Minnesota (March 1986).

[2] G. Lame - B.P. Clayperon, "Memoire sur la solidification par refroidissement d'un globe liquide". Annales Chimie Physique, 47(1831), 250--256.

[3] J.L. Menaldi - D.A. Tarzia, "Generalized Lamé-Clayperon solution for a one-phase source Stefan problem", Preprint (January 1990).