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Numerical analysis of a family of simultaneous distributed-boundary mixed elliptic optimal control problems and their asymptotic behaviour through a commutative diagram and error estimates

Carolina M. Bollo [a], Claudia M. Gariboldi [a], Domingo A. Tarzia [b,c,*]

[a] *Depto. Matemática, FCEFQyN, Universidad Nacional de Río Cuarto, Ruta 36 Km 601, 5800 Río Cuarto, Argentina*

[b] *Depto. Matemática, FCE, Universidad Austral, Paraguay 1950, S2000FZF Rosario, Argentina*

[c] *CONICET, Argentina*

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Dedicated to Professor Weimin Han on the occasion of his 60th birthday

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ABSTRACT

In this paper, we consider a family of simultaneous distributed-boundary optimal control problems (P_α) on the internal energy and the heat flux for a system governed by a mixed elliptic variational equality with a parameter $\alpha > 0$ (the heat transfer coefficient on a portion of the boundary of the domain) and a simultaneous distributed-boundary optimal control problem (P) governed also by an elliptic variational equality with a Dirichlet boundary condition on the same portion of the boundary. We formulate discrete approximations $(P_{h\alpha})$ and (P_h) of the optimal control problems (P_α) and (P) respectively, for each $h > 0$ and for each $\alpha > 0$, through the finite element method with Lagrange's triangles of type 1 with parameter h (the longest side of the triangles). The goal of this paper is to study the convergence of this family of discrete simultaneous distributed-boundary mixed elliptic optimal control problems $(P_{h\alpha})$ when the parameters α goes to infinity and the parameter h goes to zero simultaneously. We prove the convergence of the family of discrete problems $(P_{h\alpha})$ to the discrete problem (P_h) when $\alpha \to +\infty$, for each $h > 0$, in adequate functional spaces. We study the convergence of the discrete problems $(P_{h\alpha})$ and (P_h), for each $\alpha > 0$, when $h \to 0^+$ obtaining a commutative diagram which relates the continuous and discrete simultaneous distributed-boundary mixed elliptic optimal control problems $(P_{h\alpha}), (P_\alpha), (P_h)$ and (P) by taking the limits $h \to 0^+$ and $\alpha \to +\infty$ respectively. We also study the double convergence of $(P_{h\alpha})$ to (P) when $(h, \alpha) \to (0^+, +\infty)$ which represents the diagonal convergence in the above commutative diagram.

* Corresponding author at: Depto. Matemática, FCE, Universidad Austral, Paraguay 1950, S2000FZF Rosario, Argentina.
E-mail addresses: cbollo@exa.unrc.edu.ar (C.M. Bollo), cgariboldi@exa.unrc.edu.ar (C.M. Gariboldi), DTarzia@austral.edu.ar (D.A. Tarzia).

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1. Introduction

We consider a bounded domain Ω in $\mathbb{R}^d$ whose regular boundary Γ consists of the union of two disjoint portions Γ_i, $i = 1, 2$, with $|\Gamma_i| > 0$, where $|\Gamma_i|$ denotes the $(d-1)$-dimensional Hausdorff measure of the portion Γ_i on Γ. The outward normal vector on the boundary is denoted by n. We formulate the following classical steady-state heat conduction problems with mixed boundary conditions [1–5]:

$$-\Delta u = g \text{ in } \Omega, \qquad u\big|_{\Gamma_1} = b, \qquad -\frac{\partial u}{\partial n}\Big|_{\Gamma_2} = q, \tag{1.1}$$

$$-\Delta u = g \text{ in } \Omega, \qquad -\frac{\partial u}{\partial n}\Big|_{\Gamma_1} = \alpha(u-b), \qquad -\frac{\partial u}{\partial n}\Big|_{\Gamma_2} = q, \tag{1.2}$$

where u is the temperature in Ω, g is the internal energy in Ω, $b = Const. > 0$ is the temperature on Γ_1 for the system (1.1) and the temperature of the external neighbourhood on Γ_1 for the system (1.2) respectively, q is the heat flux on Γ_2 and $\alpha > 0$ is the heat transfer coefficient on Γ_1, which satisfy the hypothesis: $g \in H = L^2(\Omega)$ and $q \in Q = L^2(\Gamma_2)$.

Throughout the paper we use the following notation:

$$\begin{aligned}
&V = H^1(\Omega), \quad V_0 = \{v \in V / v = 0 \text{ on } \Gamma_1\},\\
&K = \{v \in V / v = b \text{ on } \Gamma_1\} = b + V_0,\\
&a(u,v) = \int_\Omega \nabla u \,\nabla v\, dx, \quad L(v) = \int_\Omega g v\, dx - \int_{\Gamma_2} q\gamma(v)\, d\Gamma,\\
&a_\alpha(u,v) = a(u,v) + \alpha \int_{\Gamma_1} \gamma(u)\gamma(v)\, d\Gamma, \quad L_\alpha(v) = L(v) + \alpha \int_{\Gamma_1} b\gamma(v)\, d\Gamma,
\end{aligned}$$

where $\gamma : V \to L^2(\Gamma)$ denotes the trace operator on Γ. In what follows, we write u for the trace of a function $u \in V$ on the boundary. In a standard way, we obtain the following variational formulations of (1.1) and (1.2), [6]:

$$\text{find } u \in K \text{ such that } a(u,v) = L(v) \text{ for all } v \in V_0, \tag{1.3}$$

$$\text{find } u_\alpha \in V \text{ such that } a_\alpha(u_\alpha,v) = L_\alpha(v) \text{ for all } v \in V. \tag{1.4}$$

The standard norms on V and V_0 are denoted by

$$\begin{aligned}
&\|v\|_V = \left(\|v\|^2_{L^2(\Omega)} + \|\nabla v\|^2_{L^2(\Omega;\mathbb{R}^d)}\right)^{1/2} \text{ for } v \in V,\\
&\|v\|_{V_0} = \|\nabla v\|_{L^2(\Omega;\mathbb{R}^d)} \text{ for } v \in V_0.
\end{aligned}$$

It is well known by the Poincaré inequality, see [7,8], that on V_0 the above two norms are equivalent. Note that the bilinear, symmetric and continuous forms a and a_α are coercive on V_0 and V respectively, that is, [9]:

$$\exists \lambda > 0 \quad \text{such that} \quad a(v,v) = \|v\|^2_{V_0} \geq \lambda \|v\|^2_V \text{ for all } v \in V_0, \tag{1.5}$$

$$\exists \lambda_\alpha > 0 \quad \text{such that} \quad a_\alpha(v,v) = \|v\|^2_{V_0} \geq \lambda_\alpha \|v\|^2_V \text{ for all } v \in V \tag{1.6}$$

where $\lambda_\alpha = \lambda_1 \min\{1, \alpha\}$, with $\lambda_1 > 0$ the coerciveness constant for the bilinear form a_1, [9,10].

We remark that, under additional hypotheses on the data g, q and b, problem (1.1) can be considered as steady-state two-phase Stefan problem, see [5,6,10,11].

We consider the following continuous optimal control problems [12–14]:

(P) A simultaneous distributed and Neumann boundary optimal control problem, given by:

$$\text{find} \quad (\overline{g}, \overline{q}) \in H \times Q \quad \text{such that} \quad J(\overline{g}, \overline{q}) = \min_{(g,q)\in H\times Q} J(g,q) \tag{1.7}$$

with

$$J(g,q) = \frac{1}{2}\|u_{gq} - z_d\|_H^2 + \frac{M_1}{2}\|g\|_H^2 + \frac{M_2}{2}\|q\|_Q^2 \tag{1.8}$$

where u_{gq} is the unique solution to the variational equality (1.3) for $g \in H$ and $q \in Q$, $z_d \in H$ given and M_1 and M_2 are positive constants given.

(P_α) For each $\alpha > 0$, the simultaneous distributed and Neumann boundary optimal control problem:

$$\text{find} \quad (\overline{g}_\alpha, \overline{q}_\alpha) \in H \times Q \quad \text{such that} \quad J_\alpha(\overline{g}_\alpha, \overline{q}_\alpha) = \min_{(g,q)\in H\times Q} J_\alpha(g,q) \tag{1.9}$$

with

$$J_\alpha(g,q) = \frac{1}{2}\|u_{\alpha gq} - z_d\|_H^2 + \frac{M_1}{2}\|g\|_H^2 + \frac{M_2}{2}\|q\|_Q^2 \tag{1.10}$$

where $u_{\alpha gq}$ is a solution to the variational equality (1.4) for $g \in H$, $q \in Q$ and $\alpha > 0$, $z_d \in H$ is given and M_1 and M_2 are positive constants.

In relation with the simultaneous optimal control problems (1.7) and (1.9), we define the adjoint states, as the unique solutions of the variational equalities, [12]:

$$\text{find} \;\; p_{gq} \in V_0 \;\; \text{such that} \;\; a(p_{gq}, v) = (u_{gq} - z_d, v)_H \;\; \text{for all} \;\; v \in V_0, \tag{1.11}$$

$$\text{find} \;\; p_{\alpha gq} \in V \;\; \text{such that} \;\; a_\alpha(p_{\alpha gq}, v) = (u_{\alpha gq} - z_d, v)_H \;\; \text{for all} \;\; v \in V. \tag{1.12}$$

The unique continuous simultaneous vectorial optimal controls $(\overline{g}, \overline{q})$ and $(\overline{g}_\alpha, \overline{q}_\alpha)$ can be characterized, following [12,15], as a fixed point on $H \times Q$ for suitable operators W and W_α over their optimal adjoint system states $p_{\overline{g}\,\overline{q}} \in V_0$ and $p_{\alpha\overline{g}_\alpha\,\overline{q}_\alpha} \in V$, defined by:

$$W : H \times Q \to H \times Q \quad \text{such that} \quad W(g,q) = (-\frac{1}{M_1}p_{gq}, \frac{1}{M_2}p_{gq})$$
$$W_\alpha : H \times Q \to H \times Q \quad \text{such that} \quad W_\alpha(g,q) = (-\frac{1}{M_1}p_{\alpha gq}, \frac{1}{M_2}p_{\alpha gq}).$$

The limit of the optimal control problems (1.9) when $\alpha \to +\infty$ was studied in [12] and it was proved that:

$$\lim_{\alpha\to+\infty} \left\|u_{\alpha\overline{g}_\alpha\,\overline{q}_\alpha} - u_{\overline{g}\,\overline{q}}\right\|_V = 0, \quad \lim_{\alpha\to+\infty} \left\|p_{\alpha\overline{g}_\alpha\,\overline{q}_\alpha} - p_{\overline{g}\,\overline{q}}\right\|_V = 0,$$

$$\lim_{\alpha\to+\infty} \|(\overline{g}_\alpha, \overline{q}_\alpha) - (\overline{g}, \overline{q})\|_{H\times Q} = 0$$

where the norm in $H \times Q$ is defined by:

$$\|(g,q)\|_{H\times Q}^2 = \|(g)\|_H^2 + \|q\|_Q^2, \quad \forall (g.q) \in H \times Q.$$

Now, we consider the finite element method and a polygonal domain $\Omega \subset \mathbb{R}^n$ with a regular triangulation with Lagrange triangles of type 1, constituted by affine-equivalent finite element of class C^0 being h the parameter of the finite element approximation which goes to zero [16,17]. Then, we discretize the elliptic variational equalities for the system states (1.3) and (1.4), the adjoint system states (1.11) and (1.12), and the cost functional (1.8) and (1.10), respectively. In general, the solution of a mixed elliptic boundary problem belongs to $H^r(\Omega)$ with $1 < r \le 3/2 - \epsilon$ ($\epsilon > 0$), but there exist some examples which solutions belong to $H^r(\Omega)$ with $2 \le r$ [1,4,18].

The goal of this paper is to study the numerical analysis, by using the finite element method, of the convergence results corresponding to the continuous simultaneous distributed-boundary elliptic optimal control problems (1.7) and (1.9) when $\alpha \to +\infty$. Moreover, the following commutative diagram which relates the continuous simultaneous distributed-boundary mixed optimal control problems (P_α) and (P), with the discrete simultaneous distributed-boundary mixed optimal control problems $(P_{h\alpha})$ and (P_h) is obtained by taking the limits $h \to 0^+$, $\alpha \to +\infty$ and $(h,\alpha) \to (0^+,+\infty)$ as follows:

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\textbf{Problem } (P_\alpha) \quad (\overline{g}_\alpha,\overline{q}_\alpha),\ u_{\alpha\overline{g}_\alpha\overline{q}_\alpha},\ p_{\alpha\overline{g}_\alpha\overline{q}_\alpha} \arrow[r, "\alpha\to+\infty"] & \textbf{Problem } (P) \quad (\overline{g},\overline{q}),\ u_{\overline{g}\,\overline{q}},\ p_{\overline{g}\,\overline{q}} \\
(\overline{g}_{h\alpha},\overline{q}_{h\alpha}),\ u_{h\alpha\overline{g}_{h\alpha}\overline{q}_{h\alpha}},\ p_{h\alpha\overline{g}_{h\alpha}\overline{q}_{h\alpha}} \quad \textbf{Problem } (P_{h\alpha}) \arrow[u, "h\to0^+"'] \arrow[ur, "(h,\alpha)\to(0^+,+\infty)"] \arrow[r, "\alpha\to+\infty"] & (\overline{g}_h,\overline{q}_h),\ u_{h\overline{g}_h\overline{q}_h},\ p_{h\overline{g}_h\overline{q}_h} \quad \textbf{Problem } (P_h) \arrow[u, "h\to0^+"']
\end{tikzcd}$$

where $(\overline{g}_h,\overline{q}_h)$, $u_{h\overline{g}_h\overline{q}_h}$ and $p_{h\overline{g}_h\overline{q}_h}$ are the optimal control, system state and adjoint state of the discrete simultaneous distributed-boundary optimal control problem (P_h) for each $h>0$, and $(\overline{g}_{h\alpha},\overline{q}_{h\alpha})$, $u_{h\alpha\overline{g}_{h\alpha}\overline{q}_{h\alpha}}$ and $p_{h\alpha\overline{g}_{h\alpha}\overline{q}_{h\alpha}}$ are the optimal control, the system state and adjoint state of the discrete simultaneous distributed-boundary optimal control problem $(P_{h\alpha})$ for each $h>0$ and $\alpha>0$, respectively.

The study of the limit $h\to0^+$ of the discrete solutions of optimal control problems can be considered as a classical limit, see [19–30] but the double limit $(h,\alpha)\to(0^+,+\infty)$ can be considered as a new ones for a vectorial control problem.

The paper is structured as follows. In Section 2, we formulate the discrete elliptic variational equalities for the system states u_{hgq} and $u_{h\alpha gq}$, we define the discrete cost functional J_h and $J_{h\alpha}$, we formulate the discrete simultaneous distributed-boundary optimal control problems (P_h) and $(P_{h\alpha})$, and the discrete elliptic variational equalities for the adjoint states p_{hgq} and $p_{h\alpha gq}$ for each $\alpha>0$ and $h>0$. We obtain properties for the discrete optimal control problems and we define contraction operators W_h and $W_{h\alpha}$ which allows obtain the optimal controls $(\overline{g}_h,\overline{q}_h)$ and $(\overline{g}_{h\alpha},\overline{q}_{h\alpha})$ as fixed points. In Section 3, we study the convergences of the discrete optimal control problems (P_h) to (P), and $(P_{h\alpha})$ to (P_α) when $h\to0^+$ (for each $\alpha>0$). In Section 4, we study the convergence of the discrete optimal control problems $(P_{h\alpha})$ to (P_h) when $\alpha\to+\infty$ (for each $h>0$) and we obtain a commutative diagram which relates the continuous and discrete optimal control problems by taking the limits $h\to0^+$ and $\alpha\to+\infty$. In Section 5, we study the double convergence of the discrete optimal control problems $(P_{h\alpha})$ to (P) when $(h,\alpha)\to(0^+,+\infty)$ and we obtain the diagonal convergence in the previous commutative diagram. In Section 6, we obtain the relationship and estimations among the optimal values $J(\overline{g},\overline{q})$, $J(\overline{g}_h,\overline{q}_h)$, $J_h(\overline{g}_h,\overline{q}_h)$ and $J_h(\overline{g},\overline{q})$ corresponding to the optimal control problems (P) and (P_h) and the same estimations corresponding to the optimal control problems (P_α) and $(P_{h\alpha})$. In Section 7, we formulate the conclusions of this paper.

2. Discretization by finite element method and properties

In this section, we consider the finite element method and a polygonal domain $\Omega\subset\mathbb{R}^n$ with a regular triangulation with Lagrange triangles of type 1, constituted by affine-equivalent finite element of class C^0 being h the parameter of the finite element approximation which goes to zero [16,17]. We can take h equal to the longest side of the triangles $T \in \tau_h$ and we can approximate the sets V, V_0 and K by:

$$V_h = \left\{ v_h \in C^0(\bar{\Omega}) / v_h|_T \in P_1(T), \forall T \in \tau_h \right\},$$

$$V_{0h} = \{ v_h \in V_h / v_h = 0 \text{ on } \Gamma_1 \}, \quad K_h = b + V_{0h}$$

where P_1 is the set of the polynomials of degree less than or equal to 1. Let $\pi_h : C^0(\bar{\Omega}) \to V_h$ be the corresponding linear interpolation operator. Then there exists a constant $c_0 > 0$ (independent of h) such that $\forall v \in H^r(\Omega)$, $1 < r \leq 2$, [16]:

$$\|v - \pi_h(v)\|_H \leq c_0 h^r \|v\|_r \tag{2.1}$$

$$\|v - \pi_h(v)\|_V \leq c_0 h^{r-1} \|v\|_r. \tag{2.2}$$

The discrete cost functional $J_h, J_{h\alpha} : H \times Q \to \mathbb{R}_0^+$ are defined by:

$$J_h(g,q) = \frac{1}{2} \|u_{hgq} - z_d\|_H^2 + \frac{M_1}{2} \|g\|_H^2 + \frac{M_2}{2} \|q\|_Q^2 \tag{2.3}$$

$$J_{h\alpha}(g,q) = \frac{1}{2} \|u_{h\alpha gq} - z_d\|_H^2 + \frac{M_1}{2} \|g\|_H^2 + \frac{M_2}{2} \|q\|_Q^2. \tag{2.4}$$

where u_{hgq} and $u_{h\alpha gq}$ are the discrete system states defined as the solution of the following discrete elliptic variational equalities [9,31]:

$$u_{hgq} \in K_h : \quad a(u_{hgq}, v_h) = (g, v_h)_H - (q, v_h)_Q, \quad \forall v_h \in V_{0h}, \tag{2.5}$$

$$u_{h\alpha gq} \in V_h : \quad a_\alpha(u_{h\alpha gq}, v_h) = (g, v_h)_H - (q, v_h)_Q + \alpha \int_{\Gamma_1} b v_h d\gamma, \quad \forall v_h \in V_h. \tag{2.6}$$

The corresponding discrete distributed optimal control problems consist in finding $(\overline{g}_h, \overline{q}_h)$, $(\overline{g}_{h\alpha}, \overline{q}_{h\alpha}) \in H \times Q$ such that:

$$\text{Problem}\,(P_h): \quad J_h(\overline{g}_h, \overline{q}_h) = \min_{(g,q) \in H \times Q} J_h(g,q), \tag{2.7}$$

$$\text{Problem}\,(P_{h\alpha}): \quad J_{h\alpha}(\overline{g}_{h\alpha}, \overline{q}_{h\alpha}) = \min_{(g,q) \in H \times Q} J_{h\alpha}(g,q) \tag{2.8}$$

and their corresponding discrete adjoint states p_{hgq} and $p_{h\alpha gq}$ are defined respectively as the solution of the following discrete elliptic variational equalities:

$$p_{hgq} \in V_{0h} : \quad a(p_{hgq}, v_h) = (u_{hgq} - z_d, v_h)_H, \quad \forall v_h \in V_{0h} \tag{2.9}$$

$$p_{h\alpha gq} \in V_h : \quad a_\alpha(p_{h\alpha gq}, v_h) = (u_{h\alpha gq} - z_d, v_h)_H, \quad \forall v_h \in V_h. \tag{2.10}$$

Remark 2.1. We note that the discrete (in the d-dimensional space) distributed optimal control problem (P_h) and $(P_{h\alpha})$ are still an infinite dimensional optimal control problem since the control space is not discretized.

Lemma 2.2.

(i) For all $(g,q) \in H \times Q$, $b > 0$ on Γ_1, there exist unique solutions $u_{hgq} \in K_h$ and $p_{hgq} \in V_{0h}$ of the elliptic variational equalities (2.5) and (2.9)*, respectively, and $u_{h\alpha gq} \in V_h$ and $p_{h\alpha gq} \in V_h$ of the elliptic variational equalities* (2.6) and (2.10)*, respectively.*

(ii) The operators $(g,q) \in H \times Q \to u_{hgq} \in V$, and $(g,q) \in H \times Q \to u_{h\alpha gq} \in V$ are Lipschitzians, i.e., $\forall (g_1,q_1),(g_2,q_2) \in H \times Q, \forall h > 0$

$$\|u_{hg_2q_2} - u_{hg_1q_1}\|_V \leq \frac{(1+\|\gamma\|)\sqrt{2}}{\lambda} \|(g_2,q_2) - (g_1,q_1)\|_{H\times Q},$$

$$\|u_{h\alpha g_2q_2} - u_{h\alpha g_1q_1}\|_V \leq \frac{(1+\|\gamma\|)\sqrt{2}}{\lambda_\alpha} \|(g_2,q_2) - (g_1,q_1)\|_{H\times Q}.$$

where $\|\gamma\|$ is the norm of the trace operator.

(iii) We have, $\forall (f,\eta) \in H \times Q$ the following equalities:

$$a(p_{hgq}, u_{hf\eta} - u_{h00}) = (f, p_{hgq})_H - (\eta, p_{hgq})_Q$$

$$a_\alpha(p_{h\alpha gq}, u_{h\alpha f\eta} - u_{h\alpha 00}) = (f, p_{h\alpha gq})_H - (\eta, p_{h\alpha gq})_Q$$

where u_{h00} and $u_{h\alpha 00}$ are the unique solutions for data $g = 0$ and $q = 0$, to the problems (2.5) and (2.6), *respectively.*

(iv) The operators $(g,q) \in H \times Q \to p_{hgq} \in V_{0h}$, and $(g,q) \in H \times Q \to p_{h\alpha gq} \in V_h$ are Lipschitzians and strictly monotones, i.e., $\forall (g_1,q_1),(g_2,q_2) \in H \times Q, \forall h > 0$, we have:

$$(a)\ (p_{hg_2q_2} - p_{hg_1q_1}, g_2 - g_1)_H - (p_{hg_2q_2} - p_{hg_1q_1}, q_2 - q_1)_Q = \|u_{hg_2q_2} - u_{hg_1q_1}\|_H^2 \geq 0,$$

$$(b)\ (p_{h\alpha g_2q_2} - p_{h\alpha g_1q_1}, g_2 - g_1)_H - (p_{h\alpha g_2q_2} - p_{h\alpha g_1q_1}, q_2 - q_1)_Q = \|u_{h\alpha g_2q_2} - u_{h\alpha g_1q_1}\|_H^2 \geq 0,$$

$$(c)\ \|p_{hg_2q_2} - p_{hg_1q_1}\|_V \leq \frac{(1+\|\gamma\|)\sqrt{2}}{\lambda^2} \|(g_2,q_2) - (g_1,q_1)\|_{H\times Q},$$

$$(d)\ \|p_{h\alpha g_2q_2} - p_{h\alpha g_1q_1}\|_V \leq \frac{(1+\|\gamma\|)\sqrt{2}}{\lambda_\alpha^2} \|(g_2,q_2) - (g_1,q_1)\|_{H\times Q}.$$

Proof. We use the Lax–Milgram Theorem, the variational equalities (2.5), (2.6), (2.9) and (2.10), the coerciveness (1.5) and (1.6) and following [12,13,32,33]. □

Theorem 2.3.

(i) The discrete cost functionals J_h and $J_{h\alpha}$ are H-elliptic and strictly convex applications, that is, $\forall (g_1,q_1),(g_2,q_2) \in H \times Q, \forall t \in [0,1]$, we have:

$$\begin{aligned}(1-t)J_h(g_2,q_2) + tJ_h(g_1,q_1) - J_h((1-t)(g_2,q_2) + t(g_1,q_1))\\ = \frac{t(1-t)}{2}\|u_{hg_2q_2} - u_{hg_1q_1}\|_H^2 + M_1\frac{t(1-t)}{2}\|g_2 - g_1\|_H^2 + M_2\frac{t(1-t)}{2}\|q_2 - q_1\|_Q^2\\ \geq m\frac{t(1-t)}{2}\|(g_2,q_2) - (g_1,q_1)\|_{H\times Q}^2,\end{aligned}$$

and

$$\begin{aligned}(1-t)J_{h\alpha}(g_2,q_2) + tJ_{h\alpha}(g_1,q_1) - J_{h\alpha}((1-t)(g_2,q_2) + t(g_1,q_1))\\ = \frac{t(1-t)}{2}\|u_{h\alpha g_2q_2} - u_{h\alpha g_1q_1}\|_H^2 + M_1\frac{t(1-t)}{2}\|g_2 - g_1\|_H^2 + M_2\frac{t(1-t)}{2}\|q_2 - q_1\|_Q^2\\ \geq m\frac{t(1-t)}{2}\|(g_2,q_2) - (g_1,q_1)\|_{H\times Q}^2,\end{aligned}$$

where $m = \min\{M_1, M_2\}$.

(ii) There exist unique optimal controls $(\overline{g}_h, \overline{q}_h) \in H \times Q$ and $(\overline{g}_{h\alpha}, \overline{q}_{h\alpha}) \in H \times Q$ that satisfy the optimization problems (2.7) and (2.8), *respectively.*

(iii) *J_h and $J_{h\alpha}$ are Gâteaux differentiable applications and their derivatives are given by the following expressions, $\forall (f,\eta) \in H \times Q$, $\forall h > 0$:*

$$J_h'(g,q)(f-g,\eta-q) = (f-g, p_{hgq} + M_1\ g)_H + (\eta - q, M_2 q - p_{hgq})_Q,$$

$$J_{h\alpha}'(g,q)(f-g,\eta-q) = (f-g, p_{h\alpha gq} + M_1\ g)_H + (\eta - q, M_2 q - p_{h\alpha gq})_Q.$$

(iv) *The optimality conditions for the problems* (2.7) and (2.8) *are given by, $\forall (f,\eta) \in H \times Q$:*

$$J_h'(\overline{g}_h, \overline{q}_h)(f,\eta) = 0 \Leftrightarrow (f, p_{h\overline{g}_h\overline{q}_h} + M_1\overline{g}_h)_H + (\eta, M_2\overline{q}_h - p_{h\overline{g}_h\overline{q}_h})_Q = 0$$

$$J_{h\alpha}'(\overline{g}_{h\alpha}, \overline{q}_{h\alpha})(f,\eta) = 0 \Leftrightarrow (f, p_{h\overline{g}_{h\alpha}\overline{q}_{h\alpha}} + M_1\overline{g}_{h\alpha})_H + (\eta, M_2\overline{q}_{h\alpha} - p_{h\overline{g}_{h\alpha}\overline{q}_{h\alpha}})_Q = 0.$$

(v) *J_h' and $J_{h\alpha}'$ are Lipschitzian and strictly monotone operators, i.e., $\forall (g_1,q_1), (g_2,q_2) \in H \times Q, \forall h > 0$, we have:*

$$\left\| J_h'(g_2,q_2) - J_h'(g_1,q_1) \right\|_{H\times Q} \le \left(M + \frac{(1+\|\gamma\|)^2}{\lambda^2} \right) \sqrt{2} \left\| (g_2,q_2) - (g_1,q_1) \right\|_{H\times Q},$$

$$\begin{aligned} \langle J_h'(g_2,q_2) - J_h'(g_1,q_1), (g_2,q_2) - (g_1,q_1) \rangle &= \left\| u_{hg_2q_2} - u_{hg_1q_1} \right\|_H^2 \\ &\quad + M_1 \left\| g_2 - g_1 \right\|_H^2 + M_2 \left\| q_2 - q_1 \right\|_Q^2 \\ &\ge m \left\| (g_2,q_2) - (g_1,q_1) \right\|_{H\times Q}^2, \end{aligned}$$

$$\left\| J_{h\alpha}'(g_2,q_2) - J_{h\alpha}'(g_1,q_1) \right\|_{H\times Q} \le \left(M + \frac{(1+\|\gamma\|)^2}{\lambda_\alpha^2} \right) \sqrt{2} \left\| (g_2,q_2) - (g_1,q_1) \right\|_{H\times Q},$$

$$\begin{aligned} \langle J_{h\alpha}'(g_2,q_2) - J_{h\alpha}'(g_1,q_1), (g_2,q_2) - (g_1,q_1) \rangle &= \left\| u_{h\alpha g_2q_2} - u_{h\alpha g_1q_1} \right\|_H^2 \\ &\quad + M_1 \left\| g_2 - g_1 \right\|_H^2 + M_2 \left\| q_2 - q_1 \right\|_Q^2 \\ &\ge m \left\| (g_2,q_2) - (g_1,q_1) \right\|_{H\times Q}^2 \end{aligned}$$

where $M = \max\{M_1, M_2\}$ and $m = \min\{M_1, M_2\}$.

Proof. We use the definitions (2.3) and (2.4), the elliptic variational equalities (2.5) and (2.6) and the coerciveness (1.5) and (1.6), following [12,13,32–34]. $\square$

We define the operators

$$\begin{aligned} W_h : H \times Q \to V_{0h} \times Q \subset V_0 \times Q \subset H \times Q \quad &\text{such that} \\ W_h(g,q) = \left(-\frac{1}{M_1} p_{hgq}, \frac{1}{M_2}\gamma(p_{hgq})\right) \end{aligned} \tag{2.11}$$

$$\begin{aligned} W_{h\alpha} : H \times Q \to V_h \times Q \subset V \times Q \subset H \times Q \quad &\text{such that} \\ W_{h\alpha}(g,q) = \left(-\frac{1}{M_1} p_{h\alpha gq}, \frac{1}{M_2}\gamma(p_{h\alpha gq})\right). \end{aligned} \tag{2.12}$$

and we prove the following result.

Theorem 2.4. *We have that:*

(i) *W_h and $W_{h\alpha}$ are Lipschitzian operators, that is, $\forall (g_1,q_1), (g_2,q_2) \in H \times Q, h > 0$:*

$$\left\| W_h(g_2,q_2) - W_h(g_1,q_1) \right\|_{H\times Q} \le C_0 \left\| (g_2,q_2) - (g_1,q_1) \right\|_{H\times Q},$$

$$\left\| W_{h\alpha}(g_2,q_2) - W_{h\alpha}(g_1,q_1) \right\|_{H\times Q} \le C_{0\alpha} \left\| (g_2,q_2) - (g_1,q_1) \right\|_{H\times Q}$$

with $C_0 = \frac{\sqrt{2}}{\lambda^2}\sqrt{\frac{1}{M_1^2} + \frac{\|\gamma\|^2}{M_2^2}}(1+\|\gamma\|)$ and $C_{0\alpha} = \frac{\sqrt{2}}{\lambda_\alpha^2}\sqrt{\frac{1}{M_1^2} + \frac{\|\gamma\|^2}{M_2^2}}(1+\|\gamma\|)$.

(ii) W_h $(W_{h\alpha})$ *is a contraction operator if and only if* $C_0 < 1$ $(C_{0\alpha} < 1)$.

(iii) If data satisfy inequality $C_0 < 1$ $(C_{0\alpha} < 1)$, *then the unique solution* $(\overline{g}_h, \overline{q}_h)$ $((\overline{g}_{h\alpha}, \overline{q}_{h\alpha}))$ *to the discrete optimal control* P_h $(P_{h\alpha})$ *can be obtained as the unique fixed point of the operator* W_h $(W_{h\alpha})$, *that is:*

$$W_h(\overline{g}_h, \overline{q}_h) = (\overline{g}_h, \overline{q}_h) \quad \textit{and} \quad W_{h\alpha}(\overline{g}_{h\alpha}, \overline{q}_{h\alpha}) = (\overline{g}_{h\alpha}, \overline{q}_{h\alpha}).$$

Proof. This results by using the definitions (2.11) and (2.12), and following [12]. □

3. Convergence of the discrete distributed-boundary optimal control problems (P_h) to (P), and $(P_{h\alpha})$ to (P_α) when $h \to 0^+$

In this section, we obtain error estimates between the optimal controls, system and adjoint states of the discrete simultaneous distributed-boundary optimal control problems (P_h) and $(P_{h\alpha})$ and convergence results of the discrete optimal control problems (P_h) to (P) and $(P_{h\alpha})$ to (P_α) when $h \to 0^+$, for each $\alpha > 0$.

Lemma 3.1 (i). *If the continuous system states and the continuous adjoint states have the regularity* u_{gq}, $u_{\alpha gq}$, p_{gq}, $p_{\alpha gq} \in H^r(\Omega)$ $(1 < r \leq 2)$, *then* $\forall \alpha > 0$, $\forall (g,q) \in H \times Q$, $h > 0$, *we have the following estimations:*

$$\|u_{gq} - u_{hgq}\|_V \leq \frac{c_0}{\sqrt{\lambda}} \|u_{gq}\|_r h^{r-1}, \quad \|p_{gq} - p_{hgq}\|_V \leq c_1 h^{r-1} \tag{3.1}$$

$$\|u_{h\alpha gq} - u_{\alpha gq}\|_V \leq c_{0\alpha} h^{r-1}, \quad \|p_{h\alpha gq} - p_{\alpha gq}\|_V \leq c_{1\alpha} h^{r-1} \tag{3.2}$$

where c_0 *(given in* (2.1) and (2.2)*),* c_1, $c_{0\alpha}$ *and* $c_{1\alpha}$ *are constants independents of* h.

(ii) We have the following convergences, $\forall (g,q) \in H \times Q$:

$$\lim_{h\to 0^+} \|u_{gq} - u_{hgq}\|_V = 0, \quad \lim_{h\to 0^+} \|p_{gq} - p_{hgq}\|_V = 0,$$

$$\lim_{h\to 0^+} \|u_{h\alpha gq} - u_{\alpha gq}\|_V = 0, \quad \lim_{h\to 0^+} \|p_{h\alpha gq} - p_{\alpha gq}\|_V = 0, \quad \forall \alpha > 0.$$

Proof. By using the variational equalities (1.3), (1.4), (1.11), (1.12), (2.5), (2.6), (2.9) and (2.10), the coerciveness properties (1.5) and (1.6), the estimations (2.1) and (2.2) and the following properties, $\forall (g,q) \in H \times Q$:

$$a\left(p_{gq} - p_{hgq}, \pi_h\left(p_{gq}\right) - p_{hgq}\right) = \left(u_{gq} - u_{hgq}, \pi_h\left(p_{gq}\right) - p_{hgq}\right)$$

$$a_\alpha\left(p_{\alpha gq} - p_{h\alpha gq}, \pi_h\left(p_{\alpha gq}\right) - p_{h\alpha gq}\right) = \left(u_{h\alpha gq} - u_{\alpha gq}, \pi_h\left(p_{\alpha gq}\right) - p_{h\alpha gq}\right)_H$$

following a similar method given in [32,33], the thesis holds. □

Theorem 3.2. *We consider the continuous system states and adjoint states have the regularities* $u_{\overline{g}\,\overline{q}}, u_{\alpha\overline{g}_\alpha\overline{q}_\alpha}$, $p_{\overline{g}\,\overline{q}}, p_{\alpha\overline{g}_\alpha\overline{q}_\alpha} \in H^r(\Omega)$ $(1 < r \leq 2)$:

(i) We have the following limits, $\forall \alpha > 1$:

$$\lim_{h\to 0^+} \|(\overline{g}_h, \overline{q}_h) - (\overline{g}, \overline{q})\|_{H\times Q} = 0 \tag{3.3}$$

$$\lim_{h\to 0^+} \left\|u_{h\overline{g}_h\overline{q}_h} - u_{\overline{g}\,\overline{q}}\right\|_V = 0, \quad \lim_{h\to 0^+} \left\|p_{h\overline{g}_h\overline{q}_h} - p_{\overline{g}\,\overline{q}}\right\|_V = 0 \tag{3.4}$$

$$\lim_{h\to 0^+} \|(\overline{g}_{h\alpha}, \overline{q}_{h\alpha}) - (\overline{g}_\alpha, \overline{q}_\alpha)\|_{H\times Q} = 0 \tag{3.5}$$

$$\lim_{h\to 0^+} \left\|u_{h\alpha\overline{g}_{h\alpha}\overline{q}_{h\alpha}} - u_{\alpha\overline{g}_\alpha\overline{q}_\alpha}\right\|_V = 0, \quad \lim_{h\to 0^+} \left\|p_{h\alpha\overline{g}_{h\alpha}\overline{q}_{h\alpha}} - p_{\alpha\overline{g}_\alpha\overline{q}_\alpha}\right\|_V = 0. \tag{3.6}$$

(ii) If data M_1 and M_2 satisfy the following inequalities

$$\frac{\sqrt{2}}{\lambda^2}\sqrt{\frac{1}{M_1^2}+\frac{\|\gamma\|^2}{M_2^2}}(1+\|\gamma\|)<1 \quad \text{and} \quad \frac{\sqrt{2}}{\lambda_\alpha^2}\sqrt{\frac{1}{M_1^2}+\frac{\|\gamma\|^2}{M_2^2}}(1+\|\gamma\|)<1 \tag{3.7}$$

we have the following error bonds:

$$\|(\overline{g}_h,\overline{q}_h)-(\overline{g},\overline{q})\|_{H\times Q}\le ch^{r-1} \tag{3.8}$$

$$\left\|u_{h\overline{g}_h\overline{q}_h}-u_{\overline{g}\,\overline{q}}\right\|_V\le ch^{r-1}, \quad \left\|p_{h\overline{g}_h\overline{q}_h}-p_{\overline{g}\,\overline{q}}\right\|_V\le ch^{r-1} \tag{3.9}$$

$$\|(\overline{g}_{h\alpha},\overline{q}_{h\alpha})-(\overline{g}_\alpha,\overline{q}_\alpha)\|_{H\times Q}\le c_\alpha h^{r-1} \tag{3.10}$$

$$\left\|u_{h\alpha\overline{g}_{h\alpha}\overline{q}_{h\alpha}}-u_{\alpha\overline{g}_\alpha\,\overline{q}_\alpha}\right\|_V\le c_\alpha h^{r-1}, \quad \left\|p_{h\alpha\overline{g}_{h\alpha}\overline{q}_{h\alpha}}-p_{\alpha\overline{g}_\alpha\overline{q}_\alpha}\right\|_V\le c_\alpha h^{r-1} \tag{3.11}$$

where c and c_α are different constants independents of h.

Proof. We follow a similar method to the one developed in [32,33].

(i) From the definition of the functional (2.3), we obtain, $\forall h>0$:

$$\frac{1}{2}\left\|u_{h\overline{g}_h\overline{q}_h}-z_d\right\|_H^2+\frac{M_1}{2}\|\overline{g}_h\|_H^2+\frac{M_2}{2}\|\overline{q}_h\|_Q^2\le\frac{1}{2}\|u_{h00}-z_d\|_H^2\le c$$

where u_{h00} is the unique solution of the variational equality (2.5) for $g=0$ and $q=0$. That is,

$$\left\|u_{h\overline{g}_h\overline{q}_h}\right\|_H\le c \quad \|\overline{g}_h\|_H\le c \quad \text{and} \quad \|\overline{q}_h\|_Q\le c$$

with c different positive constants independent of h. Moreover, by using the variational equality (2.5), we obtain

$$\left\|u_{h\overline{g}_h\overline{q}_h}-b\right\|_V\le\frac{1}{\lambda}(\|\overline{g}_h\|_H+\|\overline{q}_h\|_Q\|\gamma\|)\le c$$

then

$$\left\|u_{h\overline{g}_h\overline{q}_h}\right\|_V\le c.$$

Next, by using the variational equality (2.9), we have

$$\left\|p_{h\overline{g}_h\overline{q}_h}\right\|_V\le\frac{1}{\lambda}\left\|u_{h\overline{g}_h\overline{q}_h}-z_d\right\|_H\le c, \quad \forall h>0.$$

Now, from the above estimations we obtain, when $h\to 0^+$:

$$\exists f\in H\,:\,\overline{g}_h\to f \text{ weakly in } H$$

$$\exists \rho\in Q\,:\,\overline{q}_h\to \rho \text{ weakly in } Q$$

$$\exists \eta\in V\,:\,u_{h\overline{g}_h\overline{q}_h}\to \eta \text{ weakly in } V (\text{in } H \text{ strong})$$

$$\exists \xi\in V\,:\,p_{h\overline{g}_h\overline{q}_h}\to \xi \text{ weakly in } V (\text{in } H \text{ strong}).$$

By using the above weak convergences, we can pas to the limit as $h\to 0^+$, and by uniqueness of the variational equalities (1.3) and (1.11), we obtain that

$$\eta=u_{f\rho}, \quad \xi=p_{f\rho}.$$

Next, by the weak lower semicontinuity of the functional J_h and the uniqueness of the solution of the optimal control problem (1.7), we have that

$$f = \overline{g} \quad \text{and} \quad \rho = \overline{q}.$$

By the following inequalities

$$\begin{aligned}\lambda \left\| u_{h\overline{g}_h\overline{q}_h} - u_{\overline{g}\,\overline{q}} \right\|_V^2 \leq (\overline{g}_h - \overline{g}, u_{h\overline{g}_h\overline{q}_h} - b)_H - (\overline{q}_h - \overline{q}, u_{h\overline{g}_h\overline{q}_h} - b)_Q \\ + (\overline{g}, u_{\overline{g}\,\overline{q}} - u_{h\overline{g}_h\overline{q}_h})_H - (\overline{q}, u_{\overline{g}\,\overline{q}} - u_{h\overline{g}_h\overline{q}_h})_Q\end{aligned}$$

and

$$\lambda \left\| p_{h\overline{g}_h\overline{q}_h} - p_{\overline{g}\,\overline{q}} \right\|_V^2 \leq a(p_{\overline{g}\,\overline{q}}, p_{\overline{g}\,\overline{q}} - p_{h\overline{g}_h\overline{q}_h})_H - (p_{h\overline{g}_h\overline{q}_h}, u_{\overline{g}\,\overline{q}} - u_{h\overline{g}_h\overline{q}_h})_Q$$

we obtain the strong convergences (3.4). Next, from the definition (2.3), we have

$$\lim_{h\to 0^+} \|\overline{g}_h\|_H = \|\overline{g}\|_H \quad \text{and} \quad \lim_{h\to 0^+} \|\overline{q}_h\|_Q = \|\overline{q}\|_Q$$

and (3.3) holds. In a similar way, by using the elliptic variational equalities (2.6) and (2.10), we prove (3.5) and (3.6).

(ii) Following [12], we obtain that

$$\|(\overline{g}_h, \overline{q}_h) - (\overline{g}, \overline{q})\|_{H\times Q} \leq \frac{c_1\sqrt{\frac{1}{M_1^2} + \frac{\|\gamma\|^2}{M_2^2}}}{1 - \frac{\sqrt{2}}{\lambda^2}\sqrt{\frac{1}{M_1^2} + \frac{\|\gamma\|^2}{M_2^2}}(1 + \|\gamma\|)} h^{r-1}$$

$$\|(\overline{g}_{h\alpha}, \overline{q}_{h\alpha}) - (\overline{g}_\alpha, \overline{q}_\alpha)\|_{H\times Q} \leq \frac{c_{1\alpha}\sqrt{\frac{1}{M_1^2} + \frac{\|\gamma\|^2}{M_2^2}}}{1 - \frac{\sqrt{2}}{\lambda_\alpha^2}\sqrt{\frac{1}{M_1^2} + \frac{\|\gamma\|^2}{M_2^2}}(1 + \|\gamma\|)} h^{r-1}$$

where c_1 and $c_{1\alpha}$ are constants given in (3.1) and (3.2), respectively. □

4. Convergence of the discrete optimal control problems $(P_{h\alpha})$ to (P_h) when $\alpha \to +\infty$

In this section, for each $h > 0$, we obtain convergence results of the discrete simultaneous distributed-boundary optimal control problems $(P_{h\alpha})$ to (P_h) when the parameter $\alpha \to +\infty$. For fixed $h > 0$, we have the following convergences.

Lemma 4.1. *For fixed $(g, q) \in H \times Q$, $h > 0$, we have the following limits:*

$$\lim_{\alpha\to+\infty} \|u_{h\alpha gq} - u_{hgq}\|_V = 0. \tag{4.1}$$

$$\lim_{\alpha\to+\infty} \|p_{h\alpha gq} - p_{hgq}\|_V = 0. \tag{4.2}$$

Proof. For fixed $(g, q) \in H \times Q$, $h > 0$, and by using the variational equalities (2.5) and (2.6), and taking into account that for $\alpha > 1$ we can split

$$a_\alpha(u, v) = a_1(u, v) + (\alpha - 1)\int_{\Gamma_1} uv d\gamma$$

we obtain the following estimations

$$\|u_{h\alpha gq} - u_{hgq}\|_V \leq c, \quad (\alpha - 1)\int_{\Gamma_1} (u_{h\alpha gq} - b)^2\, d\gamma \leq c, \quad \forall \alpha > 1.$$

From the above inequalities, we deduce that

$$\exists \eta_{hgq} \in V / u_{h\alpha gq} \longrightarrow \eta_{hgq} \text{ in } V \text{ weakly (in } H \text{ strong) as } \alpha \to +\infty \text{ with } \eta_{hgq}\big|_{\Gamma_1} = b.$$

By using the variational equality (2.6), we can pass to the limit when $\alpha \to +\infty$, and by uniqueness of the solution to the variational equality (2.5) we obtain that $\eta_{hgq} = u_{hgq}$. By using the above properties, and the variational equalities (2.5) and (2.6), we deduce (4.1) and by using a similar method we can obtain the limit (4.2) for the discrete adjoint system state. □

Theorem 4.2. *We have the following limits:*

$$\lim_{\alpha\to+\infty} \|(\overline{g}_{h\alpha}, \overline{q}_{h\alpha}) - (\overline{g}_h, \overline{q}_h)\|_{H\times Q} = 0. \tag{4.3}$$

$$\lim_{\alpha\to+\infty} \|u_{h\alpha\overline{g}_{h\alpha}\overline{q}_{h\alpha}} - u_{h\overline{g}_h\overline{q}_h}\|_V = 0, \qquad \lim_{\alpha\to+\infty} \|p_{h\alpha\overline{g}_{h\alpha}\overline{q}_{h\alpha}} - p_{h\overline{g}_h\overline{q}_h}\|_V = 0. \tag{4.4}$$

Proof. For each fixed $h > 0$, the thesis holds in similar way that Theorem 7 in [12]. □

5. Double convergence of the discrete distributed-boundary optimal control problems $(P_{h\alpha})$ to (P) when $(h, \alpha) \to (0^+, +\infty)$

In this section, we prove the main result of the paper.

Theorem 5.1. *We have the following limits:*

$$\lim_{(h,\alpha)\to(0^+,+\infty)} \|(\overline{g}_{h\alpha}, \overline{q}_{h\alpha}) - (\overline{g}, \overline{q})\|_{H\times Q} = 0. \tag{5.1}$$

$$\lim_{(h,\alpha)\to(0^+,+\infty)} \|u_{h\alpha\overline{g}_{h\alpha}\overline{q}_{h\alpha}} - u_{\overline{g}\,\overline{q}}\|_V = 0, \qquad \lim_{(h,\alpha)\to(0^+,+\infty)} \|p_{h\alpha\overline{g}_{h\alpha}\overline{q}_{h\alpha}} - p_{\overline{g}\,\overline{q}}\|_V = 0. \tag{5.2}$$

Proof. We obtain the proof in two steps.

Step 1. We show a sketch of the proof by obtaining the following estimations, for $h > 0$ and $\alpha > 1$:

$$\|u_{h00}\|_V \leq c_1 = b\sqrt{|\Omega|}$$

$$\|u_{h\alpha 00}\|_V \leq c_2 = \left(1 + \frac{1}{\lambda_1}\right) c_1$$

$$(\alpha - 1)\int_{\Gamma_1} (u_{h\alpha 00} - b)^2 d\gamma \leq c_3 = \frac{c_1^2}{\lambda_1}$$

$$\|(\overline{g}_{h\alpha}, \overline{q}_{h\alpha})\|_{H\times Q} \leq c_4 = \frac{1}{\sqrt{\min\{M_1, M_2\}}}(c_2 + \|z_d\|_H)$$

$$\|u_{h\alpha\overline{g}_{h\alpha}\overline{q}_{h\alpha}}\|_H \leq c_5 = c_2 + 2\|z_d\|_H$$

$$\|(\overline{g}_h, \overline{q}_h)\|_{H\times Q} \leq c_6 = \frac{1}{\sqrt{\min\{M_1, M_2\}}}(c_1 + \|z_d\|_H)$$

$$\|u_{h\overline{g}_h\overline{q}_h}\|_V \leq c_7 = \sqrt{2}(1 + \|\gamma\|)c_6 + c_1$$

$$\|u_{h\alpha\overline{g}_{h\alpha}\overline{q}_{h\alpha}}\|_V \leq c_8 = \sqrt{2}(1 + \|\gamma\|)c_4 + \left(1 + \frac{1}{\lambda_1}\right) c_7$$

$$(\alpha - 1)\int_{\Gamma_1} (u_{h\alpha\overline{g}_{h\alpha}\overline{q}_{h\alpha}} - b)^2 d\gamma \leq c_9 = \frac{1}{\lambda_1}(\sqrt{2}(1 + \|\gamma\|)c_4 + c_7)^2$$

$$\|p_{h\overline{g}_h\overline{q}_h}\|_V \leq c_{10} = \frac{1}{\lambda}(c_7 + \|z_d\|_H)$$
$$\|p_{h\alpha\overline{g}_{h\alpha}\overline{q}_{h\alpha}}\|_V \leq c_{11} = \frac{1}{\lambda_1}(c_8 + \|z_d\|_H) + \left(1 + \frac{1}{\lambda_1}\right) c_{10}$$
$$(\alpha - 1)\int_{\Gamma_1} (p_{h\alpha\overline{g}_{h\alpha}\overline{q}_{h\alpha}} - b)^2 d\gamma \leq c_{12} = \frac{1}{\lambda_1}(c_5 + \|z_d\|_H + c_{10})^2.$$

Therefore, from the above estimations we have that

$$\exists f \in H \,:\, \overline{g}_{h\alpha} \to f \text{ weakly in } H, \text{ when } (h,\alpha) \to (0^+, +\infty)$$
$$\exists \rho \in Q \,:\, \overline{q}_{h\alpha} \to \rho \text{ weakly in } Q, \text{ when } (h,\alpha) \to (0^+, +\infty)$$
$$\exists \eta \in V \,:\, u_{h\alpha\overline{g}_{h\alpha}\overline{q}_{h\alpha}} \to \eta \text{ weakly in } V (\text{in } H \text{ strong}), \text{ when } (h,\alpha) \to (0^+, +\infty)$$
$$\exists \xi \in V \,:\, p_{h\alpha\overline{g}_{h\alpha}\overline{q}_{h\alpha}} \to \xi \text{ weakly in } V (\text{in } H \text{ strong}), \text{ when } (h,\alpha) \to (0^+, +\infty)$$

and

$$\exists f_h \in H \,:\, \overline{g}_{h\alpha} \to f_h \text{ weakly in } H, \text{ when } \alpha \to +\infty$$
$$\exists \rho_h \in Q \,:\, \overline{q}_{h\alpha} \to \rho_h \text{ weakly in } Q, \text{ when } \alpha \to +\infty$$
$$\exists \eta_h \in V \,:\, u_{h\alpha\overline{g}_{h\alpha}\overline{q}_{h\alpha}} \to \eta_h \text{ weakly in } V (\text{in } H \text{ strong}), \text{ when } \alpha \to +\infty$$
$$\exists \xi_h \in V \,:\, p_{h\alpha\overline{g}_{h\alpha}\overline{q}_{h\alpha}} \to \xi_h \text{ weakly in } V (\text{in } H \text{ strong}), \text{ when } \alpha \to +\infty$$

and

$$\exists f_\alpha \in H \,:\, \overline{g}_{h\alpha} \to f_\alpha \text{ weakly in } H, \text{ when } h \to 0^+$$
$$\exists \rho_\alpha \in Q \,:\, \overline{q}_{h\alpha} \to \rho_\alpha \text{ weakly in } Q, \text{ when } h \to 0^+$$
$$\exists \eta_\alpha \in V \,:\, u_{h\alpha\overline{g}_{h\alpha}\overline{q}_{h\alpha}} \to \eta_\alpha \text{ weakly in } V (\text{in } H \text{ strong}), \text{ when } h \to 0^+$$
$$\exists \xi_\alpha \in V \,:\, p_{h\alpha\overline{g}_{h\alpha}\overline{q}_{h\alpha}} \to \xi_\alpha \text{ weakly in } V (\text{in } H \text{ strong}), \text{ when } h \to 0^+.$$

Step 2. Now, taking into account that

$$\eta = \eta_h = b \text{ on } \Gamma_1,$$
$$\xi = \xi_h = 0 \text{ on } \Gamma_1,$$

by the uniqueness of the solutions of the simultaneous distributed-boundary optimal control problems $(P_{h\alpha})$, (P_h), (P_α) and (P), and the uniqueness of the solutions of the elliptic variational equalities corresponding to their state systems, we obtain that

$$\eta_h = u_{hf_h\rho_h} = u_{h\overline{g}_h\overline{q}_h}, \quad \xi_h = p_{hf_h\rho_h} = p_{h\overline{g}_h\overline{q}_h}, \quad f_h = \overline{g}_h, \quad \rho_h = \overline{q}_h$$
$$\eta_\alpha = u_{\alpha f_\alpha\rho_\alpha} = u_{\alpha\overline{g}_\alpha\overline{q}_\alpha}, \quad \xi_\alpha = p_{\alpha f_\alpha\rho_\alpha} = p_{\alpha\overline{g}_\alpha\overline{q}_\alpha}, \quad f_\alpha = \overline{g}_\alpha, \quad \rho_\alpha = \overline{q}_\alpha$$

and the limits (3.3) and (3.4). Next, by using [12], we get

$$\lim_{\alpha\to+\infty} \|f_\alpha - \overline{g}\|_H = 0, \qquad \lim_{\alpha\to+\infty} \|\rho_\alpha - \overline{q}\|_Q = 0$$
$$\lim_{\alpha\to+\infty} \|\eta_\alpha - u_{\overline{g}\,\overline{q}}\|_V = 0, \qquad \lim_{\alpha\to+\infty} \|\xi_\alpha - p_{\overline{g}\,\overline{q}}\|_V = 0$$

and therefore the double limits (5.1) and (5.2) hold, when $(h,\alpha) \to (0^+, +\infty)$. $\square$

6. Relationship among the optimal values corresponding to the optimal control problems (P), (P_h), (P_α) and $(P_{h\alpha})$

In this section, we obtain the estimates given below.

Lemma 6.1. *If M_1 and M_2 satisfy the inequalities (3.7) and the continuous system states and adjoint states have the regularity $u_{\overline{g}\,\overline{q}}, u_{\alpha\overline{g}_\alpha\overline{q}_\alpha}, p_{\overline{g}\,\overline{q}}, p_{\alpha\overline{g}_\alpha\overline{q}_\alpha} \in H^r(\Omega)$ $(1 < r \leq 2)$, we have the following error bounds:*

$$0 \leq J(\overline{g}_h, \overline{q}_h) - J(\overline{g}, \overline{q}) \leq ch^{2(r-1)} \tag{6.1}$$

$$0 \leq J_h(\overline{g}, \overline{q}) - J_h(\overline{g}_h, \overline{q}_h) \leq ch^{2(r-1)} \tag{6.2}$$

$$J(\overline{g}, \overline{q}) - J_h(\overline{g}_h, \overline{q}_h) \leq ch^{r-1} \tag{6.3}$$

and, for each $\alpha > 0$:

$$0 \leq J_\alpha(\overline{g}_{h\alpha}, \overline{q}_{h\alpha}) - J_\alpha(\overline{g}_\alpha, \overline{q}_\alpha) \leq c_\alpha h^{2(r-1)} \tag{6.4}$$

$$0 \leq J_{h\alpha}(\overline{g}_\alpha, \overline{q}_\alpha) - J_{h\alpha}(\overline{g}_{h\alpha}, \overline{q}_{h\alpha}) \leq c_\alpha h^{2(r-1)} \tag{6.5}$$

$$J_\alpha(\overline{g}_\alpha, \overline{q}_\alpha) - J_{h\alpha}(\overline{g}_{h\alpha}, \overline{q}_{h\alpha}) \leq c_\alpha h^{r-1} \tag{6.6}$$

where c and c_α are different constants independents of h.

Proof. Estimations (6.1), (6.2), (6.4) and (6.5) follow from the estimations (3.1), (3.2), (3.9), (3.11) and the equalities:

$$J(\overline{g}_h, \overline{q}_h) - J(\overline{g}, \overline{q}) = \frac{1}{2}\left\|u_{\overline{g}_h\overline{q}_h} - u_{\overline{g}\,\overline{q}}\right\|_H^2 + \frac{M_1}{2}\left\|\overline{g}_h - \overline{g}\right\|_H^2 + \frac{M_2}{2}\left\|\overline{q}_h - \overline{q}\right\|_Q^2$$
$$J_h(\overline{g}, \overline{q}) - J_h(\overline{g}_h, \overline{q}_h) = \frac{1}{2}\left\|u_{h\overline{g}\,\overline{q}} - u_{h\overline{g}_h\overline{q}_h}\right\|_H^2 + \frac{M_1}{2}\left\|\overline{g} - \overline{g}_h\right\|_H^2 + \frac{M_2}{2}\left\|\overline{q} - \overline{q}_h\right\|_Q^2$$

$$J_\alpha(\overline{g}_{h\alpha}, \overline{q}_{h\alpha}) - J_\alpha(\overline{g}_\alpha, \overline{q}_\alpha) = \frac{1}{2}\left\|u_{h\alpha\overline{g}_{h\alpha}\overline{q}_{h\alpha}} - u_{\alpha\overline{g}_\alpha\overline{q}_\alpha}\right\|_H^2 + \frac{M_1}{2}\left\|\overline{g}_{h\alpha} - \overline{g}_\alpha\right\|_H^2 + \frac{M_2}{2}\left\|\overline{q}_{h\alpha} - \overline{q}_\alpha\right\|_Q^2$$

$$J_{h\alpha}(\overline{g}_\alpha, \overline{q}_\alpha) - J_{h\alpha}(\overline{g}_{h\alpha}, \overline{q}_{h\alpha}) = \frac{1}{2}\left\|u_{h\alpha\overline{g}_\alpha\overline{q}_\alpha} - u_{h\alpha\overline{g}_{h\alpha}\overline{q}_{h\alpha}}\right\|_H^2 + \frac{M_1}{2}\left\|\overline{g}_\alpha - \overline{g}_{h\alpha}\right\|_H^2 + \frac{M_2}{2}\left\|\overline{q}_\alpha - \overline{q}_{h\alpha}\right\|_Q^2.$$

Estimations (6.3) and (6.6) follow from estimations (3.1) and (3.2), taking into account that

$$\begin{aligned} J(\overline{g}, \overline{q}) - J_h(\overline{g}_h, \overline{q}_h) &\leq J(\overline{g}_h, \overline{q}_h) - J_h(\overline{g}_h, \overline{q}_h) \\ &= \frac{1}{2}\left(\left\|u_{\overline{g}_h\overline{q}_h} - z_d\right\|_H^2 - \left\|u_{h\overline{g}_h\overline{q}_h} - z_d\right\|_H^2\right) \\ &= \frac{1}{2}\left(u_{\overline{g}_h\overline{q}_h} - u_{h\overline{g}_h\overline{q}_h}, u_{h\overline{g}_h\overline{q}_h} + u_{\overline{g}_h\overline{q}_h} - 2z_d\right)_H \\ &\leq \frac{1}{2}\left\|u_{\overline{g}_h\overline{q}_h} - u_{h\overline{g}_h\overline{q}_h}\right\|_H\left(\left\|u_{h\overline{g}_h\overline{q}_h} - z_d\right\|_H + \left\|u_{\overline{g}_h\overline{q}_h} - z_d\right\|_H\right) \\ &\leq ch^{r-1} \end{aligned}$$

and

$$\begin{aligned} &J_\alpha(\overline{g}_\alpha, \overline{q}_\alpha) - J_{h\alpha}(\overline{g}_{h\alpha}, \overline{q}_{h\alpha}) \leq \frac{1}{2}\left(\left\|u_{\alpha\overline{g}_{h\alpha}\overline{q}_{h\alpha}} - z_d\right\|_H^2 - \left\|u_{h\alpha\overline{g}_{h\alpha}\overline{q}_{h\alpha}} - z_d\right\|_H^2\right) \\ &\leq \frac{1}{2}\left\|u_{\alpha\overline{g}_{h\alpha}\overline{q}_{h\alpha}} - u_{h\alpha\overline{g}_{h\alpha}\overline{q}_{h\alpha}}\right\|_H\left(\left\|u_{h\alpha\overline{g}_{h\alpha}\overline{q}_{h\alpha}} - z_d\right\|_H + \left\|u_{\alpha\overline{g}_{h\alpha}\overline{q}_{h\alpha}} - z_d\right\|_H\right) \leq c_\alpha h^{r-1}. \end{aligned} \quad \square$$

Remark 6.2. In a forthcoming paper, we will do the numerical analysis and its corresponding error estimates when we replace the condition (1.2)ii by the following

$$-\frac{\partial u}{\partial n}|_{\Gamma_1} \in \alpha \partial j(u).$$

Here $j(x,.)$ is locally Lipschitz for a.e. $x \in \Gamma_1$ and not necessary differentiable following [35–38]. Therefore, the variational formulation, for the system state, will be given by an elliptic hemivariational inequality, and the corresponding control variable can be the energy g, or the heat flux q or the vectorial control (g,q).

7. Conclusions

For two vectorial continuous optimal control problems (P_α) and (P), and for the corresponding two vectorial discrete optimal control problems $(P_{h\alpha})$ and (P_h) we have obtained a commutative diagram when $h \to 0^+$ and $\alpha \to +\infty$, with $\alpha \to +\infty$ and $h \to 0^+$, and the corresponding double convergence when $(h,\alpha) \to (0^+,+\infty)$ simultaneously for the optimal controls, for the optimal system states and for the optimal adjoint states. The parameter α can be considered as the heat transfer coefficient on a portion of the boundary of a material, and h is the parameter of the finite element approximation.

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