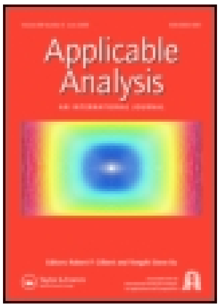

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Critical outflow for a steady-state stefan problem

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Critical Outflow for a Steady-State Stefan Problem

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Abstract: The problem of the steady temperature distribution in a container of fluid that is kept at a given positive temperature over a part of its boundary and is cooled with a given rate q on the rest of the boundary is considered. It is shown that there exists a critical $q_c > 0$ such that for $q > q_c$ the temperature is negative in a part of the fluid and this is interpreted as the existence of a solid phase together with the liquid fase. For $q < q_c$ the temperature is positive and there is only liquid in the container. Various estimates for q_c are given in terms of the geometry.

KEY WORDS: Stefan problem, variational formulation, heat flux, phase change.

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1. INTRODUCTION

We consider the problem of the steady temperature distribution of a body or a container with a fluid. Our main concern is the critical cooling rate q_c, on a part of the boundary that is being cooled, such that for larger cooling rates a solid phase appears in the system while for lower cooling rates the system remains liquid everywhere.

We assume the body to be a bounded domain $\Omega \subset R^n$, with a sufficiently regular boundary $\partial\Omega = \Gamma_1 \cup \Gamma_2$, Γ_1 and Γ_2 being disjoint portions of $\partial\Omega$ of positive $(n-1)$-dimensional measure. Assuming a phase-change temperature of 0°C for the material occupying Ω, keep Γ_1 at the temperature $\theta = b > 0$ and maintain a heat flux $q > 0$ on Γ_2. Assuming a steady-state problem, we can expect a phase change to take place in Ω if the outflow of heat $q > 0$ through Γ_2 is large enough: this paper is devoted to obtain estimates for the critical flux q_c such that

for $q < q_c$, $\theta > 0$ in Ω (no phase change), and

for $q > q_c$, θ takes negative and positive values in Ω (two phases are present).

The temperature $\theta = \theta(x)$ can be represented in the following way:

$$\theta(x) = \begin{cases} \theta_1(x) < 0, & x \in \Omega_1 \text{ (solid phase)} \\ 0, & x \in \mathcal{L} \text{ (free boundary)} \\ \theta_2(x) > 0, & x \in \Omega_2 \text{ (liquid phase)} \end{cases} \tag{1.1}$$

where $\Omega = \Omega_1 \cup \Omega_2 \cup \mathcal{L}$, and satisfies the condition below

$$\begin{aligned} &\Delta\theta_i = 0, \quad \text{in } \Omega_i \ (i = 1,2) \\ &\theta_1 = \theta_2 = 0, \quad k_1 \frac{\partial\theta_1}{\partial n} = k_2 \frac{\partial\theta_2}{\partial n} \quad \text{on } \mathcal{L}, \\ &\theta_2 = b, \quad \text{on } \Gamma_1, \end{aligned} \tag{1.2}$$

$$\begin{cases} -k_2 \dfrac{\partial \theta_2}{\partial n} = q & \text{if } \theta > 0 \text{ on } \Gamma_2 \\ -k_1 \dfrac{\partial \theta_1}{\partial n} = q & \text{if } \theta < 0 \text{ on } \Gamma_2 \end{cases}$$

where $k_i > 0$ is the thermal conductivity of phase i ($i = 1$ solid phase, $i = 2$ liquid phase).

If we define the new unknown function u as follows [5,6]

$$u = k_2\theta^+ - k_1\theta^- \quad \text{in } \Omega \tag{1.3}$$

we obtain the problem $(B = k_2 b > 0)$:

$$\begin{aligned} &\Delta u = 0, \quad \text{in } \Omega \\ &u|\Gamma_1 = B, \quad -\frac{\partial u}{\partial n}\Big|_{\Gamma_2} = q. \end{aligned} \tag{1.4}$$

The notation above and in the sequel is the following: n is the outer normal to Γ_2. $|\Gamma|$ denotes $(n-1)$-dimensional Lebesgue measure of Γ.

Γ_1, Γ_2 are assumed to be smooth (say C^1), but $\bar{\Gamma}_1 \cap \bar{\Gamma}_2$ may represent edges (R^3) or corners (R^2) of $\partial\Omega$.

On occasions (Example 1), $\partial\Omega = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ where Γ_3 is a part of the boundary where $\frac{\partial u}{\partial n} \equiv 0$; our analysis applies with minor modifications to this case as well.

In Section 2 we present the variational formulation of (1.4). The existence of a classical solution to this problem is well known. We introduce several comparison theorems that will allow us to estimate, from above and below (Section 3) the critical flux q_c. Several examples illustrate this method.

Some of the results were presented by D.A. Tarzia in [7]. Some related questions on the optimization of q can be found in [1].

A.A. B

2. SOME GENERAL RESULTS

Following the presentation in Section 1, we are led to the following problem in a bounded domain $\Omega \subset R^n$, $\partial\Omega = \Gamma_1 \cup \Gamma_2$ (disjoint), $|\Gamma_1| > 0$, $|\Gamma_2| > 0$:

$$\begin{aligned} &\Delta u = 0 \quad \text{in } \Omega , \\ &u|_{\Gamma_1} = B > 0 \quad \text{on } \Gamma_1, \\ &\left.\frac{\partial u}{\partial n}\right|_{\Gamma_2} = -q, \quad q > 0, \quad \text{on } \Gamma_2. \end{aligned} \tag{2.1}$$

That is, we have a boundary condition of the first kind on Γ_1, and of the second kind on the portion Γ_2 of $\partial\Omega$. B and q are positive constants, although many of our arguments apply to functions of $x \in \partial\Omega$.

We recall the variational formulations of (2.1): Put $a(u,v) := \int_\Omega \nabla u . \nabla v \, dx$,

$$J(v) := \frac{1}{2} a(v,v) + \int_{\Gamma_2} qv \, d\gamma ,$$

$$K = \{v \in H^1(\Omega) \; : \; v|_{\Gamma_1} = B\} .$$

The unique solution $u = u_{B,q}$ of (2.1) is characterized [3,4] by

$$\begin{cases} u \in K, \\ a(u,v-u) = - \int_{\Gamma_2} q(v-u)d\gamma \;, \quad \forall\, v \in K \; ; \end{cases} \tag{2.2}$$

and also by

$$\begin{cases} u \in K, \\ J(u) \leqslant J(v), \quad \forall v \in K \; ; \end{cases} \tag{2.3}$$

In the present setting the existence of a phase change in Ω is equivalent to $u = u_{B,q}$ taking negative values in Ω. The purpose of this note is to find conditions on Ω, B, and q under which this change of sign of u takes place.

Our techniques rely on the variational formulations (2.2), (2.3). We begin with the statement of a sequence of results that will be useful in the sequel: their proofs are fairly standard and can be found elsewhere (cf. [6]).

Consider, for fixed $B > 0$, the unique solution $u = u_q = u_{B,q}$ to (2.2). We have:

$$a(u_q^-, u_q^-) = \int_{\Gamma_2} q\, u_q^- \, d\gamma . \tag{2.4}$$

$$u_q^- \not\equiv 0 \text{ in } \Omega \text{ if and only if } u_q^- \not\equiv 0 \text{ on } \Gamma_2 . \tag{2.5}$$

In other words, there will be a change of phase in Ω if and only if u_q takes negative values on Γ_2: if u_q is going to change sign at all, u_q will take negative values at points of $\partial\Omega$ where the outflow of heat $q > 0$ takes place.

The function $0 < q \to u_q \in H^1(\Omega)$ is strictly decreasing; (2.6)

The function $0 < q \to \int_{\Gamma_2} u_q \, d\gamma$ is strictly decreasing. (2.7)

Let $f : R^+ \to R$ be the real function defined by

$$f(q) := J(u_q) = \frac{1}{2} a(u_q, u_q) + \int_{\Gamma_2} q\, u_q \, d\gamma . \tag{2.8}$$

Then

f is differentiable, f' is a continuous and strictly decreasing function, and is given by (2.9)

$$f'(q) = \int_{\Gamma_2} u_q \, d\gamma .$$

There exists a constant $C > 0$ such that $a(u_q,u_q) = C\,q^2$, and

$$f(q) = -\frac{C}{2}q^2 + B.|\Gamma_2|.q \tag{2.10}$$

The constant $C = C(\Omega,\Gamma_1,\Gamma_2) > 0$ is given by the following expressions:

$$\text{i)}\quad C = \frac{1}{q}\int_{\Gamma_2}(B-u_{qB})\,d\gamma\ ,\ \text{for some } q > 0,\ \text{or}$$

$$\text{ii)}\quad C = a(\bar{u},\bar{u}) = \int_{\Gamma_2}\bar{u}\,d\gamma\ , \tag{2.11}$$

where $\bar{u}$ is the solution of the following problem

$$\begin{cases} a(\bar{u},v) = \int_{\Gamma_2} v\,d\gamma\ , & \forall v \in V_o \\ \bar{u} \in V_o \end{cases}$$

with $V_o = \{v \in H^1(\Omega)/v|\Gamma_1 = 0\}$.

Let $q_o = \frac{B}{C}.|\Gamma_2|$; then if $q > q_o$, $u_q = u_{B,q}$ changes sign in Ω. (2.12)

The proof of this statement follows from the fact that

$$0 = f'(q_o) = \int_{\Gamma_2} u_{q_o}\,d\gamma\ .$$

The difficulty here lies with the fact that the constant C is unknown 'a priori'.

As said before, the main purpose of this note is, for a given Ω, $\partial\Omega = \Gamma_1 \cup \Gamma_2$, $B > 0$, to estimate the critical heat flux q_c such that

$$
\begin{aligned}
&q > q_c \quad \text{if and only if } u_q \text{ changes sign in } \Omega \\
&\qquad\quad \text{(two-phase Stefan problem)}, \\
&q \leqslant q_c \quad \text{if and only if } u_q \geqslant 0 \text{ in } \Omega \\
&\qquad\quad \text{(heat conduction problem)}.
\end{aligned} \tag{2.13}
$$

With this definition, $q_o \geqslant q_c$.

The following theorems give comparison results for solutions $u = u_{B,q}$. These results will be applied in the sequel. We shall only include a proof for the second and third. The first is proved using similar techniques.

Theorem 1

(a) Let $u_{B,q}$ be the solution to (2.1) in the sense (2.2), then $u_{B,q}(x) \leqslant \max_{x \in \Gamma_1} B(x)$, $x \in \Omega$;

(b) Let u_{B_1,q_1} and u_{B_2,q_2} be the solutions to (2.1) in the sense (2.2) for $B_1 = B_1(x)$, $q_1 = q_1(x)$ and $B_2 = B_2(x)$, $q_2 = q_2(x)$ respectively. Then if $B_1 \leqslant B_2$ on Γ_1, $\frac{\partial u_1}{\partial n} = -q_1(x) \leqslant -q_2(x) = \frac{\partial u_2}{\partial n}$ on Γ_2, it follows that $u_{B_1,q_1} \leqslant u_{B_2,q_2}$ in Ω.

Remark: We are clearly allowing functions $B(x)$ defined on Γ_1 (as traces) and $q(x)$ on Γ_2.

Corollary: A strict inequality is obtained in the theorem above if either of the inequalities on Γ_1 or Γ_2 is strict.

Theorem 2. Assume $\partial\Omega = \Gamma_1 \cup \Gamma_2 = \tilde{\Gamma}_1 \cup \tilde{\Gamma}_2$, with $\tilde{\Gamma}_1 \subset \Gamma_1$, $\Gamma_2 \subset \tilde{\Gamma}_2$, and let $u, \tilde{u}$ be the solutions, in the sense (2.2), of

$$\Delta u = 0 \quad \text{in} \quad \Omega \;, \quad u|\Gamma_1 = B, \quad \frac{\partial u}{\partial n}\Big|_{\Gamma_2} = -q(x), \quad \text{and}$$

$$\Delta \tilde{u} = 0 \quad \text{in} \quad \Omega \;, \quad \tilde{u}|\tilde{\Gamma}_1 = B, \quad \frac{\partial \tilde{u}}{\partial n}\Big|_{\tilde{\Gamma}_2} = -\tilde{q}(x),$$

with $\frac{\partial u}{\partial n} \geq \frac{\partial \tilde{u}}{\partial n}$ on $\Gamma_2 \subset \tilde{\Gamma}_2$, i.e. $q(x) \leq \tilde{q}(x)$ on Γ_2. Then $u \geq \tilde{u}$ in Ω.

<u>Proof</u>. Put $z = (\tilde{u} - u)^+$. We shall show that $z = 0$.
First we show that $v = \tilde{u} + z \in \tilde{K} = \{v \in H^1(\Omega) : v|\tilde{\Gamma}_1 = B\}$ and $v = u + z \in K$.
As $\tilde{u} \leq B$ in Ω (Theorem 1.(a)) we have $\tilde{u}|\Gamma_1 \setminus \tilde{\Gamma}_1 \leq B$ and, obviously, $u|\Gamma_1 \setminus \tilde{\Gamma}_1 = B$.
Hence $\tilde{u} - u|\Gamma_1 \setminus \tilde{\Gamma}_1 \leq 0$ and therefore $z|\Gamma_1 \setminus \tilde{\Gamma}_1 = 0$. We clearly have $\tilde{u} - u \mid \tilde{\Gamma}_1 = 0$, whence we conclude that $z|\Gamma_1 = 0$.
Take now $\tilde{v} = \tilde{u} + z : \tilde{v} \in H^1(\Omega)$ and $\tilde{v}|\tilde{\Gamma}_1 = \tilde{u}|\tilde{\Gamma}_1 + z|\tilde{\Gamma}_1 = B + 0 = B$, therefore $\tilde{v} \in \tilde{K}$. Similar arguments show that $v = u + z \in K$. Replacing these $v, \tilde{v}$ in the variational formulation (2.2) for $\tilde{u}$ and u gives

$$0 \leq a(z,z) = a((\tilde{u}-u)^+, (\tilde{u}-u)^+) = a(\tilde{u}-u, z) =$$

$$= -\int_{\tilde{\Gamma}_2} \tilde{q}\; z\; d\gamma + \int_{\Gamma_2} q\; z\; d\gamma =$$

$$= \int_{\Gamma_2} (-\tilde{q} + q) z\; d\gamma - \int_{\tilde{\Gamma}_2 \setminus \Gamma_2} \tilde{q}\; z\; d\gamma \leq 0 \;.$$

Thus $\int_\Omega (\nabla(\tilde{u}-u)^+)^2\, dx = 0$, and as $(\tilde{u}-u)^+|\Gamma_1 = 0$ it follows that $(\tilde{u}-u)^+ = 0$ in Ω, giving $u \geq \tilde{u}$ in Ω, as desired.

We shall now consider $q_c = q_c(\Omega)$ as a function of the domain Ω. Let Ω_1 and Ω_2 be two bounded domains, with regular boundaries, such that

$$\begin{cases} \Omega_1 \subset \Omega_2 \,, \\ \partial\Omega_1 = \Gamma_1^{(1)} \cup \Gamma_2 \,, \\ \partial\Omega_2 = \Gamma_1^{(2)} \cup \Gamma_2 \,. \end{cases} \tag{2.15}$$

where the boundary conditions on $\Gamma_1^{(i)}$ $(i = 1,2)$ and Γ_2 are of the same type as the ones defined before.

Let u_i $(i = 1,2)$ be the solution to problem (2.2) for the domain Ω_i with data B on $\Gamma_1^{(i)}$ and $q_i = q_i(x)$ on Γ_2 $(i = 1,2)$, that is

$$\begin{cases} a_i(u_i, v - u_i) = -\int_{\Gamma_2} q_i(v - u_i)\,d\gamma \,, \quad \forall v \in K_i \,, \\ u_i \in K_i \quad (i = 1,2) \end{cases} \tag{2.16}$$

where

$$\begin{cases} K_i = \{v \in H^1(\Omega_i) \,/\, v|\Gamma_1^{(i)} = B\} \,, \\ a_i(u,v) = \int_{\Omega_i} \nabla u.\nabla v \, dx \quad (i = 1,2). \end{cases} \tag{2.17}$$

Theorem 3. Under the hypotheses above, we obtain:

$$q_1 \leqslant q_2 \quad \text{on} \quad \Gamma_2 \Rightarrow u_2 \leqslant u_1 \quad \text{in} \quad \bar{\Omega}_1 \,. \tag{2.18}$$

Proof. To prove (2.18) we must show that $z = 0$ in Ω, where $z = (u_2 - u_1)^+ \in H^1(\Omega_1)$. Clearly, $z|\Gamma_1^{(1)} = 0$ because $\Gamma_1^{(1)} \in \bar{\Omega}_2$. By using (2.16) with $v_1 = u_1 + z \in K_1$, we have

$$a_1(u_1, z) = -\int_{\Gamma_2} q_1 \, z \, d\gamma \,. \tag{2.19}$$

If we extend by 0 the function z to the whole set Ω_2 and we put $v_2 = u_2 + z \in K_2$ in (2.16), we obtain

$$- \int_{\Gamma_2} q_2 \, z \, d\gamma = a_2(u_2,z) = \int_{\Omega_2} \nabla u_2 . \nabla z \, dx =$$

$$= \int_{\Omega_1} \nabla u_2 . \nabla z \, dx = a_1(u_2,z) \,. \tag{2.20}$$

From (2.19) and (2.20) we obtain

$$0 \leq a_1(z,z) = a_1(u_2 - u_1, z) = \int_{\Gamma_2} (q_1 - q_2) z \, d\gamma \leq 0 \;,$$

that is $z = 0$ in $\bar{\Omega}_1$.

Corollary. Let Ω_1 and Ω_2 be as above, i.e. they satisfy the conditions in (2.15). Then, we have that

$$q_c(\Omega_2) \leq q_c(\Omega_1) \tag{2.21}$$

that is, $q_c = q_c(\Omega)$ is a non-increasing function of the domain Ω where the order is represented by conditions (2.15).

Proof. It is enough to put $q_1 \equiv q_2$ (=q) in Theorem 3.

Remark. Let $\Omega \subset R^2$ be the set defined, in polar coordinates, by

$$\begin{cases} \Omega = \{(r,\phi) \;/\; r_o(\phi) < r < R, \quad 0 \leq \phi < 2\pi\} \;, \\ 0 < R_2 \leq r_o(\phi) \leq R_1 < R, \quad \forall\phi \in [0,2\pi) \quad (R_2 < R_1) . \end{cases} \tag{2.22}$$

Consider the following annular domains

$$\begin{cases} \Omega_1 = \{(r,\phi) \;/\; R_1 < r < R \;, \quad 0 \leq \phi < 2\pi\} \\ \Omega_2 = \{(r,\phi) \;/\; R_2 < r < R \;, \quad 0 \leq \phi < 2\pi\} \end{cases}$$

which satisfy the inclusions $\Omega_1 \subset \Omega \subset \Omega_2$: employing the values for $q_c(\Omega_1)$ and $q_c(\Omega_2)$ (cf. Example 2, Sect. 3) we obtain

$$\frac{B}{R \log \frac{R}{R_2}} \leq q_c(\Omega) \leq \frac{B}{R \log \frac{R}{R_1}} \,. \tag{2.23}$$

3. ESTIMATES FOR q_c

Two estimates for the critical flux q_c can be obtained using Theorem 1. Sect.2:

Theorem 4. (i) Let ω denote the solution of $\Delta\omega = 0$ in Ω,

$$\omega|\Gamma_1 = B, \ \omega|\Gamma_2 = 0; \tag{3.1}$$

$$\text{Define } q_i := \inf_{\Gamma_2} \left(- \frac{\partial\omega}{\partial n}\Big|_{\Gamma_2}\right) . \tag{3.2}$$

Then $q \leqslant q_i$ implies $u_q \geqslant \omega \geqslant 0$ in Ω.
As $\omega \geqslant 0$ in Ω, this implies $q_i \leqslant q_c$.

(ii) Let $p_2 \in \Gamma_2$ and the affine function π be such that

$$\pi|\Gamma_1 \geqslant B, \quad \pi(p_2) = 0, \quad \pi|\Gamma_2 \geqslant 0 \tag{3.3}$$

and put

$$q_s = \sup_{x \in \Gamma_2} \left(- \frac{\partial\pi}{\partial n}(x)\right). \tag{3.4}$$

Then $q > q_s$ implies $u_q < \pi$ in Ω.

(iii) $u_q(p_2) < 0$ for all $q > q_s$, and therefore $q_c \leqslant q_s$.
On the other hand, $\omega \leqslant \pi$ in Ω and, if $\omega \neq \pi$ we have $q_i < q_s$.

Remark. A sufficient condition for such π to exist is the existence of supporting hyperplanes σ to Ω at $p_2 \in \Gamma_2$ which are a positive distance away from $\bar{\Gamma}_1$: construct an affine function π vanishing on σ (and at p_2), and such that $\pi|\Gamma_1 \geqslant B$, and there is $p_1 \in \bar{\Gamma}_1$ with $\pi(p_1) = B$. The optimal q_s can be obtained by selecting p_2, $\sigma = \sigma_{p_2}$ such that $\mathrm{dist}(\sigma,\bar{\Gamma}_1)$ is largest.

This construction fails if Γ_2 is a flat portion of $\partial\Omega$, e.g. the side of a triangle $\Omega \subset R^2$, Γ_1 being formed by the other two sides.

The fact that $u_q(p_2) < 0$ suggests that the second phase appears at $p_2 \in \Gamma_2$, the point "farthest" from Γ_1 (cf. (2.5)).

In many cases (cf. Examples below) the function π can be obtained by satisfying (3.3) and $\pi(p_1) = B$, where $p_1 \in \bar{\Gamma}_1$, $p_2 \in \Gamma_2$ and $\text{dist}(p_1,p_2) = \sup_{x \in \Gamma_2} \text{dist}(x,\bar{\Gamma}_1)$. There is no uniqueness in general for the points $p_1 \in \bar{\Gamma}_1$, $p_2 \in \Gamma_2$. For instance, in Example 1. there are many $p_1 = (0,y)$ and $p_2 = (x_o,y)$, with $y \in [0,y_o]$.

In the following examples the functions ω and/or π can be found explicity, leading to the determination of q_i and/or q_s. The straightforward computations are omitted.

Example 1. $\Omega = (0,x_o) \times (0,y_o) \subset R^2$, $\Gamma_1 = \{0\} \times [0,y_o]$, $\Gamma_2 = \{x_o\} \times [0,y_o]$, $\Gamma_3 = (0,x_o) \times \{0\} \cup (0,x_o) \times \{y_o\}$. We obtain

$$\pi(x) \equiv \omega(x) = B - \frac{B}{x_o} x .$$

Therefore $q_s = q_i = q_c = B/x_o$.

Example 2. Anular region: $\Omega = \{(r,\phi): r_1 < r := (x^2+y^2)^{1/2} < r_2, 0 \leqslant \phi < 2\pi\} \subset R^2$

$$\Gamma_1 = \{(r,\phi): r = r_1, \quad 0 \leqslant \phi < 2\pi\} ;$$

$$\Gamma_2 = \{(r,\phi): r = r_2, \quad 0 \leqslant \phi < 2\pi\} ;$$

We obtain $\omega(r) = B \dfrac{\log \frac{r_2}{r}}{\log \frac{r_2}{r_1}}$

$$q_i = -\omega'(r_2) = \frac{B}{r^2 \log \frac{r_2}{r_1}} = q_c ,$$

$$\pi(x,y) = \pi(x) = \frac{B(r_2 - x)}{r_2 - r_1}$$

$$q_s = \max_{0 \leq \phi < 2\pi} \frac{B \cos \phi}{r_2 - r_1} = \frac{B}{r_2 - r_1} \quad ;$$

obviously, $q_s > q_c$.

Example 3. Spherical shell: $\Omega = \{(r,\phi,\lambda) : r_1 < r := (x^2 + y^2 + z^2)^{1/2} < r_2\} \subset R^3$; $\Gamma_1 = \{r = r_1\}$; $\Gamma_2 = \{r = r_2\}$.

$$\omega(r) = B \frac{r_1 r_2}{r_1 - r_2} \left(\frac{1}{r} - \frac{1}{r_2} \right) ,$$

$$q_i = -\omega' (r_2) = \frac{B r_1}{(r_2 - r_1) r_2} = q_c .$$

$$\pi(x) = B \frac{r_2 - x}{r_2 - r_1} \quad \text{(as in Example 2.)}.$$

Therefore, $q_s = \dfrac{B}{r_2 - r_1} > q_c$.

Example 4. Let $R > 0$, $0 < \theta_o < \pi/2$:

$$\Omega = \{(r,\phi) : r := (x^2 + y^2)^{1/2} < R\} \subset R^2,$$

$$\Gamma_1 = \{(r,\phi) : r = R , \ -\theta_o < \phi < \theta_o\} ,$$

$$\Gamma_2 = \{(r,\phi) : r = R , \ \theta_o \leq \phi \leq 2\pi - \theta_o\} .$$

$$\pi(r,\phi) = \pi(x) = \frac{B}{R(1 + \cos \theta_o)} (x + R)$$

$$q_s = \max_{(x,y)=(r,\phi)\in\Gamma_2}(\pi'(x).\cos\phi) = \frac{B}{R(1+\cos\theta_o)} \geqslant q_c \ .$$

The upper estimate q_s for q_c relies on the construction of a linear "barrier" π to the solution $u(x)$ of (2.1); as remarked before this in turn depends strongly on the geometry of the portion Γ_2 of $\partial\Omega$ (e.g. "strict convexity"). We discuss now the use of Poincaré type barriers to compute a value q_s. Let $\xi \in \Gamma_2$ be such that there exists $x_o \notin \bar{\Omega}$, $|x_o - \xi| = a > 0$, $\{x : |x - x_o| \leqslant a\} \cap \bar{\Omega} = \{\xi\}$. The Poincaré barriers at $\xi \in \Gamma_2$ are (cf. [2])

$$V(x,\xi) = C\left(\frac{1}{a} - \frac{1}{|x-x_o|^{n-2}}\right), \quad C > 0, \quad n \geqslant 3 \ ;$$

$$= C \log\left(\frac{|x - x_o|}{a}\right), \quad C > 0, \quad n = 2 \ .$$

The following properties of V are straightforward

Lemma. (i) $\Delta_x V(x,\xi) = 0$ in Ω ;

(ii) $\nabla_x V(x,\xi) = (n-2)\, C \dfrac{x - x_o}{|x-x_o|^n}$, $n \geqslant 3$;

$= C \dfrac{x - x_o}{|x-x_o|^2}$, $n = 2$;

(iii) $\forall x \in \Gamma_2$, $\dfrac{\partial V(x,\xi)}{\partial n(x)} = C \,.\dfrac{(x-x_o).n(x)}{|x - x_o|^n}$;

(iv) $\forall x \in \Gamma_2$, $\dfrac{\partial V(x,\xi)}{\partial n(x)} \geqslant \dfrac{\partial V(\xi,\xi)}{\partial n(\xi)} = -\dfrac{C}{a^{n-1}}$ (< 0) .

Let $p_1 \in \bar{\Gamma}_1$, $p_2 \in \Gamma_2$ be such that $d = \sup_{x\in\Gamma_2} \operatorname{dist}(x,\bar{\Gamma}_1) = |p_2 - p_1| > 0$. Denote $V(x,\xi) = V_{a,c}(x,\xi)$, with $\xi = p_2$.

Theorem 5. Assume

(a) $V|\Gamma_1 \geqslant B$ if and only if $V(p_1,\xi) \geqslant B$;

(b) Let $$q_V = \inf_{V_{a,c}(p_1,\xi)=B} \left[- \inf_{x\in\Gamma_2} \frac{\partial V_{a,c}(x,\xi)}{\partial n(x)} \right]$$

$$= \inf_{V_{a,c}(p_1,\xi)=B} \left(\frac{C}{a^{n-1}}\right). \tag{3.5}$$

Then if u_q is the solution to (2.1), $q > q_V$ implies $V(x,\xi) > u_q(x)$ in Ω. Therefore $u_q(\xi) < 0$, and if q_c is the critical value for Ω,B, then $q_c < q_V$.

Remark. (a) is an immediate consequence of the monotonicity of V on $|x - x_o|$, for special domains (cf. Examples).

Proof. Apply Theorem 1, part (b) to $u_q(x)$ and $V(x,\xi)$: Assuming $V(p_1,\xi) = B$, we have $V \geqslant u_q$ on Γ_1 by (a).

Now $$\left.\frac{\partial u_q}{\partial n}\right|_{\Gamma_2} = - q < - q_V \leqslant \frac{\partial V(x,\xi)}{\partial n(x)}, \quad \forall x \in \Gamma_2,$$ after selection of the parameters a,C in the definition of V.

Example 5. As an application we shall compute $q_V(\geqslant q_c)$ for the anular region in Example 2 above: Put $x_o = (x_o,0)$, $p_1 = (r_1,0)$, $p_2 = \xi = (r_2,0)$, $x_o > r_2 > r_1$, $a = x_o - r_2$. Then, $V(x,\xi) = C \log \frac{|x-x_o|}{a}$ and the condition $V(p_1,\xi) = B$ is $C.\log(\frac{a+r_2-r_1}{a}) = B$.

Hence $q_V = \inf_{C\log\left(\frac{a+r_2-r_1}{a}\right)=B} \frac{C}{a} = \inf_{a>0} \frac{B}{a\log\left(\frac{a+r_2-r_1}{a}\right)}$

$$= B \cdot \frac{1}{\sup_{a>0} f(a)}, \quad \text{with} \quad f(a) = a\log\left(1+\frac{r_2-r_1}{a}\right).$$

The analysis of $f(a)$ is easily accomplished in the usual way, yielding $\sup_{a>0} f(a) = r_2 - r_1$, whence

$$q_V = \frac{B}{r_2 - r_1}, \text{ same value as in Example 2.}$$

As the previous example shows, letting $a \to \infty$ when Γ_2 has convexity properties "flattens" the Poincaré barrier yielding the same upper bound given by the plane barrier. It is precisely the case when this plane barrier is not available that suggests employing the functions V.

Consider a domain $\Omega \subset \{(x,y): x < 0\} \subset R^2$, such that $\Gamma_2 \subset \partial\Omega$ be the segment $\{(0,y): -1 < y < 1\}$ and such that there exists an open triangle T with vertices $(0,1)$, $(0,-1)$, $(-A,0)$, $A > 0$, contained in Ω. Notice that Γ_2 is a common boundary to both regions, so if we define $\Gamma_1^{(T)}$ as being composed of the two sides joining $(0,-1)$, $(-A,0)$ and $(0,1)$ of T, we can apply Theorem 3 to obtain

$$q_c(\Omega) \leq q_c(T) \leq q_V = q_V(T).$$

Therefore we will compute q_V for T:

Example 6. By symmetry we take $\xi = p_2 = (0,0) \in \Gamma_2$. In order to find the distance between $\bar{\Gamma}_1$ and $x_o = (a,0)$ in the definition

$$V(x,\xi) = C \cdot \log\frac{|x-x_o|}{a},$$

we easily compute

$$\text{distance } (x_o, \bar{\Gamma}_1) = \sqrt{1 + a^2} \quad \text{if} \quad a \geq \frac{1}{A}$$

$$= \frac{a + A}{\sqrt{1 + A^2}} \quad \text{if} \quad a < \frac{1}{A}$$

so $V \geq B$ on Γ_1, provided $C \,.\, \log \dfrac{\text{distance}(x_o, \bar{\Gamma}_1)}{a} = B$.

The reasoning now parallels that of Example 5 and

$$q_V = \min \left(\frac{B}{\sup\limits_{a \geq \frac{1}{A}} \left(a \log \frac{\sqrt{1 + a^2}}{a}\right)}, \frac{B}{\sup\limits_{0 < a < \frac{1}{A}} \left(a \log \left(\frac{a + A}{a\sqrt{1+A^2}}\right)\right)} \right). \tag{3.7}$$

1: Case $a \geq 1/A$:
The analysis of the function $f(a) = \frac{a}{2} \log \left(1 + \frac{1}{a^2}\right)$ leads to solving $1 + a^2 = a^2 \, e^{(2/(1+a^2))}$: this gives a critical point $a_c \sim 0.5049$ where $f'(a_c) = 0$, with $f' < 0$ for $a > a_c$. Therefore,

$$\sup_{a \geq \frac{1}{A}} \left(a \log \sqrt{1 + \frac{1}{a^2}}\right) \sim 0.4026 = f(a_c) \quad \text{if} \quad a_c \geq 1/A,$$

$$= f(1/A) = \frac{1}{A} \log \sqrt{1 + A^2} \quad \text{if} \quad a_c < 1/A.$$

2: Case $a < 1/A$:
The discussion of $g(a) = a \log \left(\frac{a + A}{a\sqrt{1+A^2}}\right) = a \log \left(\alpha + \frac{\beta}{a}\right)$,

$\alpha = \frac{1}{\sqrt{1+A^2}}$, $\beta = \frac{A}{\sqrt{1+A^2}}$, leads to a critical point a^c that satisfies $\left(\alpha + \frac{\beta}{a^c}\right) \log \left(\alpha + \frac{\beta}{a^c}\right) = \frac{\beta}{a^c}$ or, putting $z = \alpha + \frac{\beta}{a}$

$z \log z = z - \alpha$, $0 < z < 1$. Standard considerations (namely, secant and tangent line approximations to z^c) give, for $z^c :=$ $\alpha + \frac{\beta}{a^c}$, the estimates

$$e - \alpha(e - 1) < z^c < e - \alpha$$

whence for a^c one has

$$\frac{A}{e\sqrt{1+A^2} - 2} < a^c < \frac{A}{e(\sqrt{1+A^2} - 1)} < \frac{1}{A} .$$

Moreover, the graph of $g(a)$ shows that $g(a^c) > g(\frac{1}{A}) = \frac{1}{A} \log \sqrt{1+A^2}$ and therefore, if $a_c \sim 0.5049 < 1/A$ in Case 1 (i.e. $A < 2$) we find by (3.7)

$$q_V = \frac{B}{g(a^c)} < \frac{B(e\sqrt{1+A^2} - 2)}{A \log \frac{(e\sqrt{1+A^2} - 1)}{\sqrt{1+A^2}}} , \text{ for instance.}$$

Remark. A lower bound for q_c is readily found observing that the plane π such that $\pi|\Gamma_2 = 0$, $\pi(-A,0) = B$ satisfies also $\pi|\Gamma_1 \leq B$. We therefore have $\frac{B}{A} \leq q_c$.

We introduce now a variant of the Poincaré barriers used above to examine our last

Example 7. Put $\Omega = \{(x,y): -E \leq x \leq E, -h \leq y \leq h\}$, $E > 0$, $h > 0$. Define Γ_1 as being composed of top and bottom sides of this rectangle, and Γ_2 as being made of the two vertical sides. We maintain a temperature $B > 0$ on Γ_1 and ask for the minimum heat flux q on Γ_2 for which the zone $\{(x,y) \in \Omega : u(x,y) > 0\}$ (whose boundary obviously contains Γ_1) be disconnected, a region where $u < 0$ joining the two components of Γ_2.

Clearly, if is enough to consider $\Omega \cap \{x > 0, y > 0\}$, set $x_o = (E + a, 0)$, $\xi = (E,0)$, observe that the point in Γ_1 nearest to x_o is $p_1 = (E,h)$, put $d := \sqrt{h^2 + a^2}$ and define, with $1 < \delta < \frac{d}{a}$,

$$V(x,\xi) = B. \frac{\log\left(\frac{|x-x_o|}{\delta a}\right)}{\log\left(\frac{d}{\delta a}\right)} .$$

With the choice of constants we have $V(p_1,\xi) = B$, whence $V|\Gamma_1 \geqslant B$. With an argument already employed, select

$$q_V := \frac{B}{\sup\limits_{0<a<\frac{d}{\delta}} \left(a \log\left(\frac{d}{\delta a}\right)\right)} = e\, B\, \frac{\delta}{d} = \frac{e\, B\delta}{\sqrt{h^2 + a^2}}$$

and according to Theorem 1 (b). Sect. 2, if $q > q_V$, $u_q \leqslant V(x,\xi)$ in Ω. But $V(x,\xi) = 0$ if $|x-x_o| = \delta\, a > a$, and therefore $u_q < 0$ in a circular segment that penetrates $\delta a - a$ into Ω. Hence if $a(\delta-1) > E$ the zone $u_q > 0$ is not connected. We now optimize the choice of a,δ.

We want to have $a(\delta-1) > E$ with the restrictions $1 < \delta < \frac{\sqrt{h^2+a^2}}{a}$. Putting $\eta = \delta a$ reduces the problem to find a,η so that $\eta - a > E$ and $a < \eta < \sqrt{h^2+a^2}$ or

$$E + a < \eta < \sqrt{h^2+a^2} .$$

A discussion of the functions of $a > 0$, $\eta = E + a$ and $\eta = \sqrt{h^2+a^2}$ yields

(i) No solution if $h < E$ (which is obvious);

(ii) if $h > E$, the region

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$$D = \{(a,\eta):\ 0 < a < \frac{h^2 - E^2}{2E}\ ,\quad a + E < \eta < \sqrt{h^2 + a^2}\}\ .$$

In terms of a, η we need, for $h > E$,

$$\inf q_V = \inf_{(a,\eta)\in D} \frac{e\,B\,\eta}{a\sqrt{h^2+a^2}}\ .$$

The minimum value is attained at $a = \frac{h^2 - E^2}{2E}$, giving

$$\inf q_V = \frac{2e\,E\,B}{h^2 - E^2}$$

with $a = \frac{h^2 - E^2}{2E}$, $\delta = \frac{E^2 + h^2}{E^2 - h^2} = \frac{d}{a}$. For this value δ, $V(x,\xi)$ is not defined.

By theorem 3 Sect. 2, the same phenomenon is found for any domain $\tilde{\Omega} \supset \Omega$ such that $\tilde{\Gamma}_2 = \Gamma_2$, if temperature $B > 0$ is maintained on $\partial\tilde{\Omega}\setminus\Gamma_2 = \tilde{\Gamma}_1$.

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