J. Nonlinear Var. Anal. 6 (2022), No. 5, pp. 535-549
Available online at http://jnva.biemdas.com
https://doi.org/10.23952/jnva.6.2022.5.07

SIMULTANEOUS DISTRIBUTED AND NEUMANN BOUNDARY OPTIMAL CONTROL PROBLEMS FOR ELLIPTIC HEMIVARIATIONAL INEQUALITIES

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Abstract. In this paper, we study boundary optimal control problems on the heat flux and simultaneous distributed-boundary optimal control problems on the internal energy and the heat flux for a system governed by a class of elliptic boundary hemivariational inequalities with a parameter. The system was originated by a steady-state heat conduction problem with non-monotone multivalued subdifferential boundary condition on a portion of the boundary of the domain described by the Clarke generalized gradient of a locally Lipschitz function. We prove an existence result for the boundary optimal control problem and simultaneous distributed-boundary optimal control problems. We show an asymptotic behavior result for the optimal controls and the system states for both optimal control problems, when the parameter, like a heat transfer coefficient, tends to infinity on a portion of the boundary.
Keywords. Asymptotic behavior; Clarke generalized gradient; Elliptic hemivariational inequality; Mixed elliptic problem; Simultaneous optimal control problems.

1. INTRODUCTION

We consider a bounded domain Ω in $\mathbb{R}^d$ whose regular boundary Γ consists of the union of three disjoint portions Γ_i, $i = 1, 2, 3$ with $|\Gamma_i| > 0$, where $|\Gamma_i|$ denotes the $(d-1)$-dimensional Hausdorff measure of the portion Γ_i on Γ. The outward normal vector on the boundary is denoted by n. We formulate the following classical steady-state heat conduction problem with mixed boundary conditions [1, 2, 3, 4, 5, 6]:

$$-\Delta u = g \ \text{ in } \ \Omega, \qquad u\big|_{\Gamma_1} = 0, \qquad -\frac{\partial u}{\partial n}\big|_{\Gamma_2} = q, \qquad u\big|_{\Gamma_3} = b, \tag{1.1}$$

where u is the temperature in Ω, g is the internal energy in Ω, b is the temperature on Γ_3 and q is the heat flux on Γ_2, which satisfy the hypothesis: $g \in H = L^2(\Omega)$, $q \in Q = L^2(\Gamma_2)$, and $b \in H^{\frac{1}{2}}(\Gamma_3)$.

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Received June 11, 2022; Accepted June 16, 2022.

Throughout the paper, we use the following notation:

$$V = H^1(\Omega), \quad V_0 = \{v \in V \mid v = 0 \ \text{ on } \ \Gamma_1\},$$
$$K = \{v \in V \mid v = 0 \ \text{ on } \ \Gamma_1, \ v = b \ \text{ on } \ \Gamma_3\}, \quad K_0 = \{v \in V \mid v = 0 \ \text{ on } \ \Gamma_1 \cup \Gamma_3\},$$
$$a(u,v) = \int_\Omega \nabla u \nabla v \, dx, \quad L(v) = \int_\Omega g v \, dx - \int_{\Gamma_2} q\gamma(v)\, d\Gamma,$$

where $\gamma\colon V \to L^2(\Gamma)$ denotes the trace operator on Γ. In what follows, we write u for the trace of a function $u \in V$ on the boundary. In a standard way, we obtain the following variational formulation of (1.1), [7]:

$$\text{find } \ u_\infty \in K \ \text{ such that } \ a(u_\infty, v) = L(v) \ \text{ for all } \ v \in K_0. \tag{1.2}$$

The standard norms on V and V_0 are denoted by

$$\|v\|_V = \left(\|v\|^2_{L^2(\Omega)} + \|\nabla v\|^2_{L^2(\Omega;\mathbb{R}^d)}\right)^{1/2} \ \text{ for } \ v \in V,$$
$$\|v\|_{V_0} = \|\nabla v\|_{L^2(\Omega;\mathbb{R}^d)} \ \text{ for } \ v \in V_0.$$

It is well known by the Poincaré inequality (see, e.g., [5, 8, 9]) that on V_0 the above two norms are equivalent. Note that the form a is bilinear, symmetric, continuous, and coercive with constant $m_a > 0$, i.e.,

$$a(v,v) = \|v\|^2_{V_0} \geq m_a \|v\|^2_V \ \text{ for all } \ v \in V_0.$$

We remark that, under additional hypotheses on the data g, q, and b, problem (1.1) can be considered as steady-state two-phase Stefan problem; see, e.g., [6, 7, 10, 11].

Now, in this paper, we consider the mixed nonlinear boundary value problem for an elliptic equation as follows:

$$-\Delta u = g \ \text{ in } \ \Omega, \quad u\big|_{\Gamma_1} = 0, \quad -\frac{\partial u}{\partial n}\Big|_{\Gamma_2} = q, \quad -\frac{\partial u}{\partial n}\Big|_{\Gamma_3} \in \alpha\, \partial j(u), \tag{1.3}$$

which was recently studied in [12, 13] (see also [14]).

Here α is a positive constant which can be considered as the heat transfer coefficient on the boundary while the function $j\colon \Gamma_3 \times \mathbb{R} \to \mathbb{R}$, called a superpotential (nonconvex potential), is such that $j(x,\cdot)$ locally Lipschitz for a.e. $x \in \Gamma_3$ and not necessary differentiable. Since in general $j(x,\cdot)$ is nonconvex, so the multivalued condition on Γ_3 in problem (1.3) is described by a nonmonotone relation expressed by the generalized gradient of Clarke [15]. Such multivalued relation in problem (1.3) is met in certain types of steady-state heat conduction problems (the behavior of a semipermeable membrane of finite thickness, a temperature control problems, etc.). Further, problem (1.3) can be considered as a prototype of several boundary semipermeability models, see [16, 17, 18, 19], which are motivated by problems arising in hydraulics, fluid flow problems through porous media, and electrostatics, where the solution represents the pressure and the electric potentials. Note that the analogous problems with maximal monotone multivalued boundary relations (that is the case when $j(x,\cdot)$ is a convex function) were considered in [20, 21], see also the references therein.

Under the above notation, the weak formulation of the elliptic problem (1.3) becomes the following elliptic boundary hemivariational inequality [13]:

$$\text{find } u \in V_0 \text{ such that } a(u,v) + \alpha \int_{\Gamma_3} j^0(u;v)\, d\Gamma \geq L(v) \text{ for all } v \in V_0. \tag{1.4}$$

Here and in what follows, we often omit the variable x, and we simply write $j(r)$ instead of $j(x,r)$. The stationary heat conduction models with nonmonotone multivalued subdifferential interior and boundary semipermeability relations can not be described by convex potentials. They use locally Lipschitz potentials and their weak formulations lead to hemivariational inequalities, see [17, Chapter 5.5.3] and [18].

We mention that theory of hemivariational and variational inequalities was proposed in the 1980s by Panagiotopoulos, see [17, 22, 23], as variational formulations of important classes of inequality problems in mechanics. In the last few years, new kinds of variational, hemivariational, and variational-hemivariational inequalities were investigated, see recent monographs [8, 24, 25], and the theory has emerged today as a new and interesting branch of applied mathematics.

We formulate the following optimal control problems:

- A problem of the type studied in [26, 27, 28] given by:

$$\text{find} \quad q^* \in Q \quad \text{such that} \quad J(q^*) = \min_{q \in Q} J(q) \tag{1.5}$$

  with

$$J(q) = \frac{1}{2}||u_q - z_d||_H^2 + \frac{M_2}{2}||q||_Q^2$$

  and, for each $\alpha > 0$, the problem

$$\text{find} \quad q_\alpha^* \in Q \quad \text{such that} \quad J_\alpha(q_\alpha^*) = \min_{q \in Q} J_\alpha(q) \tag{1.6}$$

  with

$$J_\alpha(q) = \frac{1}{2}||u_{\alpha q} - z_d||_H^2 + \frac{M_2}{2}||q||_Q^2 \tag{1.7}$$

  where u_q is the unique solution to the variational equality (1.2), $u_{\alpha q}$ is a solution to the hemivariational inequality (1.4), $z_d \in H$ given, and M_2 a positive constant. This complement to the one studied in [12, 13, 29].

- A simultaneous distributed and Neumann boundary optimal control problem, given by [13, 28, 30]:

$$\text{find} \quad (\overline{g}, \overline{q}) \in H \times Q \quad \text{such that} \quad J(\overline{g}, \overline{q}) = \min_{(g,q) \in H \times Q} J(g,q) \tag{1.8}$$

  with

$$J(g,q) = \frac{1}{2}||u_{gq} - z_d||_H^2 + \frac{M_1}{2}||g||_H^2 + \frac{M_2}{2}||q||_Q^2 \tag{1.9}$$

  where u_{gq} is the unique solution to the variational equality (1.2), $z_d \in H$ given and M_1 and M_2 are positive constants given. For each $\alpha > 0$, the following simultaneous distributed and Neumann boundary optimal control problem

$$\text{find} \quad (\overline{g}_\alpha, \overline{q}_\alpha) \in H \times Q \quad \text{such that} \quad J_\alpha(\overline{g}_\alpha, \overline{q}_\alpha) = \min_{(g,q) \in H \times Q} J_\alpha(g,q) \tag{1.10}$$

with

$$J_\alpha(g,q) = \frac{1}{2}||u_{\alpha gq} - z_d||_H^2 + \frac{M_1}{2}||g||_H^2 + \frac{M_2}{2}||q||_Q^2 \tag{1.11}$$

where $u_{\alpha gq}$ is a solution to the hemivariational inequality (1.4), $z_d \in H$ is given and M_1 and M_2 are positive constants.

The paper is structured as follows. In Section 2, we establish preliminaries concepts of the hemivariational inequalities theory, which are necessary for the development of the following sections. In Section 3, we prove existence of the boundary optimal controls and asymptotic behavior of the boundary optimal controls and the system states (1.6)-(1.7), when $\alpha \to \infty$. In Section 4, for each $\alpha > 0$, we obtain an existence result of solution to the simultaneous distributed-boundary optimal control problem (1.10). In Section 5, the strong convergence of a sequence of optimal controls and the system states to the problems (1.10)-(1.11) to the unique optimal control and the system state to the problem (1.8)-(1.9), are obtained when the parameter α goes to infinity. A novelty of this work is that we can obtain the asymptotic behavior of the optimal system states $u_{\alpha\overline{g}_\alpha\overline{q}_\alpha}$ as $\alpha \to \infty$ without any prior knowledge on the monotonicity on α as it was given in [13, 14]. In Section 6, we formulate the conclusions of this paper.

2. Preliminaries

In this section, we recall standard notation and preliminary concepts, which are necessary for the development of this paper.

Let $(X, \|\cdot\|_X)$ be a reflexive Banach space, X^* be its dual, and $\langle\cdot,\cdot\rangle$ denote the duality between X^* and X. For a real valued function defined on X, we have the following definitions [15, Section 2.1] and [24, 31].

Definition 2.1. A function $\varphi\colon X \to \mathbb{R}$ is said to be locally Lipschitz if, for every $x \in X$, there exist U_x a neighborhood of x and a constant $L_x > 0$ such that

$$|\varphi(y) - \varphi(z)| \leq L_x\|y - z\|_X \text{ for all } y, z \in U_x.$$

For such a function, the generalized (Clarke) directional derivative of φ at the point $x \in X$ in the direction $v \in X$ is defined by

$$\varphi^0(x;v) = \limsup_{y\to x,\,\lambda\to 0^+} \frac{\varphi(y+\lambda v) - \varphi(y)}{\lambda}.$$

The generalized gradient (subdifferential) of φ at x is a subset of the dual space X^* given by

$$\partial\varphi(x) = \{\zeta \in X^* \mid \varphi^0(x;v) \geq \langle\zeta, v\rangle \text{ for all } v \in X\}.$$

We consider the following hypothesis.

$\underline{H(j)}$: $j\colon \Gamma_3 \times \mathbb{R} \to \mathbb{R}$ is such that

(a) $j(\cdot, r)$ is measurable for all $r \in \mathbb{R}$,

(b) $j(x,\cdot)$ is locally Lipschitz for a.e. $x \in \Gamma_3$,

(c) there exist c_0, $c_1 \geq 0$ such that $|\partial j(x,r)| \leq c_0 + c_1|r|$ for all $r \in \mathbb{R}$, a.e. $x \in \Gamma_3$,

(d) $j^0(x,r;b-r) \leq 0$ for all $r \in \mathbb{R}$, a.e. $x \in \Gamma_3$ with a constant $b \in \mathbb{R}$.

Note that the existence results for elliptic hemivariational inequalities can be found in several contributions; see, e.g., [8, 17, 24, 32, 33]. In [13, Theorem 4], the hypothesis $H(j)$(d) is considered in order to obtain the existence of a solution to problem (1.4). Moreover, under this condition, the authors studied the asymptotic behavior when $\alpha \to \infty$ (see [13, Theorem 7]).

We note that, if the hypothesis $H(j)$(d) is replaced by the relaxed monotonicity condition (see [13, Remark 10] for details)

$$(e) \qquad j^0(x,r;s-r)+j^0(x,s;r-s) \le m_j|r-s|^2$$

for all r, $s \in \mathbb{R}$, a.e. $x \in \Gamma_3$ with $m_j \ge 0$, and the following smallness condition

$$(f) \qquad m_a > \alpha m_j \|\gamma\|^2$$

is assumed, then problem (1.4) is uniquely solvable, see [33, Lemma 20] for the proof. However, this smallness condition is not suitable in the study to problem (1.4) since for a sufficiently large value of α, it is not satisfied. Finally, in [13], we can find several examples of locally Lipschitz (nondifferentiable and nonconvex) functions which satisfy the above hypothesis.

3. Existence and Asymptotic Behavior of the Boundary Optimal Controls

In this section, we study the existence of solutions of problem (1.6) and its asymptotic behavior when the parameter α goes to infinity.

We know, by [26], that there exists a unique optimal solution $q^* \in Q$ of the boundary optimal control problem (1.5). Here, we prove the existence of solution to the optimal control problem (1.6) in which the system is governed by the hemivariational inequality (1.4).

Theorem 3.1. *For each $\alpha > 0$, if $H(j)(a)-(d)$ holds, then the boundary optimal control problems (1.6) has a solution.*

Proof. We denote, for each $\alpha > 0$ and each $q \in Q$, by $T_\alpha(q)$ the set of solutions of (1.4), and we have that

$$m = \inf\{J_\alpha(q), q \in Q, u_{\alpha q} \in T_\alpha(q)\} \ge 0. \tag{3.1}$$

Next, for each $\alpha > 0$, we consider $q_n^\alpha \in Q$ a minimizing sequence to (3.1), and we prove that there exist $\xi_\alpha \in Q$ and $\eta_\alpha \in V_0$ such that, when $n \to \infty$

$$u_{\alpha q_n^\alpha} \rightharpoonup \eta_\alpha \text{ in } V_0 \text{ weakly} \qquad \text{and} \qquad q_n^\alpha \rightharpoonup \xi_\alpha \text{ in } Q \text{ weakly}.$$

After that, we obtain that $\eta_\alpha = u_{\alpha\xi_\alpha}$, where $u_{\alpha\xi_\alpha}$ is a solution to the hemivariational inequality (1.4) for data $\xi_\alpha \in Q$ and $g \in H$. Finally, we prove that $m \ge J_\alpha(\xi_\alpha)$. Thus ξ_α is an optimal solution to optimal control problem (1.6). □

Now, we can prove the following asymptotic result by using the following additional hypothesis on the superpotential j.

$\underline{(H_1)}$: if $j^0(x,r;b-r)=0$ for all $r \in \mathbb{R}$, a.e. $x \in \Gamma_3$, then $r=b$.

Theorem 3.2. *Assume $H(j)$ and (H_1). If q_α^* is a optimal solution to problem (1.6) and q^* is the unique solution to problem (1.5), then $q_\alpha^* \to q^*$ in Q strongly and $u_{\alpha q_\alpha^*} \to u_{\infty q^*}$ in V strongly, when $\alpha \to \infty$.*

Proof. We make the prove in three steps.

Step 1 Since q_α^* is a optimal solution to problem (1.6), we deduce that there exist positive constants k_1 and k_2, independent of α, such that

$$||q_\alpha^*||_Q \le k_1 \quad \text{and} \quad ||u_{\alpha q_\alpha^*}||_H \le k_2. \tag{3.2}$$

Now, if we choose $v = u_{\infty q^*} - u_{\alpha q^*_\alpha} \in V_0$ as a test function in the elliptic boundary hemivariational inequality (1.4) and by $H(j)$(d), we have

$$a(u_{\infty q^*} - u_{\alpha q^*_\alpha}, u_{\infty q^*} - u_{\alpha q^*_\alpha}) \le a(u_{\infty q^*}, u_{\infty q^*} - u_{\alpha q^*_\alpha}) - L(u_{\infty q^*} - u_{\alpha q^*_\alpha}).$$

Next, we obtain that $\|u_{\alpha q^*_\alpha}\|_V \le k_3$, with $k_3 = \frac{1}{m_a}(M_a\|u_{\infty q^*}\|_V + \|L\|_{V^*}) + \|u_{\infty q^*}\|_V$ and $M_a > 0$. Moreover, there exists $k_4 > 0$ such that $-\alpha \int_{\Gamma_3} j^0(u_{\alpha q^*_\alpha}; u_{\infty q^*} - u_{\alpha q^*_\alpha})\, d\Gamma \le k_4$. Therefore, there exists $\eta \in V$ such that

$$u_{\alpha q^*_\alpha} \rightharpoonup \eta \ \text{ weakly in } \ V, \text{ as } \alpha \to \infty \tag{3.3}$$

and we have from (3.2) that there exists $\xi \in Q$ such that

$$q^*_\alpha \rightharpoonup \xi \ \text{ weakly in } \ Q, \text{ as } \alpha \to \infty.$$

Step 2 Here, we show that $\xi = q^*$ and $\eta = u_{\infty q^*}$. Taking into account that $\eta \in V_0$, for $w \in K$, if we consider $v = w - u_{\alpha q^*_\alpha} \in V_0$ in (1.4), since $w = b$ on Γ_3, by $H(j)$(d), we obtain

$$L(w - u_{\alpha q^*_\alpha}) \le a(u_{\alpha q^*_\alpha}, w - u_{\alpha q^*_\alpha}). \tag{3.4}$$

Next, we use the weak lower semicontinuity of the functional $V \ni v \mapsto a(v,v) \in \mathbb{R}$ and (3.4) to deduce that

$$\eta \in V_0 \ \text{ satisfies } \ L(w-\eta) \le a(\eta, w-\eta) \ \text{ for all } \ w \in K.$$

Subsequently, in a similar way to [12, 26] (see more details in Theorem 5.1, Step 2), from (3.3), by the compactness of the trace operator, the upper semicontinuity of the function $\mathbb{R}\times\mathbb{R} \ni (r,s) \mapsto j^0(x,r;s) \in \mathbb{R}$ for a.e. $x \in \Gamma_3$ and $H(j)$(d), we prove that $j^0(x,\eta; b-\eta) = 0$ a.e. $x \in \Gamma_3$. By using the hypothesis (H_1), we have $\eta(x) = b$ for a.e. $x \in \Gamma_3$, that is, $\eta \in K$. Therefore, we obtain that

$$\eta \in K \ \text{ satisfies } \ L(w-\eta) \le a(\eta, w-\eta) \ \text{ for all } \ w \in K.$$

Next, we have that

$$\eta \in K \ \text{ satisfies } \ a(\eta, v) = L(v) \ \text{ for all } \ v \in K_0,$$

i.e., $\eta \in K$ is a solution to problem (1.2) and by the uniqueness of solution to problem (1.2), we have $\eta = u_{\infty\xi}$. Hence $u_{\alpha q^*_\alpha} \rightharpoonup u_{\infty\xi}$ weakly in V as $\alpha \to \infty$. Now $J_\alpha(q^*_\alpha) \le J_\alpha(q)$, $\forall q \in Q$. Next

$$\begin{aligned} J(\xi) &= \frac{1}{2}||u_{\infty\xi} - z_d||^2_H + \frac{M_2}{2}||\xi||^2_Q = \frac{1}{2}||\eta - z_d||^2_H + \frac{M_2}{2}||\xi||^2_Q \\ &\le \liminf_{\alpha\to\infty} J_\alpha(q^*_\alpha) \le \liminf_{\alpha\to\infty} J_\alpha(q) \\ &= \lim_{\alpha\to\infty} J_\alpha(q) = J(q), \quad \forall q \in Q. \end{aligned}$$

From the uniqueness of the optimal control problem (1.5) (see [26]), we obtain that $\xi = q^*$. Thus $u_{\infty\xi} = u_{\infty q^*}$. Next, we have that, when $\alpha \to \infty$,

$$q^*_\alpha \rightharpoonup q^* \ \text{ weakly in } \ Q \quad \text{and} \quad u_{\alpha q^*_\alpha} \rightharpoonup u_{\infty q^*} \ \text{ weakly in } \ V.$$

Step 3 Taking $v = u_{\infty q^*} - u_{\alpha q^*_\alpha} \in V_0$ in problem (1.4), since $u_{\infty q^*} = b$ on Γ_3, by $H(j)$(d) and the coerciveness of the form a, we have

$$m_a\|u_{\infty q^*} - u_{\alpha q^*_\alpha}\|^2_V \le a(u_{\infty q^*}, u_{\infty q^*} - u_{\alpha q^*_\alpha}) + L(u_{\alpha q^*_\alpha} - u_{\infty q^*}).$$

Next, by the weak continuity of $a(u_{\infty g^*},\cdot)$, the compactness of the trace operator and the fact that $u_{\alpha q_\alpha^*} \to u_{\infty q^*}$ strongly in H, we conclude that $u_{\alpha q_\alpha^*} \to u_{\infty q^*}$ strongly in V as $\alpha \to \infty$. Now, from $u_{\alpha q_\alpha^*} \to u_{\infty q^*}$ strongly in H, we deduce

$$\lim_{\alpha\to\infty} \frac{1}{2}||u_{\alpha q_\alpha^*} - z_d||_H^2 = \frac{1}{2}||u_{\infty q^*} - z_d||_H^2. \tag{3.5}$$

As $q_\alpha^* \rightharpoonup q^*$ weakly in Q, one has

$$||q^*||_Q^2 \le \liminf_{\alpha\to\infty} ||q_\alpha^*||_Q^2. \tag{3.6}$$

Next, from (3.5) and (3.6), we obtain

$$\frac{1}{2}||u_{\infty q^*} - z_d||_H^2 + \frac{M_2}{2}||q^*||_Q^2 \le \liminf_{\alpha\to\infty}\left(\frac{1}{2}||u_{\alpha q_\alpha^*} - z_d||_H^2 + \frac{M_2}{2}||q_\alpha^*||_Q^2\right),$$

that is, $J(q^*) \le \liminf_{\alpha\to\infty} J_\alpha(q_\alpha^*)$.

On the other hand, from the definition of q_α^* and $u_{\alpha q^*} \to u_{\infty q^*}$ strongly in H (see [13, Theorem 7]), we obtain

$$\limsup_{\alpha\to\infty} J_\alpha(q_\alpha^*) \le \limsup_{\alpha\to\infty} J_\alpha(q^*) = J(q^*).$$

Thus

$$\lim_{\alpha\to\infty}\left(\frac{1}{2}||u_{\alpha q_\alpha^*} - z_d||_H^2 + \frac{M_2}{2}||q_\alpha^*||_Q^2\right) = \frac{1}{2}||u_{\infty q^*} - z_d||_H^2 + \frac{M_2}{2}||q^*||_Q^2. \tag{3.7}$$

Finally, from (3.5) and (3.7), when $\alpha \to \infty$, we have $||q_\alpha^*||_Q^2 \to ||q^*||_Q^2$. As $q_\alpha^* \rightharpoonup q^*$ weakly in Q, we deduce that $q_\alpha^* \to q^*$ strongly in Q. □

4. Existence of the Simultaneous Optimal Controls

We know, by [30], that there exists a unique optimal pair $(\overline{g},\overline{q}) \in H \times Q$ of the simultaneous distributed-boundary optimal control problem (1.8). In similar way to [12], we have a result on existence of solution to the simultaneous optimal control problem (1.10) in which the system is governed by the hemivariational inequality (1.4).

Theorem 4.1. *For each $\alpha > 0$, if $H(j)(a)-(d)$ holds, then the simultaneous distributed-boundary optimal control problem (1.10) governed by the hemivariational inequality (1.4) has a solution.*

Proof. By definition, for each $\alpha > 0$, the functional J_α is bounded from bellow. Next, taking into account that the hemivariational inequality (1.4) has a solution (see [13, Theorem 4]), for each $\alpha > 0$ and each $(g,q) \in H \times Q$, we denote by $T_\alpha(g,q)$ the set of solutions of (1.4) and have that

$$m = \inf\{J_\alpha(g,q), (g,q) \in H \times Q, u_{\alpha gq} \in T_\alpha(g,q)\} \ge 0. \tag{4.1}$$

For each $\alpha > 0$, let $(g_n^\alpha, q_n^\alpha) \in H \times Q$ be with $n \in \mathbb{N}$ a minimizing sequence to (4.1) such that

$$m \le J_\alpha(g_n^\alpha, q_n^\alpha) \le m + \frac{1}{n}. \tag{4.2}$$

Taking into account that the functional J_α satisfies

$$\lim_{||(g,q)||_{H\times Q}\to+\infty} J_\alpha(g,q) = +\infty$$

where $||(g,q)||^2_{H\times Q} = ||g||^2_H + ||q||^2_Q$, we obtain that there exists $C_1 > 0$, independent of α, such that

$$||g_n^\alpha||_H \le C_1 \quad \text{and} \quad ||q_n^\alpha||_Q \le C_1.$$

Moreover, from (1.4), we can prove that there exists $C_2 > 0$, independent of α, such that

$$||u_{\alpha g_n^\alpha q_n^\alpha}||_{V_0} \le C_2. \tag{4.3}$$

In fact, let $u_\infty \in K$ be the solution to problem (1.2). We have

$$a(u_{\alpha g_n^\alpha q_n^\alpha}, u_\infty - u_{\alpha g_n^\alpha q_n^\alpha}) + \alpha \int_{\Gamma_3} j^0(u_{\alpha g_n^\alpha q_n^\alpha}; u_\infty - u_{\alpha g_n^\alpha q_n^\alpha})\, d\Gamma \ge \int_\Omega g_n^\alpha (u_\infty - u_{\alpha g_n^\alpha q_n^\alpha})\, dx - \int_{\Gamma_2} q_n^\alpha (u_\infty - u_{\alpha g_n^\alpha q_n^\alpha})\, d\Gamma.$$

Hence

$$a(u_\infty - u_{\alpha g_n^\alpha q_n^\alpha}, u_\infty - u_{\alpha g_n^\alpha q_n^\alpha}) \le a(u_\infty, u_\infty - u_{\alpha g_n^\alpha q_n^\alpha}) + \alpha \int_{\Gamma_3} j^0(u_{\alpha g_n^\alpha q_n^\alpha}; b - u_{\alpha g_n^\alpha q_n^\alpha})\, d\Gamma + \int_\Omega g_n^\alpha (u_{\alpha g_n^\alpha q_n^\alpha} - u_\infty)\, dx - \int_{\Gamma_2} q_n^\alpha (u_{\alpha g_n^\alpha q_n^\alpha} - u_\infty)\, d\Gamma.$$

From hypothesis $H(j)$(d), since the form a is bounded (with positive constant M_a), we have

$$\begin{aligned}
\|u_\infty - u_{\alpha g_n^\alpha q_n^\alpha}\|^2_{V_0} &\le a(u_\infty, u_\infty - u_{\alpha g_n^\alpha q_n^\alpha}) + \int_\Omega g_n^\alpha (u_{\alpha g_n^\alpha q_n^\alpha} - u_\infty)\, dx - \int_{\Gamma_2} q_n^\alpha (u_{\alpha g_n^\alpha q_n^\alpha} - u_\infty)\, d\Gamma \\
&\le M_a \|u_\infty\|_V \|u_\infty - u_{\alpha g_n^\alpha q_n^\alpha}\|_V + (||g_n^\alpha||_H + ||q_n^\alpha||_Q ||\gamma||)\, \|u_\infty - u_{\alpha g_n^\alpha q_n^\alpha}\|_V \\
&\le C_3\, (M_a \|u_\infty\|_V + C_1 + C_1 ||\gamma||)\, \|u_\infty - u_{\alpha g_n^\alpha q_n^\alpha}\|_{V_0}
\end{aligned}$$

where $||\gamma||$ denotes the norm of trace operator, and C_3 is a positive constant due to the equivalence of norms. Subsequently, we obtain (4.3). Therefore, for each $\alpha > 0$, there exist $f_\alpha \in H$, $\xi_\alpha \in Q$ and $\eta_\alpha \in V_0$ such that, as $n \to \infty$,

$$u_{\alpha g_n^\alpha q_n^\alpha} \rightharpoonup \eta_\alpha \quad \text{in} \quad V_0, \qquad g_n^\alpha \rightharpoonup f_\alpha \quad \text{in} \quad H \qquad \text{and} \qquad q_n^\alpha \rightharpoonup \xi_\alpha \quad \text{in} \quad Q.$$

Now, for each $\alpha > 0$ and for all $(g_n^\alpha, q_n^\alpha) \in H \times Q$, we have

$$a(u_{\alpha g_n^\alpha q_n^\alpha}, v) + \alpha \int_{\Gamma_3} j^0(u_{\alpha g_n^\alpha q_n^\alpha}; v)\, d\Gamma \ge \int_\Omega g_n^\alpha v\, dx - \int_{\Gamma_2} q_n^\alpha v\, d\Gamma \ \text{ for all } \ v \in V_0$$

and taking the upper limit, we obtain

$$a(\eta_\alpha, v) + \alpha \limsup_{n\to\infty} \int_{\Gamma_3} j^0(u_{\alpha g_n^\alpha q_n^\alpha}; v)\, d\Gamma \ge \int_\Omega f_\alpha v\, dx - \int_{\Gamma_2} \xi_\alpha v\, d\Gamma \ \text{ for all } \ v \in V_0. \tag{4.4}$$

By the compactness of the trace operator from V into $L^2(\Gamma_3)$, we have $u_{\alpha g_n^\alpha q_n^\alpha}\big|_{\Gamma_3} \to \eta_\alpha\big|_{\Gamma_3}$ in $L^2(\Gamma_3)$ as $n \to +\infty$, and at least for a subsequence, $u_{\alpha g_n^\alpha q_n^\alpha}(x) \to \eta_\alpha(x)$ for a.e. $x \in \Gamma_3$ and $|u_{\alpha g_n^\alpha q_n^\alpha}(x)| \le h_\alpha(x)$ a.e. $x \in \Gamma_3$, where $h_\alpha \in L^2(\Gamma_3)$. Since the function $\mathbb{R} \times \mathbb{R} \ni (r,s) \mapsto j^0(x,r;s) \in \mathbb{R}$ a.e. is upper semicontinuous on Γ_3 (see [13, Proposition 3]), we obtain

$$\limsup_{n\to\infty} j^0(x, u_{\alpha g_n^\alpha q_n^\alpha}(x); v(x)) \le j^0(x, \eta_\alpha(x); v(x)) \ \text{ a.e. } \ x \in \Gamma_3.$$

Next, from $H(j)(c)$, we deduce the estimate

$$|j^0(x, u_{\alpha g_n^\alpha q_n^\alpha}(x); v(x))| \le (c_0 + c_1 |u_{\alpha g_n^\alpha q_n^\alpha}(x)|)\, |v(x)| \le k_\alpha(x) \ \text{ a.e. } \ x \in \Gamma_3$$

where $k_\alpha \in L^1(\Gamma_3)$ and $k_\alpha(x) = (c_0 + c_1 h_\alpha(x))|v(x)|$. From the dominated convergence theorem (see [31]), we have

$$\limsup_{n\to\infty} \int_{\Gamma_3} j^0(u_{\alpha g_n^\alpha q_n^\alpha}; v)\, d\Gamma \le \int_{\Gamma_3} \limsup_{n\to\infty} j^0(u_{\alpha g_n^\alpha q_n^\alpha}; v)\, d\Gamma \le \int_{\Gamma_3} j^0(\eta_\alpha; v)\, d\Gamma.$$

Using the latter in (4.4), we obtain

$$a(\eta_\alpha, v) + \alpha \int_{\Gamma_3} j^0(\eta_\alpha; v)\, d\Gamma \ge \int_\Omega f_\alpha v\, dx - \int_{\Gamma_2} \xi_\alpha v\, d\Gamma \quad \text{for all} \quad v \in V_0$$

that is, $\eta_\alpha \in V_0$ is a solution to the hemivariational inequality (1.4). Next, we prove that

$$\eta_\alpha = u_{\alpha f_\alpha \xi_\alpha}$$

where $u_{\alpha f_\alpha \xi_\alpha}$ is a solution to the hemivariational inequality (1.4) for data $f_\alpha \in H$ and $\xi_\alpha \in Q$, for each $\alpha > 0$. Finally, from (4.2) and the weak lower semicontinuity of J_α, we have

$$\begin{aligned} m &\ge \liminf_{n\to\infty} J_\alpha(g_n^\alpha, q_n^\alpha) \\ &\ge \frac{1}{2} \liminf_{n\to\infty} ||u_{\alpha g_n^\alpha q_n^\alpha} - z_d||_H^2 + \frac{M_1}{2} \liminf_{n\to\infty} ||g_n^\alpha||_H^2 + \frac{M_2}{2} \liminf_{n\to\infty} ||q_n^\alpha||_Q^2 \\ &\ge \frac{1}{2} ||u_{\alpha f_\alpha \xi_\alpha} - z_d||_H^2 + \frac{M_1}{2} ||f_\alpha||_H^2 + \frac{M_2}{2} ||\xi_\alpha||_Q^2 = J_\alpha(f_\alpha, \xi_\alpha), \end{aligned}$$

and therefore, for each $\alpha > 0$, (f_α, ξ_α) is an optimal pair to simultaneous distributed-boundary optimal control problem (1.10). □

5. Asymptotic Behavior of the Simultaneous Optimal Controls

In this section, we investigate the asymptotic behavior of the optimal solutions to problem (1.10) when $\alpha \to \infty$.

Theorem 5.1. *Assume $H(j)$ and (H_1). If $(\overline{g}_\alpha, \overline{q}_\alpha)$ is a optimal solution to simultaneous distributed and Neumann boundary optimal control problem (1.10) and $(\overline{g}, \overline{q})$ is the unique solution to simultaneous optimal control problem (1.8), then $(\overline{g}_\alpha, \overline{q}_\alpha) \to (\overline{g}, \overline{q})$ in $H \times Q$ strongly and $u_{\alpha \overline{g}_\alpha \overline{q}_\alpha} \to u_{\infty \overline{g}\,\overline{q}}$ in V strongly, when $\alpha \to \infty$.*

Proof. As in Theorem 3.2, we make the prove in three steps.
Step 1. Since, for each $\alpha > 0$, $(\overline{g}_\alpha, \overline{q}_\alpha)$ is a optimal solution to problem (1.10), we have the following inequality, for all $(g, q) \in H \times Q$

$$\frac{1}{2} ||u_{\alpha \overline{g}_\alpha \overline{q}_\alpha} - z_d||_H^2 + \frac{M_1}{2} ||\overline{g}_\alpha||_H^2 + \frac{M_2}{2} ||\overline{q}_\alpha||_Q^2 \le \frac{1}{2} ||u_{\alpha g q} - z_d||_H^2 + \frac{M_1}{2} ||g||_H^2 + \frac{M_2}{2} ||q||_Q^2.$$

Taking $g = 0$ in Ω and $q = 0$ on Γ_2, we obtain that there exists a positive constant C_1 such that

$$\frac{1}{2} ||u_{\alpha \overline{g}_\alpha \overline{q}_\alpha} - z_d||_H^2 + \frac{M_1}{2} ||\overline{g}_\alpha||_H^2 + \frac{M_2}{2} ||\overline{q}_\alpha||_Q^2 \le \frac{1}{2} ||u_{\alpha 00} - z_d||_H^2 \le C_1$$

because $\{u_{\alpha 00}\}$ is convergent when $\alpha \to \infty$ (see [13, Theorem 7]). Therefore, there exist positive constants C_2, C_3, and C_4, independent of α, such that

$$||\overline{g}_\alpha||_H \le C_2, \qquad ||\overline{q}_\alpha||_Q \le C_3 \quad \text{and} \quad ||u_{\alpha \overline{g}_\alpha \overline{q}_\alpha}||_H \le C_4. \tag{5.1}$$

Now, we choose $v = u_{\infty\overline{g}\,\overline{q}} - u_{\alpha\overline{g}_\alpha\overline{q}_\alpha} \in V_0$ as a test function in elliptic boundary hemivariational inequality (1.4) to obtain

$$a(u_{\alpha\overline{g}_\alpha\overline{q}_\alpha}, u_{\infty\overline{g}\,\overline{q}} - u_{\alpha\overline{g}_\alpha\overline{q}_\alpha}) + \alpha \int_{\Gamma_3} j^0(u_{\alpha\overline{g}_\alpha\overline{q}_\alpha}, u_{\infty\overline{g}\,\overline{q}} - u_{\alpha\overline{g}_\alpha\overline{q}_\alpha})\, d\Gamma \geq L(u_{\infty\overline{g}\,\overline{q}} - u_{\alpha\overline{g}_\alpha\overline{q}_\alpha}).$$

From the equality

$$a(u_{\alpha\overline{g}_\alpha\overline{q}_\alpha}, u_{\infty\overline{g}\,\overline{q}} - u_{\alpha\overline{g}_\alpha\overline{q}_\alpha}) = -a(u_{\infty\overline{g}\,\overline{q}} - u_{\alpha\overline{g}_\alpha\overline{q}_\alpha}, u_{\infty\overline{g}\,\overline{q}} - u_{\alpha\overline{g}_\alpha\overline{q}_\alpha}) + a(u_{\infty\overline{g}\,\overline{q}}, u_{\infty\overline{g}\,\overline{q}} - u_{\alpha\overline{g}_\alpha\overline{q}_\alpha}),$$

we obtain

$$\begin{aligned} &a(u_{\infty\overline{g}\,\overline{q}} - u_{\alpha\overline{g}_\alpha\overline{q}_\alpha}, u_{\infty\overline{g}\,\overline{q}} - u_{\alpha\overline{g}_\alpha\overline{q}_\alpha}) - \alpha \int_{\Gamma_3} j^0(u_{\alpha\overline{g}_\alpha\overline{q}_\alpha}; u_{\infty\overline{g}\,\overline{q}} - u_{\alpha\overline{g}_\alpha\overline{q}_\alpha})\, d\Gamma \\ &\leq a(u_{\infty\overline{g}\,\overline{q}}, u_{\infty\overline{g}\,\overline{q}} - u_{\alpha\overline{g}_\alpha\overline{q}_\alpha}) - L(u_{\infty\overline{g}\,\overline{q}} - u_{\alpha\overline{g}_\alpha\overline{q}_\alpha}). \end{aligned} \tag{5.2}$$

Taking into account that $j^0(u_{\alpha\overline{g}_\alpha\overline{q}_\alpha}; u_{\infty\overline{g}\,\overline{q}} - u_{\alpha\overline{g}_\alpha\overline{q}_\alpha}) = j^0(u_{\alpha\overline{g}_\alpha\overline{q}_\alpha}; b - u_{\alpha\overline{g}_\alpha\overline{q}_\alpha})$ on Γ_3, and $H(j)$(d), we have $j^0(u_{\alpha\overline{g}_\alpha\overline{q}_\alpha}; u_{\infty\overline{g}\,\overline{q}} - u_{\alpha\overline{g}_\alpha\overline{q}_\alpha}) \leq 0$ on Γ_3. Hence

$$a(u_{\infty\overline{g}\,\overline{q}} - u_{\alpha\overline{g}_\alpha\overline{q}_\alpha}, u_{\infty\overline{g}\,\overline{q}} - u_{\alpha\overline{g}_\alpha\overline{q}_\alpha}) \leq a(u_{\infty\overline{g}\,\overline{q}}, u_{\infty\overline{g}\,\overline{q}} - u_{\alpha\overline{g}_\alpha\overline{q}_\alpha}) - L(u_{\infty\overline{g}\,\overline{q}} - u_{\alpha\overline{g}_\alpha\overline{q}_\alpha}).$$

By the boundedness and coerciveness of a, we infer

$$m_a \|u_{\infty\overline{g}\,\overline{q}} - u_{\alpha\overline{g}_\alpha\overline{q}_\alpha}\|_V^2 \leq (M_a \|u_{\infty\overline{g}\,\overline{q}}\|_V + \|L\|_{V^*})\, \|u_{\infty\overline{g}\,\overline{q}} - u_{\alpha\overline{g}_\alpha\overline{q}_\alpha}\|_V$$

with $M_a > 0$. Subsequently,

$$\begin{aligned} \|u_{\alpha\overline{g}_\alpha\overline{q}_\alpha}\|_V &\leq \|u_{\infty\overline{g}\,\overline{q}} - u_{\alpha\overline{g}_\alpha\overline{q}_\alpha}\|_V + \|u_{\infty\overline{g}\,\overline{q}}\|_V \\ &\leq \frac{1}{m_a}(M_a \|u_{\infty\overline{g}\,\overline{q}}\|_V + \|L\|_{V^*}) + \|u_{\infty\overline{g}\,\overline{q}}\|_V \\ &=: C_5, \end{aligned} \tag{5.3}$$

where $C_5 > 0$ is a constant independent of α. Observe that $a(u_{\infty\overline{g}\,\overline{q}} - u_{\alpha\overline{g}_\alpha\overline{q}_\alpha}, u_{\infty\overline{g}\,\overline{q}} - u_{\alpha\overline{g}_\alpha\overline{q}_\alpha}) \geq 0$. From (5.2), we have

$$\begin{aligned} -\alpha \int_{\Gamma_3} j^0(u_{\alpha\overline{g}_\alpha\overline{q}_\alpha}; u_{\infty\overline{g}\,\overline{q}} - u_{\alpha\overline{g}_\alpha\overline{q}_\alpha})\, d\Gamma &\leq (M_a \|u_{\infty\overline{g}\,\overline{q}}\|_V + \|L\|_{V^*})\, \|u_{\infty\overline{g}\,\overline{q}} - u_{\alpha\overline{g}_\alpha\overline{q}_\alpha}\|_V \\ &\leq \frac{1}{m_a}(M_a \|u_{\infty\overline{g}\,\overline{q}}\|_V + \|L\|_{V^*})^2 \\ &=: C_6, \end{aligned}$$

where $C_6 > 0$ is independent of α. Thus

$$-\int_{\Gamma_3} j^0(u_{\alpha\overline{g}_\alpha\overline{q}_\alpha}; u_{\infty\overline{g}\,\overline{q}} - u_{\alpha\overline{g}_\alpha\overline{q}_\alpha})\, d\Gamma \leq \frac{C_6}{\alpha}. \tag{5.4}$$

It follows from (5.3) that $\{u_{\alpha\overline{g}_\alpha\overline{q}_\alpha}\}$ remains in a bounded subset of V. Thus there exists $\eta \in V$ such that, by passing to a subsequence if necessary,

$$u_{\alpha\overline{g}_\alpha\overline{q}_\alpha} \rightharpoonup \eta \ \text{ weakly in } \ V, \text{ as } \alpha \to \infty. \tag{5.5}$$

Moreover. from (5.1), we have that there exists $h \in H$ and $p \in Q$ such that

$$\overline{g}_\alpha \rightharpoonup h \ \text{ weakly in } \ H, \text{ as } \alpha \to \infty$$

and

$$\overline{q}_\alpha \rightharpoonup p \ \text{ weakly in } \ Q, \text{ as } \alpha \to \infty.$$

Step 2. Next, we show that $h=\overline{g}$, $p=\overline{q}$, and $\eta=u_{\infty\overline{g}\overline{q}}$. We observe that $\eta\in V_0$ because $\{u_{\alpha\overline{g}_\alpha\overline{q}_\alpha}\}\subset V_0$ and V_0 is sequentially weakly closed in V. Let $w\in K$ and $v=w-u_{\alpha\overline{g}_\alpha\overline{q}_\alpha}\in V_0$. From (1.4), we have

$$L(w-u_{\alpha\overline{g}_\alpha\overline{q}_\alpha})\le a(u_{\alpha\overline{g}_\alpha\overline{q}_\alpha},w-u_{\alpha\overline{g}_\alpha\overline{q}_\alpha})+\alpha\int_{\Gamma_3} j^0(u_{\alpha\overline{g}_\alpha\overline{q}_\alpha};w-u_{\alpha\overline{g}_\alpha\overline{q}_\alpha})\,d\Gamma.$$

Since $w=b$ on Γ_3, by $H(j)$(d), we have

$$\alpha\int_{\Gamma_3} j^0(u_{\alpha\overline{g}_\alpha\overline{q}_\alpha};w-u_{\alpha\overline{g}_\alpha\overline{q}_\alpha})\,d\Gamma=\alpha\int_{\Gamma_3} j^0(u_{\alpha\overline{g}_\alpha\overline{q}_\alpha};b-u_{\alpha\overline{g}_\alpha\overline{q}_\alpha})\,d\Gamma\le 0$$

which implies

$$L(w-u_{\alpha\overline{g}_\alpha\overline{q}_\alpha})\le a(u_{\alpha\overline{g}_\alpha\overline{q}_\alpha},w-u_{\alpha\overline{g}_\alpha\overline{q}_\alpha}). \tag{5.6}$$

Next, we use the weak lower semicontinuity of the functional $V\ni v\mapsto a(v,v)\in\mathbb{R}$ and (5.6) to obtain

$$\eta\in V_0 \ \text{ satisfies } \ L(w-\eta)\le a(\eta,w-\eta) \ \text{ for all } \ w\in K. \tag{5.7}$$

Subsequently, we show that $\eta\in K$. In fact, from (5.5), and the compactness of the trace operator, we have $u_{\alpha\overline{g}_\alpha\overline{q}_\alpha}\big|_{\Gamma_3}\to\eta\big|_{\Gamma_3}$ in $L^2(\Gamma_3)$, as $\alpha\to\infty$. Passing to a subsequence if necessary, we may suppose that $u_{\alpha\overline{g}_\alpha\overline{q}_\alpha}(x)\to\eta(x)$ for a.e. $x\in\Gamma_3$ and there exists $f\in L^2(\Gamma_3)$ such that $|u_{\alpha\overline{g}_\alpha\overline{q}_\alpha}(x)|\le f(x)$ a.e. $x\in\Gamma_3$. Using the upper semicontinuity of the function $\mathbb{R}\times\mathbb{R}\ni(r,s)\mapsto j^0(x,r;s)\in\mathbb{R}$ for a.e. $x\in\Gamma_3$, see [13, Proposition 3 (iii)], we obtain

$$\limsup_{\alpha\to\infty} j^0(x,u_{\alpha\overline{g}_\alpha\overline{q}_\alpha}(x);b-u_{\alpha\overline{g}_\alpha\overline{q}_\alpha}(x))\le j^0(x,\eta(x);b-\eta(x)) \ \text{ a.e. } \ x\in\Gamma_3.$$

Next, taking into account the estimate

$$|j^0(x,u_{\alpha\overline{g}_\alpha\overline{q}_\alpha}(x);b-u_{\alpha\overline{g}_\alpha\overline{q}_\alpha}(x))|\le(c_0+c_1|u_{\alpha\overline{g}_\alpha\overline{q}_\alpha}(x)|)\,|b-u_{\alpha\overline{g}_\alpha\overline{q}_\alpha}(x)|\le k(x) \ \text{ a.e. } \ x\in\Gamma_3$$

with $k\in L^1(\Gamma_3)$ given by $k(x)=(c_0+c_1f(x))(|b|+f(x))$, and the dominated convergence theorem (see [31]), we obtain

$$\limsup_{\alpha\to\infty}\int_{\Gamma_3} j^0(u_{\alpha\overline{g}_\alpha\overline{q}_\alpha};b-u_{\alpha\overline{g}_\alpha\overline{q}_\alpha})\,d\Gamma\le\int_{\Gamma_3} j^0(\eta;b-\eta)\,d\Gamma.$$

Consequently, from $H(j)$(d) and (5.4), we have

$$0\le-\int_{\Gamma_3} j^0(\eta;b-\eta)\,d\Gamma\le\liminf_{\alpha\to\infty}\left(-\int_{\Gamma_3} j^0(u_{\alpha\overline{g}_\alpha\overline{q}_\alpha};b-u_{\alpha\overline{g}_\alpha\overline{q}_\alpha})\,d\Gamma\right)\le 0$$

which gives $\int_{\Gamma_3} j^0(\eta;b-\eta)\,d\Gamma=0$. Again by $H(j)$(d), we have $j^0(x,\eta(x);b-\eta(x))=0$ a.e. $x\in\Gamma_3$. By using now the hypothesis (H_1), we have $\eta(x)=b$ for a.e. $x\in\Gamma_3$, which together with (5.7) implies

$$\eta\in K \ \text{ satisfies } \ L(w-\eta)\le a(\eta,w-\eta) \ \text{ for all } \ w\in K.$$

Next, we prove that $\eta=u_{\infty hp}$. To this end, let $v:=w-\eta\in K_0$ with arbitrary $w\in K$. Hence, $L(v)\le a(\eta,v)$ for all $v\in K_0$. Recalling that $v\in K_0$ implies $-v\in K_0$, we obtain $a(\eta,v)\le L(v)$ for all $v\in K_0$. Hence, we conclude that

$$\eta\in K \ \text{ satisfies } \ a(\eta,v)=L(v) \ \text{ for all } \ v\in K_0,$$

i.e., $\eta \in K$ is a solution to problem (1.2). By the uniqueness of solution to problem (1.2), we have $\eta = u_{\infty hp}$. Hence $u_{\alpha \overline{g}_\alpha \overline{q}_\alpha} \rightharpoonup u_{\infty hp}$ weakly in V, as $\alpha \to \infty$. Now

$$J_\alpha(\overline{g}_\alpha, \overline{q}_\alpha) \leq J_\alpha(g,q), \quad \forall (g,q) \in H \times Q$$

and

$$\begin{aligned} J(h,p) &= \frac{1}{2}||u_{\infty hp} - z_d||_H^2 + \frac{M_1}{2}||h||_H^2 + \frac{M_2}{2}||p||_Q^2 \\ &= \frac{1}{2}||\eta - z_d||_H^2 + \frac{M_1}{2}||h||_H^2 + \frac{M_2}{2}||p||_Q^2 \\ &\leq \liminf_{\alpha\to\infty} J_\alpha(\overline{g}_\alpha, \overline{q}_\alpha) \leq \liminf_{\alpha\to\infty} J_\alpha(g,q) \\ &= \lim_{\alpha\to\infty} J_\alpha(g,q) = J(g,q), \quad \forall (g,q) \in H \times Q. \end{aligned}$$

From the uniqueness of the optimal control problem (1.8) (see [29]), we obtain that

$$h = \overline{g} \quad \text{and} \quad p = \overline{q}.$$

Thus $u_{\infty hp} = u_{\infty \overline{g}\,\overline{q}}$. Next, we prove that, as $\alpha \to \infty$

$$\overline{g}_\alpha \rightharpoonup \overline{g} \text{ weakly in } H, \qquad \overline{q}_\alpha \rightharpoonup \overline{q} \text{ weakly in } Q$$

and

$$u_{\alpha \overline{g}_\alpha \overline{q}_\alpha} \rightharpoonup u_{\infty \overline{g}\,\overline{q}} \text{ weakly in } V.$$

Step 3. Now, we prove the strong convergence $u_{\alpha \overline{g}_\alpha \overline{q}_\alpha} \to u_{\infty \overline{g}\,\overline{q}}$ in V, as $\alpha \to \infty$. Choosing $v = u_{\infty \overline{g}\,\overline{q}} - u_{\alpha \overline{g}_\alpha \overline{q}_\alpha} \in V_0$ in problem (1.4), we obtain

$$a(u_{\alpha \overline{g}_\alpha \overline{q}_\alpha}, u_{\infty \overline{g}\,\overline{q}} - u_{\alpha \overline{g}_\alpha \overline{q}_\alpha}) + \alpha \int_{\Gamma_3} j^0(u_{\alpha \overline{g}_\alpha \overline{q}_\alpha}; u_{\infty \overline{g}\,\overline{q}} - u_{\alpha \overline{g}_\alpha \overline{q}_\alpha})\, d\Gamma \geq L(u_{\infty \overline{g}\,\overline{q}} - u_{\alpha \overline{g}_\alpha \overline{q}_\alpha}).$$

Hence

$$\begin{aligned} a(u_{\infty \overline{g}\,\overline{q}} - u_{\alpha \overline{g}_\alpha \overline{q}_\alpha}, u_{\infty \overline{g}\,\overline{q}} - u_{\alpha \overline{g}_\alpha \overline{q}_\alpha}) &\leq a(u_{\infty \overline{g}\,\overline{q}}, u_{\infty \overline{g}\,\overline{q}} - u_{\alpha \overline{g}_\alpha \overline{q}_\alpha}) + L(u_{\alpha \overline{g}_\alpha \overline{q}_\alpha} - u_{\infty \overline{g}\,\overline{q}}) \\ &\quad + \alpha \int_{\Gamma_3} j^0(u_{\alpha \overline{g}_\alpha \overline{q}_\alpha}; u_{\infty \overline{g}\,\overline{q}} - u_{\alpha \overline{g}_\alpha \overline{q}_\alpha})\, d\Gamma. \end{aligned}$$

Since $u_{\infty \overline{g}\,\overline{q}} = b$ on Γ_3, by $H(j)$(d) and the coerciveness of the form a, we have

$$m_a \|u_{\infty \overline{g}\,\overline{q}} - u_{\alpha \overline{g}_\alpha \overline{q}_\alpha}\|_V^2 \leq a(u_{\infty \overline{g}\,\overline{q}}, u_{\infty \overline{g}\,\overline{q}} - u_{\alpha \overline{g}_\alpha \overline{q}_\alpha}) + L(u_{\alpha \overline{g}_\alpha \overline{q}_\alpha} - u_{\infty \overline{g}\,\overline{q}}).$$

Employing the weak continuity of $a(u_{\infty \overline{g}\,\overline{q}}, \cdot)$, the compactness of the trace operator, and taking into account that $u_{\alpha \overline{g}_\alpha \overline{q}_\alpha} \to u_{\infty \overline{g}\,\overline{q}}$ strongly in H, we conclude that $u_{\alpha \overline{g}_\alpha \overline{q}_\alpha} \to u_{\infty \overline{g}\,\overline{q}}$ strongly in V as $\alpha \to \infty$.

Finally, we prove the strong convergence of $\overline{g}_\alpha$ to $\overline{g}$ in H and $\overline{q}_\alpha$ to $\overline{q}$ in Q as $\alpha \to \infty$. In fact, from $u_{\alpha \overline{g}_\alpha \overline{q}_\alpha} \to u_{\infty \overline{g}\,\overline{q}}$ strongly in H, we deduce

$$\lim_{\alpha\to\infty} \frac{1}{2}||u_{\alpha \overline{g}_\alpha \overline{q}_\alpha} - z_d||_H^2 = \frac{1}{2}||u_{\infty \overline{g}\,\overline{q}} - z_d||_H^2. \tag{5.8}$$

As $\overline{g}_\alpha \rightharpoonup \overline{g}$ weakly in H and $\overline{q}_\alpha \rightharpoonup \overline{q}$ weakly in Q, we have

$$||\overline{g}||_H^2 \leq \liminf_{\alpha\to\infty} ||\overline{g}_\alpha||_H^2 \quad \text{and} \quad ||\overline{q}||_Q^2 \leq \liminf_{\alpha\to\infty} ||\overline{q}_\alpha||_Q^2. \tag{5.9}$$

Next, from (5.8) and (5.9), we obtain

$$\frac{1}{2}||u_{\infty\overline{g}\,\overline{q}}-z_d||_H^2+\frac{M_1}{2}||\overline{g}||_H^2+\frac{M_2}{2}||\overline{q}||_Q^2\leq\liminf_{\alpha\to\infty}\left(\frac{1}{2}||u_{\alpha\overline{g}_\alpha\overline{q}_\alpha}-z_d||_H^2+\frac{M_1}{2}||\overline{g}_\alpha||_H^2+\frac{M_2}{2}||\overline{q}_\alpha||_Q^2\right),$$

that is, $J(\overline{g},\overline{q})\leq\liminf_{\alpha\to\infty}J_\alpha(\overline{g}_\alpha,\overline{q}_\alpha)$.

On the other hand, from the definition of $(\overline{g}_\alpha,\overline{q}_\alpha)$, we have $J_\alpha(\overline{g}_\alpha,\overline{q}_\alpha)\leq J_\alpha(\overline{g},\overline{q})$. Taking into account that $u_{\alpha\overline{g}_\alpha\overline{q}_\alpha}\to u_{\infty\overline{g}\,\overline{q}}$ strongly in H (see [13, Theorem 7]), we obtain

$$\limsup_{\alpha\to\infty}J_\alpha(\overline{g}_\alpha,\overline{q}_\alpha)\leq\limsup_{\alpha\to\infty}J_\alpha(\overline{g},\overline{q})=J(\overline{g},\overline{q}).$$

Thus $\lim_{\alpha\to\infty}J_\alpha(\overline{g}_\alpha,\overline{q}_\alpha)=J(\overline{g},\overline{q})$ or equivalently

$$\begin{aligned}&\lim_{\alpha\to\infty}\left(\frac{1}{2}||u_{\alpha\overline{g}_\alpha\overline{q}_\alpha}-z_d||_H^2+\frac{M_1}{2}||\overline{g}_\alpha||_H^2+\frac{M_2}{2}||\overline{q}_\alpha||_Q^2\right)\\&\quad=\frac{1}{2}||u_{\infty\overline{g}\,\overline{q}}-z_d||_H^2+\frac{M_1}{2}||\overline{g}||_H^2+\frac{M_2}{2}||\overline{q}||_Q^2.\end{aligned}\tag{5.10}$$

Now, from (5.8) and (5.10), we have, as $\alpha\to\infty$,

$$||\overline{g}_\alpha||_H^2\to||\overline{g}||_H^2\quad\text{and}\quad||\overline{q}_\alpha||_Q^2\to||\overline{q}||_Q^2$$

and as $\overline{g}_\alpha\rightharpoonup\overline{g}$ weakly in H and $\overline{q}_\alpha\rightharpoonup\overline{q}$ weakly in Q, we deduce that $\overline{g}_\alpha\to\overline{g}$ strongly in H and $\overline{q}_\alpha\to\overline{q}$ strongly in Q. This completes the proof. □

6. Conclusions

We studied two optimal control problems, with a parameter, for the systems governed by elliptic boundary hemivariational inequalities with a non-monotone multivalued subdifferential boundary condition on a portion of the boundary of the domain, which is described by the Clarke generalized gradient of a locally Lipschitz function. The first optimal control problem corresponds to a family of boundary optimal control problems and the second one corresponds to a family of distributed-boundary optimal control problems. We proved an existence result for the two optimal control problems and demonstrated an asymptotic result for the optimal controls and the system states, when the parameter (the heat transfer coefficient on a portion of the boundary) tends to infinity. These results generalize for a locally Lipschitz function j, under the hypothesis $H(j)$ and (H_1) and the classical results obtained in [26, 30] for a quadratic superpotential j. We remark that, by using the hypothesis (H_1), the asymptotic behavior of the systems for both optimal control problems were obtained without any previous knowledgement on the monotonicity on the parameter like in [13, 14].

Acknowledgments
The present work was partially sponsored by the European Union's Horizon 2020 Research and Innovation Programme under the Marie Sklodowska-Curie grant agreement 823731 CONMECH and by the Project PIP No. 0275 from CONICET and Universidad Austral, Rosario, Argentina for the third author, and by the Project PPI No. 18/C555 from SECyT-UNRC, Río Cuarto, Argentina for the first and second authors.

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