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# Neumann boundary optimal control problems governed by parabolic variational equalities*

by

**Carolina M. Bollo[1], Claudia M. Gariboldi[1] and Domingo A. Tarzia[2]**

[1]Depto. Matemática, FCEFQyN, Universidad Nac. de Río Cuarto,
Ruta 36 Km 601, 5800 Río Cuarto, Argentina
cbollo@exa.unrc.edu.ar; cgariboldi@exa.unrc.edu.ar
[2]Depto. de Matemática-CONICET, FCE, Universidad Austral,
Paraguay 1950, S2000FZF Rosario, Argentina
DTarzia@austral.edu.ar

**Abstract:** We consider a heat conduction problem $S$ with mixed boundary conditions in an $n$-dimensional domain $\Omega$ with regular boundary and a family of problems $S_\alpha$ with also mixed boundary conditions in $\Omega$, where $\alpha > 0$ is the heat transfer coefficient on the portion of the boundary $\Gamma_1$. In relation to these state systems, we formulate *Neumann boundary* optimal control problems on the heat flux $q$ which is definite on the complementary portion $\Gamma_2$ of the boundary of $\Omega$. We obtain existence and uniqueness of the optimal controls, the first order optimality conditions in terms of the adjoint state and the convergence of the optimal controls, the system state and the adjoint state when the heat transfer coefficient $\alpha$ goes to infinity. Furthermore, we formulate particular *boundary* optimal control problems on a real parameter $\lambda$, in relation to the parabolic problems $S$ and $S_\alpha$ and to mixed elliptic problems $P$ and $P_\alpha$. We find a explicit form for the optimal controls, we prove monotony properties and we obtain convergence results when the parameter time goes to infinity.

**Keywords:** parabolic variational equalities, optimal control, mixed boundary conditions, optimality conditions, convergence

## 1. Introduction

Following Gariboldi and Tarzia(2008), Menaldi and Tarzia 9 2007), and Tarzia, Bollo and Gariboldi (2020), we will study some Neumann boundary parabolic and elliptic optimal control problems. We consider a bounded domain $\Omega$ in $\mathbb{R}^n$, whose regular boundary $\Gamma$ consists of the union of the two disjoint portions $\Gamma_1$ and $\Gamma_2$ with $|\Gamma_1| > 0$ and $|\Gamma_2| > 0$. We denote with $|\Gamma_i| = meas(\Gamma_i)$ (for $i = 1, 2$), the $(n-1)$-dimensional Hausdorff measure of the portion $\Gamma_i$ on

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$\Gamma$. Let $[0,T]$ be a time interval, for a $T>0$. We present the following heat conduction problems $S$ and $S_\alpha$ (for each parameter $\alpha>0$) respectively, with mixed boundary conditions (we denote by $u(t)$ to the function $u(\cdot,t)$):

$$\frac{\partial u}{\partial t}-\Delta u=g \quad \text{in } \Omega \qquad u\big|_{\Gamma_1}=b \qquad -\left.\frac{\partial u}{\partial n}\right|_{\Gamma_2}=q \qquad u(0)=v_b \tag{1}$$

$$\frac{\partial u}{\partial t}-\Delta u=g \quad \text{in } \Omega \qquad -\left.\frac{\partial u}{\partial n}\right|_{\Gamma_1}=\alpha(u-b) \qquad -\left.\frac{\partial u}{\partial n}\right|_{\Gamma_2}=q \qquad u(0)=v_b, \tag{2}$$

where $u$ is the temperature in $\Omega\times(0,T)$, $g$ is the internal energy in $\Omega$, $b$ is the temperature on $\Gamma_1$ for (1) and the temperature of the external neighborhood of $\Gamma_1$ for (2), $v_b=b$ on $\Gamma_1$, $q$ is the heat flux on $\Gamma_2$, and $\alpha>0$ is the heat transfer coefficient on $\Gamma_1$ through a Robin condition, which satisfy the hypothesis: $g\in\mathcal{H}=L^2(0,T;L^2(\Omega))$, $q\in\mathcal{Q}=L^2(0,T;L^2(\Gamma_2))$ and $b\in H^{\frac{1}{2}}(\Gamma_1)$. In addition, $v_b\in H^1(\Omega)$ is the initial temperature for (1) and (2), respectively.

Let $u$ and $u_\alpha$ be the unique solutions to the parabolic problems (1) and (2), whose variational formulations are given by (Menaldi and Tarzia, 2007):

$$\begin{cases} u-v_b\in L^2(0,T;V_0), \qquad u(0)=v_b \quad \text{and} \quad \dot{u}\in L^2(0,T;V_0') \\ \text{such that} \quad \langle \dot{u}(t),v\rangle + a(u(t),v)=L(t,v), \quad \forall v\in V_0, \end{cases} \tag{3}$$

$$\begin{cases} u_\alpha\in L^2(0,T;V), \qquad u_\alpha(0)=v_b \quad \text{and} \quad \dot{u}_\alpha\in L^2(0,T;V') \\ \text{such that} \quad \langle \dot{u}_\alpha(t),v\rangle + a_\alpha(u_\alpha(t),v)=L_\alpha(t,v), \quad \forall v\in V, \end{cases} \tag{4}$$

where $\langle\cdot,\cdot\rangle$ denotes the duality between the functional space ($V$ or $V_0$) and its dual space ($V'$ or $V_0'$) and

$$V=H^1(\Omega); \qquad V_0=\{v\in V:\, v\big|_{\Gamma_1}=0\}; \qquad Q=L^2(\Gamma_2); \quad H=L^2(\Omega)$$
$$(g,h)_H=\int_\Omega gh\,dx; \quad (q,\eta)_Q=\int_{\Gamma_2} q\eta\,d\gamma;$$
$$a(u,v)=\int_\Omega \nabla u\nabla v dx; \quad a_\alpha(u,v)=a(u,v)+\alpha\int_{\Gamma_1} uv d\gamma$$
$$L(t,v)=(g(t),v)_H-(q(t),v)_Q; \quad L_\alpha(t,v)=L(t,v)+\alpha\int_{\Gamma_1} bv d\gamma.$$

All data, $g$, $q$, $b$, $v_b$ and the domain $\Omega$ with the boundary $\partial\Omega=\Gamma_1\cup\Gamma_2$ are assumed to be sufficiently smooth so that the problems (1) and (2) admit variational solutions in Sobolev spaces. The existence and uniqueness of the solutions to the variational equalities (3) and (4), are well known, see, for example Brézis (1972), Chrysafinos and Hou (2017), Duvant and Lions (1972),or Gunzburger and Hou (1992)..

Let $\mathcal{H}=L^2(0,T;H)$ be, with norm $||.||_{\mathcal{H}}$ and internal product $(g,h)_{\mathcal{H}}=\int_0^T (g(t),h(t))_H dt$, and the space $\mathcal{Q}=L^2(0,T;Q)$, with norm $||.||_{\mathcal{Q}}$ and internal

product $(q,\eta)_{\mathcal{Q}} = \int_0^T (q(t),\eta(t))_Q dt$.
For the sake of simplicity, for a Banach space $X$ and $1 \leq p \leq \infty$, we will often use $L^p(X)$ instead of $L^p(0,T;X)$.

If we denote by $u_q$ and $u_{\alpha q}$ the unique solution to the problems (3) and (4), respectively, we formulate the following *boundary* optimal control problems for the heat flux $q$ as control variable, Gariboldi and Tarzia (2008), Lions (1968), Tröltzsch (2010):

$$\text{find} \quad \overline{q} \in \mathcal{Q} \quad \text{such that} \quad J(\overline{q}) = \min_{q\in\mathcal{Q}} J(q), \tag{5}$$

$$\text{find} \quad \overline{q}_\alpha \in \mathcal{Q} \quad \text{such that} \quad J_\alpha(\overline{q}_\alpha) = \min_{q\in\mathcal{Q}} J_\alpha(q), \tag{6}$$

where the cost functionals $J : \mathcal{Q}\to\mathbb{R}_0^+$ and $J_\alpha : \mathcal{Q}\to\mathbb{R}_0^+$ are given by:

$$\text{i) } J(q) = \frac{1}{2}\|u_q - z_d\|^2_{\mathcal{H}} + \frac{M}{2}\|q\|^2_{\mathcal{Q}}\,, \text{ ii) } J_\alpha(q) = \frac{1}{2}\|u_{\alpha q} - z_d\|^2_{\mathcal{H}} + \frac{M}{2}\|q\|^2_{\mathcal{Q}} \tag{7}$$

with $z_d \in \mathcal{H}$ given and $M$ being a positive constant.

In Garibldi and Tarzia (2008), the authors studied boundary optimal control problems on the heat flux $q$ in mixed elliptic problems and they proved existence, uniqueness and asymptotic behavior of the optimal solutions, when the heat transfer coefficient goes to infinity. Similar results were obtained in Gariboldi and Tarzia (2015) for simultaneous distributed-boundary optimal control problems on the internal energy $g$ and the heat flux $q$ in mixed elliptic problems. In Menaldi and Tarzia (2007), convergence results were proved for heat conduction problems in relation to distributed optimal control problems on the internal energy $g$ as a control variable. Parabolic control problem with Robin boundary conditions are considered in Bergounioux and Tröltzsch (1999), Boukrouche and Tarzia (2013), Chrysafinos, Gunzburger and Hou (2006), Gariboldi and Tarzia (2008), and Menaldi and Tarzia (2007). Other papers on the subject are Ben Belgacem, El Fekih and Raymond (2003), Sener and Subasi (2015), Sweilam and Abd-Elal (2003), Wang and Yan (2019). In this paper, our main goal is to study the existence and uniqueness of solutions and the asymptotic behaviour of the optimal control problems (5) and (6), when $\alpha \to \infty$. Moreover, motivated by Gonzalez and Tarzia (1990) we try find explicit solutions for the optimal controls and a relationship between elliptic and parabolic boundary optimal control problems, when the time goes to infinity. In this way, we consider the following family of optimization problems on the heat flux dependent of a real parameter: For fixed $q_0 \in \mathcal{Q}$, we define $\mathcal{Q}_0 = \{\lambda q_0 : \lambda \in \mathbb{R}\} \subset \mathcal{Q}$, and we formulate the following *real Neumann parabolic boundary* optimal control problems, for each $T > 0$ and $\alpha > 0$:

$$\text{find} \quad \overline{\lambda}(T) \in \mathbb{R} \quad \text{such that} \quad H_T(\overline{\lambda}(T)) = \min_{\lambda\in\mathbb{R}} H_T(\lambda), \tag{8}$$

$$\text{find} \quad \overline{\lambda}_\alpha(T) \in \mathbb{R} \quad \text{such that} \quad H_{\alpha T}(\overline{\lambda}_\alpha(T)) = \min_{\lambda\in\mathbb{R}} H_{\alpha T}(\lambda), \tag{9}$$

where

$$H_T(\lambda) = J(\lambda q_0) \quad \text{and} \quad H_{\alpha T}(\lambda) = J_\alpha(\lambda q_0). \tag{10}$$

Moreover, we consider the elliptic mixed problems $P$ and $P_\alpha$, for each $\alpha > 0$, Gariboldi and Tarzia (2008, 2015):

$$-\Delta u = g \ \text{ in } \Omega \qquad u\big|_{\Gamma_1} = b \qquad -\left.\frac{\partial u}{\partial n}\right|_{\Gamma_2} = q \tag{11}$$

$$-\Delta u = g \ \text{ in } \Omega \qquad -\left.\frac{\partial u}{\partial n}\right|_{\Gamma_1} = \alpha(u-b) \qquad -\left.\frac{\partial u}{\partial n}\right|_{\Gamma_2} = q, \tag{12}$$

whose variational equalities are given by

$$a(u,v) = L(v), \quad \forall v \in V_0, \ u \in K \tag{13}$$

$$a_\alpha(u_\alpha, v) = L_\alpha(v), \quad \forall v \in V, \ u_\alpha \in V \tag{14}$$

with $K = v_0 + V_0$ for a given $v_0 = b$ in $\Gamma_1$. For fixed $q_0^* \in Q$, we define $Q_0 = \{\lambda q_0^* : \lambda \in \mathbb{R}\} \subset Q$, and we formulate the following *real Neumann elliptic boundary* optimal control problems, for each $\alpha > 0$:

$$\text{find} \quad \overline{\lambda} \in \mathbb{R} \quad \text{such that} \quad H(\overline{\lambda}) = \min_{\lambda \in \mathbb{R}} H(\lambda), \tag{15}$$

$$\text{find} \quad \overline{\lambda}_\alpha \in \mathbb{R} \quad \text{such that} \quad H_\alpha(\overline{\lambda}_\alpha) = \min_{\lambda \in \mathbb{R}} H_\alpha(\lambda), \tag{16}$$

where

$$H(\lambda) = J^*(\lambda q_0^*) \quad \text{and} \quad H_\alpha(\lambda) = J_\alpha^*(\lambda q_0^*), \tag{17}$$

with $J^* : Q \to \mathbb{R}_0^+$ and $J_\alpha^* : Q \to \mathbb{R}_0^+$ given by Gariboldi and Tarzia (2008):

$$\text{i) } J^*(q) = \frac{1}{2}\|u_{\infty q} - z_d\|_H^2 + \frac{M}{2}\|q\|_Q^2 \text{ , ii) } J_\alpha^*(q) = \frac{1}{2}\|u_{\infty \alpha q} - z_d\|_H^2 + \frac{M}{2}\|q\|_Q^2 \tag{18}$$

where $u_{\infty q}$ and $u_{\infty \alpha q}$ are the unique solutions to the variational equalities (13) and (14), respectively, $z_d \in H$ is given and $M$ is a positive constant.

The paper is structured as follows. In Section 2, we consider Neumann boundary optimal control problems on the heat flux $q$ for heat conduction problems (1), (5) and (7i) and Neumann parabolic boundary optimal control problems on the heat flux $q$ for (2), (6) and (7ii), for each $\alpha > 0$. We prove existence and uniqueness of the optimal controls and we give the first order optimality conditions. In Section 3, for fixed $q$, we prove asymptotic estimates and convergence results for the system states, the adjoint states and the optimal controls, when the heat transfer coefficient goes to infinity. In Section 4, we prove estimates between the optimal controls of the problems (5) and (6) and the second component of the simultaneous optimal controls of the problems

studied in Tarzia, Bollo and Gariboldi (2020). In Section 5, for the real Neumann parabolic boundary optimal control problems (8), (9), (15) and (16) we prove the existence and uniqueness and we find explicit solutions for the optimal control $\overline{\lambda}(t)$, $\overline{\lambda_\alpha}(t)$, $\overline{\lambda}$ and $\overline{\lambda_\alpha}$, respectively. Moreover, monotonicity properties with respect to the data are also studied. Finally, in Section 6, convergence results of the solutions to the problems (3) to the solution to the problem (13) are obtained, when the parameter time $t \to \infty$.

## 2. Boundary optimal control problems for systems S and $\mathbf{S}_\alpha$

Here, we prove that the functionals $J$ and $J_\alpha$ are strictly convex and Gâteaux differentiable in $\mathcal{Q}$. Moreover, we obtain the existence and uniqueness of the boundary optimal controls $\overline{q}$ and $\overline{q}_\alpha$ and we give the optimality conditions in terms of the adjoint states, for the optimal control problems (5) and (6), respectively.

Following , Lions (1968),Menaldi and Tarzia(2007)and Tröltzsch (2010), we define the application $C : \mathcal{Q} \to L^2(V_0)$ such that $C(q) = u_q - u_0$, where $u_0$ is the solution of problem (3) for $q = 0$.

We consider $\Pi : \mathcal{Q} \times \mathcal{Q} \to \mathbb{R}$ and $\mathcal{L} : \mathcal{Q} \to \mathbb{R}$, defined by the expressions

$$\Pi(q,\eta) = (C(q), C(\eta))_{\mathcal{H}} + M(q,\eta)_{\mathcal{Q}} \quad \forall q, \eta \in \mathcal{Q}$$

$$\mathcal{L}(q) = (C(q), z_d - u_0)_{\mathcal{H}} \quad \forall q \in \mathcal{Q}$$

and we prove the following result

Lemma 1 *i) $C$ is a linear and continuous functional.*
*ii) $\Pi$ is a bilinear, symmetric, continuous form and coercive in $\mathcal{Q}$.*
*iii) $\mathcal{L}$ is linear and continuous functional in $\mathcal{Q}$.*
*iv) $J$ can be written down as:*

$$J(q) = \frac{1}{2}\Pi(q,q) - \mathcal{L}(q) + \frac{1}{2}||u_0 - z_d||^2_{\mathcal{H}} \quad \forall q \in \mathcal{Q}.$$

*v) $J$ is a strictly convex functional on $\mathcal{Q}$, that is, $\forall q_1, q_2 \in \mathcal{Q}$, $\forall t \in [0,1]$*

$$(1-t)J(q_2) + tJ(q_1) - J((1-t)q_2 + tq_1) \geq \frac{Mt(1-t)}{2}||q_2 - q_1||^2_{\mathcal{Q}}.$$

*vi) There exists a unique optimal control $\overline{q} \in \mathcal{Q}$ such that*

$$J(\overline{q}) = \min_{q \in \mathcal{Q}} J(q).$$

Proof It follows from Lions (1968) and Menaldi and Tarzia (2007), and

$$(1-t)J(q_2) + tJ(q_1) - J((1-t)q_2 + tq_1) =$$
$$\frac{t(1-t)}{2}\left[||u_{q_2} - u_{q_1}||^2_{\mathcal{H}} + M||q_2 - q_1||^2_{\mathcal{Q}}\right],$$
$$\forall q_1, q_2 \in \mathcal{Q}, \ \forall \sqcup \in [\prime, \infty] \qquad \square.$$

Now, we define the adjoint state $p_q$, corresponding to the system (1) for each $q \in \mathcal{Q}$, as the unique solution of the following mixed parabolic problem:

$$-\frac{\partial p_q}{\partial t} - \Delta p_q = u_q - z_d \text{ in } \Omega, \qquad p_q\big|_{\Gamma_1} = 0, \qquad \left.\frac{\partial p_q}{\partial n}\right|_{\Gamma_2} = 0, \qquad p_q(T) = 0,$$

whose variational formulation is given by

$$\begin{cases} p_q \in L^2(V_0),\ p_q(T) = 0 \quad \text{and } \dot{p}_q \in L^2(V_0') \text{ such that} \\ -\langle \dot{p}_q(t), v\rangle + a(p_q(t), v) = (u_q(t) - z_d(t), v)_H, \quad \forall v \in V_0, \end{cases} \tag{19}$$

and we consider the following properties of the functional $J$, following Lions (1968), Menaldi and Tarzia ( 2007) and Tarzia, Bollo and Gariboldi (2020).

Lemma 2 *i) The adjoint state $p_q$ satisfies:*

$$(C(\eta), u_q - z_d)_{\mathcal{H}} = -(\eta, p_q)_{\mathcal{Q}}, \quad \forall q, \eta \in \mathcal{Q}.$$

*ii) The functional $J$ is Gâteaux differentiable and $J'$ is given by:*

$$\begin{aligned} \langle J'(q), \eta - q\rangle &= (u_\eta - u_q, u_q - z_d)_{\mathcal{H}} + M(q, \eta - q)_{\mathcal{Q}} \\ &= \Pi(q, \eta - q) - \mathcal{L}(\eta - q), \quad \forall q, \eta \in \mathcal{Q}. \end{aligned}$$

*iii) The Gâteaux derivative of $J$ can be written as:*

$$J'(q) = Mq - p_q \quad \forall q \in \mathcal{Q}.$$

*iv) The optimality condition for the optimal control problem (5) is given by*

$$M\overline{q} - p_{\overline{q}} = 0 \quad \text{in } \mathcal{Q}.$$

Next, we define the application $C_\alpha : \mathcal{Q} \to L^2(V)$ such that $C_\alpha(q) = u_{\alpha q} - u_{\alpha 0}$, where $u_{\alpha 0}$ is the solution of the variational problem (4) for $q = 0$.

If we consider $\Pi_\alpha : \mathcal{Q} \times \mathcal{Q} \to \mathbb{R}$ and $\mathcal{L}_\alpha : \mathcal{Q} \to \mathbb{R}$, defined by

$$\Pi_\alpha(q, \eta) = (C_\alpha(q), C_\alpha(\eta))_{\mathcal{H}} + M(q, \eta)_{\mathcal{Q}} \quad \forall q, \eta \in \mathcal{Q}$$

$$\mathcal{L}_\alpha(q) = (C_\alpha(q), z_d - u_{\alpha 0})_{\mathcal{H}} \quad \forall q \in \mathcal{Q}$$

in a similar way to Lemma 1 and 1, we have the following result:

Lemma 3 *(i) There exists a unique optimal control $\overline{q}_\alpha \in \mathcal{Q}$ such that*

$$J_\alpha(\overline{q}_\alpha) = \min_{q \in \mathcal{Q}} J_\alpha(q).$$

*(ii) The Gateaux derivative of $J_\alpha$ can be written as:*

$$J'_\alpha(q) = Mq - p_{\alpha q} \quad \forall q \in \mathcal{Q}, \tag{20}$$

*and the optimality condition for the optimal control problem (6) is given by:*

$$M\overline{q}_\alpha - p_{\alpha\overline{q}_\alpha} = 0 \quad \text{in } \mathcal{Q} \tag{21}$$

*where the adjoint state $p_{\alpha q}$ corresponds to (2) for each $q \in \mathcal{Q}$, as the unique solution of*

$$-\frac{\partial p_{\alpha q}}{\partial t} - \Delta p_{\alpha q} = u_{\alpha q} - z_d \text{ in } \Omega, \ -\frac{\partial p_{\alpha q}}{\partial n}\Big|_{\Gamma_1} = \alpha p_{\alpha q}, \ \frac{\partial p_{\alpha q}}{\partial n}\Big|_{\Gamma_2} = 0, \ p_{\alpha q}(T) = 0,$$

*whose variational formulation is given by*

$$\begin{cases} p_{\alpha q} \in L^2(V), \ p_{\alpha q}(T) = 0 \quad \text{and } \dot{p}_{\alpha q} \in L^2(V') \text{ such that} \\ -\langle \dot{p}_{\alpha q}(t), v\rangle + a_\alpha(p_{\alpha q}(t), v) = (u_{\alpha q}(t) - z_d(t), v)_H, \ \forall v \in V \end{cases} \tag{22}$$

*for each $\alpha > 0$.*

## 3. Convergence of Neumann boundary optimal control problems when $\alpha \to \infty$

Now, for fixed $q \in Q$, we obtain estimates on $u_{\alpha q}$ and $p_{\alpha q}$ uniformly when $\alpha > 1$. Next, we prove strong convergence for $q_\alpha$, $u_{\alpha q}$ and $p_{\alpha q}$, when $\alpha$ goes to infinity.

PROPOSITION 1 *(i) If $u_q$ and $u_{\alpha q}$ are the unique solutions to the variational equalities (3) and (4), respectively, we have the estimation*

$$||\dot{u}_{\alpha q}||_{L^2(V_0')} + ||u_{\alpha q}||_{L^\infty(H)} + ||u_{\alpha q}||_{L^2(V)} + \sqrt{(\alpha-1)}||u_{\alpha q} - b||_{L^\infty(L^2(\Gamma_1))} \leq C \tag{23}$$

*for all $\alpha > 1$, where the constant $C$ depends only on the norms $||\dot{u}_q||_{L^2(V_0')}$, $||\dot{u}_q||_{L^2(V')}$, $||\nabla u_q||_{\mathcal{H}}$, $||u_q||_{L^2(V)}$, $||u_q||_{L^\infty(H)}$, $||g||_{\mathcal{H}}$, $||q||_{\mathcal{Q}}$ and the coerciveness constant $\lambda_1$ of the bilinear form $a_1$.*

*(ii) For fixed $q \in \mathcal{Q}$ we have $u_{\alpha q} \to u_q$ strongly in $L^2(V) \cap L^\infty(H)$ and $\dot{u}_{\alpha q} \to \dot{u}_q$ strongly in $L^2(V_0')$, when $\alpha \to \infty$.*

PROOF Taking $v = u_{\alpha q}(t) - u_q(t) \in V$ in the variational equation (4), taking into account that $u_q(t)\big|_{\Gamma_1} = b$, by using Young's inequality and integrating between $[0, T]$, we obtain

$$\begin{aligned} &\frac{1}{2}||u_{\alpha q}(T) - u_q(T)||_H^2 + \frac{\lambda_1}{2}||u_{\alpha q} - u_q||_{L^2(V)}^2 + (\alpha-1)||u_{\alpha q} - b||_{L^2(L^2(\Gamma_1))}^2 \\ &\leq \frac{2}{\lambda_1}[||g||_{\mathcal{H}}^2 + \|\gamma_0\|^2||q||_{\mathcal{Q}}^2 + ||\nabla u_q||_{\mathcal{H}}^2 + ||\dot{u}_q||_{L^2(V')}^2], \end{aligned}$$

where $\gamma_0$ is the trace operator on $\Gamma$.

Here, we prove that there exists a positive constant $K$, independent of $\alpha$, which depends on

$$K = K(\lambda_1, ||u_q||_{L^\infty(H)}, ||u_q||_{L^2(V)}, ||g||_{\mathcal{H}}||q||_{\mathcal{Q}}||\nabla u_q||_{\mathcal{H}}||\dot{u}_q||_{L^2(V')})$$

such that for all $\alpha > 1$, we have:

$$||u_{\alpha q}||_{L^\infty(H)} + ||u_{\alpha q}||_{L^2(V)} + \sqrt{(\alpha-1)}||u_{\alpha q} - b||_{L^2(L^2(\Gamma_1))} \leq K. \qquad (24)$$

Next, by taking $v \in V_0$ in the variational equality (4) and subtracting the variational equality (3), we have

$$(\dot{u}_{\alpha q}(t) - \dot{u}_q(t), v)_H \leq ||u_q(t) - u_{\alpha q}(t)||_V ||v||_{V_0} \quad \forall v \in V_0,$$

and integrating in $[0, T]$, we obtain

$$||\dot{u}_{\alpha q} - \dot{u}_q||_{L^2(V_0')} \leq ||u_q - u_{\alpha q}||_{L^2(V)}.$$

Next, by using (24), we conclude that there exists a positive constant $C = C(K, ||\dot{u}_q||_{L^2(V_0')})$ such that (23) holds.

(ii) Let there be a fixed $q \in \mathcal{Q}$, we consider a sequence $\{u_{\alpha_n q}\}$ in $L^2(V) \cap L^\infty(H)$ and by estimation (23), we have that $||u_{\alpha_n q}||_{L^2(V)} \leq C$ and $||\dot{u}_{\alpha_n q}||_{L^2(V_0')} \leq C$, therefore, there exists a subsequence $\{u_{\alpha_n q}\}$ which is weakly convergent to $w_q \in L^2(V)$ and weakly* in $L^\infty(H)$ and there exists a subsequence $\{\dot{u}_{\alpha_n q}\}$ which is weakly convergent to $\dot{w}_q \in L^2(V_0')$. Now, from the third term of left hand side of (24) and the weak lower semicontinuity of the norm in $L^2(L^2(\Gamma_1))$, we have that $w_q = b$ on $\Gamma_1$ and therefore $w_q - v_b \in L^2(V_0)$. Next, we prove that $w_q$ satisfies

$$\langle \dot{w}_q(t), v\rangle + a(w_q(t), v) = L(t, v), \quad \forall v \in V_0$$

and $w_q(0) = v_b$ with $\dot{w}_q \in L^2(V_0')$.
Therefore, by uniqueness of the solution of the variational problem (3), we obtain $w_q = u_q$. That is, when $\alpha \to \infty$ we have

$$u_{\alpha q} \rightharpoonup u_q \text{ in } L^2(V), \quad u_{\alpha q} \overset{*}{\rightharpoonup} u_q \text{ in } L^\infty(H) \quad \text{and} \quad \dot{u}_{\alpha q} \rightharpoonup \dot{u}_q \text{ in } L^2(V_0').$$

Now, we have

$$\begin{aligned} &\frac{1}{2}||u_{\alpha q}(T) - u_q(T)||_H^2 + \lambda_1 ||u_{\alpha q} - u_q||_{L^2(V)}^2 + (\alpha - 1)||u_{\alpha q} - u_q||_{L^2(L^2(\Gamma_1))}^2 \\ &\leq \int_0^T \{L(t, u_{\alpha q}(t) - u_q(t)) - a(u_q(t), u_{\alpha q}(t) - u_q(t)) - \langle \dot{u}_q(t), u_{\alpha q}(t) - u_q(t)\rangle\} dt \end{aligned}$$

and by using the weak convergence of $u_{\alpha q}$ to $u_q$, we prove the strong convergence in $L^2(V)$. Next, taking into account that

$$\begin{aligned} ||u_{\alpha q} - u_q||_{L^2(L^2(\Gamma_1))}^2 \leq \frac{1}{\alpha - 1} \int_0^T &\{L(t, u_{\alpha q}(t) - u_q(t)) - a(u_q(t), u_{\alpha q}(t) - u_q(t)) \\ &- \langle \dot{u}_q(t), u_{\alpha q}(t) - u_q(t)\rangle\} dt \end{aligned}$$

and the weak convergence of $u_{\alpha q}$ to $u_q$, we prove the strong convergence in $L^2(L^2(\Gamma_1))$. Now, from the variational equalities (3) and (4), we have

$$||\dot{u}_{\alpha q} - \dot{u}_q||_{L^2(V_0')}^2 \leq ||u_q - u_{\alpha q}||_{L^2(V)}^2 \to 0, \quad \text{when} \quad \alpha \to \infty.$$

We deduce that $\dot{u}_{\alpha q}$ is strongly convergent to $\dot{u}_q$ en $L^2(V_0')$. Finally, we have

$$||u_{\alpha q} - u_q||^2_{L^\infty(H)} \leq 2(||g||_{\mathcal{H}} + \|\gamma_0\|||q||_{\mathcal{Q}} + ||u_q||_{L^2(V_0)} + ||\dot{u}_q||_{\mathcal{H}})||u_{\alpha q} - u_q||_{L^2(V)}$$

and from the strong convergence of $u_{\alpha q}$ to $u_q$ in $L^2(V)$, we prove that $u_{\alpha q}$ is strongly convergent to $u_q$ in $L^\infty(H)$, when $\alpha \to \infty$. □

Proposition 2 *(i) If $p_q$ and $p_{\alpha q}$ are the unique solutions to the variational equalities (19) and (22), respectively, we have the estimation*

$$||\dot{p}_{\alpha q}||_{L^2(V_0')} + ||p_{\alpha q}||_{L^\infty(H)} + ||p_{\alpha q}||_{L^2(V)} + \sqrt{(\alpha - 1)}||p_{\alpha q}||_{L^2(L^2(\Gamma_1))} \leq C \quad (25)$$

*for all $\alpha > 1$, where the constant $C$ depends on the norms $||\dot{p}_q||_{L^2(V_0')}$, $||\dot{p}_q||_{L^2(V')}$, $||\nabla p_q||_{\mathcal{H}}$, $||p_q||_{L^2(V)}$, $||p_q||_{L^\infty(H)}$, $||g||_{\mathcal{H}}$, $||q||_{\mathcal{Q}}$, $||z_d||_{\mathcal{H}}$, $||\dot{u}_q||_{L^2(V')}$, $||\nabla u_q||_{\mathcal{H}}$, $||u_q||_{L^2(V)}$, $||u_q||_{L^\infty(H)}$ and of the coerciveness constant $\lambda_1$.*

*(ii) For fixed $q \in \mathcal{Q}$, we have that $p_{\alpha q} \to p_q$ strongly in $L^2(V) \cap L^\infty(H)$ and $\dot{p}_{\alpha q} \to \dot{p}_q$ strongly in $L^2(V_0')$, when $\alpha \to \infty$.*

Proof Let there be fixed $q \in \mathcal{Q}$, the estimation (25) follows from analogous reasoning to Proposition 1. We have that there exists a subsequence $\{p_{\alpha_n q}\}$, which is weakly convergent to $\eta_q \in L^2(V)$ and weakly* in $L^\infty(H)$. From the weak semicontinuity of the norm, we have that $\eta_q = 0$ on $\Gamma_1$ and therefore $\eta_q \in L^2(V_0)$. Moreover, $\eta_q$ satisfies

$$-\langle \dot{\eta}_q(t), v\rangle + a(\eta_q(t), v) = (u_q(t) - z_d(t), v)_H \quad \forall v \in V_0$$

and $\eta_q(T) = 0$ with $\dot{\eta}_q \in L^2(V_0')$. Therefore, by the uniqueness of the solution of the variational problem (19), we obtain $\eta_q = p_q$ and when $\alpha \to \infty$, we have

$$p_{\alpha q} \rightharpoonup p_q \text{ in } L^2(V), \quad p_{\alpha q} \overset{*}{\rightharpoonup} p_q \text{ in } L^\infty(H) \quad \text{and} \quad \dot{p}_{\alpha q} \rightharpoonup \dot{p}_q \text{ in } L^2(V_0').$$

Finally, the strong convergence of $p_{\alpha q}$ to $p_q$ in $L^2(V) \cap L^\infty(H)$ and of $\dot{p}_{\alpha q}$ to $\dot{p}_q$ in norm $L^2(V_0')$ is obtained in a similar way to that in Proposition 1. □

Now, we consider the boundary optimal control problems (5) and (6) and our goal is to prove the following theorem:

Theorem 1 *Let $\overline{q}$ and $\overline{q}_\alpha$ be the unique solutions of the optimal control problems (5) and (6), respectively. Then, we have that $\overline{q}_\alpha \to \overline{q}$ strongly in $\mathcal{Q}$, when the parameter $\alpha \to \infty$. Moreover, the system state and the adjoint state satisfy $(u_{\alpha\overline{q}_\alpha} \dot{u}_{\alpha\overline{q}_\alpha}) \to (u_{\overline{q}}\dot{u}_{\overline{q}})$ and $(p_{\alpha\overline{q}_\alpha}\dot{p}_{\alpha\overline{q}_\alpha}) \to (p_{\overline{q}}\dot{p}_{\overline{q}})$ strongly in $L^2(V) \times L^2(V_0')$.*

Proof We will establish the proof in three steps.

**Step 1.** From the estimation (23) for $q = 0$, there exists a constant $C_1 > 0$ such that

$$||u_{\alpha 0}||_{\mathcal{H}} \leq ||u_{\alpha 0}||_{L^2(V)} \leq C_1, \ \forall \alpha > 1,$$

and from $J_\alpha(\overline{q}_\alpha) \leq J_\alpha(0)$, we have

$$\frac{1}{2}||u_{\alpha\overline{q}_\alpha} - z_d||^2_{\mathcal{H}} + \frac{M}{2}||\overline{q}_\alpha||^2_{\mathcal{Q}} \leq \frac{1}{2}||u_{\alpha 0} - z_d||^2_{\mathcal{H}}.$$

Therefore, there exist positive constants $C_2$ and $C_3$ such that

$$||u_{\alpha\overline{q}_\alpha}||_{\mathcal{H}} \leq C_2 \quad \text{and} \quad ||\overline{q}_\alpha||_{\mathcal{Q}} \leq C_3, \quad \forall \alpha > 1.$$

Now, by an analogous reasoning to that for the estimates (23) and (25), there exist positive constants $C_4$ and $C_5$ such that, for all $\alpha > 1$, we obtain

$$||u_{\alpha\overline{q}_\alpha}||_{L^2(V)} + ||\dot{u}_{\alpha\overline{q}_\alpha}||_{L^2(V_0')} + \sqrt{(\alpha-1)}||u_{\alpha\overline{q}_\alpha} - b||_{L^2(L^2(\Gamma_1))} \leq C_4$$

$$||p_{\alpha\overline{q}_\alpha}||_{L^2(V)} + ||\dot{p}_{\alpha\overline{q}_\alpha}||_{L^2(V_0')} + \sqrt{(\alpha-1)}||p_{\alpha\overline{q}_\alpha}||_{L^2(L^2(\Gamma_1))} \leq C_5.$$

From the previous estimations, we have that there exist $f \in \mathcal{Q}$, $\mu \in L^2(V)$, $\dot{\mu} \in L^2(V_0')$, $\rho \in L^2(V)$ and $\dot{\rho} \in L^2(V_0')$ such that

$$\overline{q}_\alpha \rightharpoonup f \in \mathcal{Q}, \quad u_{\alpha\overline{q}_\alpha} \rightharpoonup \mu \in L^2(V), \quad \dot{u}_{\alpha\overline{q}_\alpha} \rightharpoonup \dot{\mu} \in L^2(V_0'),$$

$$p_{\alpha\overline{q}_\alpha} \rightharpoonup \rho \in L^2(V), \quad \dot{p}_{\alpha\overline{q}_\alpha} \rightharpoonup \dot{\rho} \in L^2(V_0').$$

**Step 2.** Taking into account the weak convergence of $u_{\alpha\overline{q}_\alpha}$ to $\mu$ in $L^2(V)$ and the estimation $\sqrt{(\alpha-1)}||u_{\alpha\overline{q}_\alpha} - b||_{L^2(L^2(\Gamma_1))} \leq C_4$, in a similar way to that of Proposition 1, we obtain that $\mu = u_f$. Moreover, for the adjoint state, we have that $p_{\alpha\overline{q}_\alpha}$ is weakly convergent to $\rho$ in $L^2(V)$ and from the estimation $\sqrt{(\alpha-1)}||p_{\alpha\overline{q}_\alpha}||_{L^2(L^2(\Gamma_1))} \leq C_5$, ina similar way to thatof Proposition 2, we obtain that $\rho = p_f$. Therefore, we have $u_{\alpha\overline{q}_\alpha} \rightharpoonup u_f$ in $L^2(V)$ and $p_{\alpha\overline{q}_\alpha} \rightharpoonup p_f$ in $L^2(V)$. Now, the optimality condition for the optimal control problem (6) is given by $(M\overline{q}_\alpha - p_{\alpha\overline{q}_\alpha}\eta)_{\mathcal{Q}} = 0$, $\forall \eta \in \mathcal{Q}$, and taking into account that

$$p_{\alpha\overline{q}_\alpha} \rightharpoonup p_f \quad \text{in} \quad L^2(V) \quad \text{and} \quad \overline{q}_\alpha \rightharpoonup f \quad \text{in} \quad \mathcal{Q},$$

we obtain $-p_f + Mf = 0$, and by the uniqueness of the optimal control we deduce that $f = \overline{q}$. Therefore, $u_f = u_{\overline{q}}$, $p_f = p_{\overline{q}}$, $\dot{u}_f = \dot{u}_{\overline{q}}$ and $\dot{p}_f = \dot{u}_{\overline{q}}$.

**Step 3.** We have, for all $q \in \mathcal{Q}$

$$J(\overline{q}) = \frac{1}{2}||u_{\overline{q}} - z_d||^2_{\mathcal{H}} + \frac{M}{2}||\overline{q}||^2_{\mathcal{Q}} \leq \liminf_{\alpha\to\infty}\left[\frac{1}{2}||u_{\alpha\overline{q}_\alpha} - z_d||^2_{\mathcal{H}} + \frac{M}{2}||\overline{q}_\alpha||^2_{\mathcal{Q}}\right] \leq$$

$$\limsup_{\alpha\to\infty}\left[\frac{1}{2}||u_{\alpha\overline{q}_\alpha} - z_d||^2_{\mathcal{H}} + \frac{M}{2}||\overline{q}_\alpha||^2_{\mathcal{Q}}\right] \leq \limsup_{\alpha\to\infty} J_\alpha(q) =$$

$$\lim_{\alpha\to\infty}\left[\frac{1}{2}||u_{\alpha q} - z_d||^2_{\mathcal{H}} + \frac{M}{2}||q||^2_{\mathcal{Q}}\right] = \frac{1}{2}||u_q - z_d||^2_{\mathcal{H}} + \frac{M}{2}||q||^2_{\mathcal{Q}} = J(q).$$

By taking infimum on $q$, all the above inequalities become equalities and therefore

$$\lim_{\alpha\to\infty}\left[||u_{\alpha\overline{q}_\alpha} - z_d||^2_{\mathcal{H}} + M||\overline{q}_\alpha||^2_{\mathcal{Q}}\right] = ||u_{\overline{q}} - z_d||^2_{\mathcal{H}} + M||\overline{q}||^2_{\mathcal{Q}}$$

that is

$$\lim_{\alpha\to\infty} ||(\sqrt{M}\overline{q}_\alpha, u_{\alpha\overline{q}_\alpha} - z_d)||^2_{\mathcal{Q}\times\mathcal{H}} = ||(\sqrt{M}\overline{q}, u_{\overline{q}} - z_d)||^2_{\mathcal{Q}\times\mathcal{H}}.$$

The previous equality, the convergence $\overline{q}_\alpha \rightharpoonup \overline{q}$ in $\mathcal{Q}$ and $u_{\alpha\overline{q}_\alpha} \rightharpoonup u_{\overline{q}}$ in $L^2(V)$ imply that $(\overline{q}_\alpha, u_{\alpha\overline{q}_\alpha}) \to (\overline{q}, u_{\overline{q}})$ strongly in $\mathcal{Q}\times\mathcal{H}$, when $\alpha\to\infty$.
Finally, if we take $v = u_{\alpha\overline{q}_\alpha}(t) - u_{\overline{q}}(t) \in V$ in (4) for $u = u_{\alpha\overline{q}_\alpha}$ and if we call $z_\alpha = u_{\alpha\overline{q}_\alpha} - u_{\overline{q}}$, we have

$$\lambda_1||z_\alpha(t)||^2_V \le (g(t) - \dot{u}_{\overline{q}}(t), z_\alpha(t))_H - (\overline{q}_\alpha(t), z_\alpha(t))_Q - a(u_{\overline{q}}(t), z_\alpha(t)).$$

Integrating between 0 and $T$ and taking into account that $z_\alpha \rightharpoonup 0$ weakly in $L^2(V)$, $z_\alpha$ is bounded independently of $\alpha$, and $\overline{q}_\alpha \to \overline{q}$ strongly in $\mathcal{Q}$ when $\alpha\to\infty$, following Boukrouche and Tarzia (2013), we obtain

$$\int_0^T [(g(t) - \dot{u}_{\overline{q}}(t), z_\alpha(t))_H - (\overline{q}_\alpha(t), z_\alpha(t))_Q - a(u_{\overline{q}}(t), z_\alpha(t))]\, dt \to 0.$$

Next, we have $\lim_{\alpha\to\infty} ||z_\alpha||_{L^2(V)} = 0$. From the variational equalities (3) and (4), we have

$$(\dot{z}_\alpha(t), v)_H + a(z_\alpha(t), v) = (\overline{q}(t) - \overline{q}_\alpha(t), v)_Q, \quad \forall v \in V_0.$$

Therefore, $||\dot{z}_\alpha(t)||^2_{V_0'} \le 2||z_\alpha(t)||^2_V + 2||\gamma_0||^2||\overline{q}(t) - \overline{q}_\alpha(t)||^2_Q$ and by integrating on $[0,T]$, we obtain

$$||\dot{z}_\alpha||^2_{L^2(V_0')} \le 2||z_\alpha||^2_{L^2(V)} + 2||\gamma_0||^2||\overline{q} - \overline{q}_\alpha||^2_{\mathcal{Q}}.$$

Since that [????]$\overline{q}_\alpha \to \overline{q}$ strongly in $\mathcal{Q}$ and $u_{\alpha\overline{q}_\alpha} \to u_{\overline{q}}$ strongly in $L^2(V)$ when $\alpha\to\infty$, we can say that $\dot{z}_\alpha \to 0$ strongly in $L^2(V_0')$, that is, $\dot{u}_{\alpha\overline{q}_\alpha} \to \dot{u}_{\overline{q}}$ strongly in $L^2(V_0')$. In similar way, we prove that $(p_{\alpha\overline{q}_\alpha}\dot{p}_{\alpha\overline{q}_\alpha}) \to (p_{\overline{q}}\dot{p}_{\overline{q}})$ strongly in $L^2(V)\times L^2(V_0')$, when $\alpha\to\infty$. □

## 4. Estimations between the optimal controls

In this section, we obtain estimations between the solutions of some Neumann boundary optimal control problems and the solutions of the simultaneous distributed-boundary optimal control problems studied in Tarzia, Bollo and Gariboldi (2020) [?? please, explain "estimations between"].

### 4.1. Estimations with respect to the problem S

We consider the Neumann boundary optimal control problem

$$\text{find} \quad \overline{q} \in \mathcal{Q} \quad \text{such that} \quad J_1(\overline{q}) = \min_{q\in\mathcal{Q}} J_1(q) \qquad \text{for fixed} \quad g \in \mathcal{H}, \qquad (26)$$

where $J_1$ is the cost functional, given in (5), plus the constant $\frac{M_1}{2}||g||^2_{\mathcal{H}}$, that is, $J_1 : \mathcal{Q} \to \mathbb{R}^+_0$ is given by

$$J_1(q) = \frac{1}{2}||u_q - z_d||^2_{\mathcal{H}} + \frac{M_1}{2}||g||^2_{\mathcal{H}} + \frac{M}{2}||q||^2_{\mathcal{Q}} \qquad (\text{fixed} \quad g \in \mathcal{H}),$$

where $u_q$ is the unique solution of the problem (3) for fixed $g$.

Remark 1 *The functional $J^+(g,q)$, defined in Tarzia, Bollo and Gariboldi (2020), see (7), and the functional $J_1$ previously defined, satisfy the following elemental estimation*

$$J^+(\overline{\overline{gq}}) \leq J_1(\overline{q}), \quad \forall g \in \mathcal{H}.$$

In the following theorem we obtain estimations between the solution of the boundary optimal control problem (26) and the second component of the solution to simultaneous distributed-boundary optimal control problem studied in Tarzia, Bollo and Gariboldi (2020).

Theorem 2 *If $(\overline{\overline{gq}}) \in \mathcal{H} \times \mathcal{Q}$ is the unique solution to the distributed-boundary optimal control problem in [?, see (7)] and $\overline{q}$ is the unique solution to the optimal control problem (26), then*

$$||\overline{q} - \overline{\overline{q}}||_{\mathcal{Q}} \leq \frac{||\gamma_0||}{\lambda_0 M}||u_{\overline{\overline{g}}\,\overline{\overline{q}}} - u_{g\,\overline{q}}||_{\mathcal{H}} \quad \forall g \in \mathcal{H} \tag{27}$$

*where $\gamma_0$ is the trace operator with $||\gamma_0|| = \sup_{v \in V - \{0\}} \frac{||\gamma_0(v)||_{\mathcal{Q}}}{||v||_V}$ and $\lambda_0$ the coercivity constant of the bilinear form $a$.*

Proof By the optimality condition for $\overline{q}$, for fixed $g \in \mathcal{H}$, we have

$$(M\overline{q} - p_{g\,\overline{q}}\eta)_{\mathcal{Q}} = 0, \quad \forall \eta \in \mathcal{Q},$$

and taking upon $\eta = \overline{\overline{q}} - \overline{q}$ we obtain

$$(M\overline{q} - p_{g\,\overline{q}}\overline{\overline{q}} - \overline{q})_{\mathcal{Q}} = 0. \tag{28}$$

On the other hand, if we take $h = 0 \in \mathcal{H}$ in the optimality condition for $(\overline{\overline{gq}})$, given in Tarzia, Bollo and Gariboldi (2020), we have $(M\overline{\overline{q}} - p_{\overline{\overline{g}}\,\overline{\overline{q}}}\eta)_{\mathcal{Q}} = 0$, $\forall \eta \in \mathcal{Q}$; next, by taking $\eta = \overline{q} - \overline{\overline{q}}$, we obtain

$$(-M\overline{\overline{q}} + p_{\overline{\overline{g}}\,\overline{\overline{q}}}\overline{\overline{q}} - \overline{q})_{\mathcal{Q}} = 0. \tag{29}$$

By adding the expressions (28) and (29), we have

$$\left(M(\overline{q} - \overline{\overline{q}}) + (p_{\overline{\overline{g}}\,\overline{\overline{q}}} - p_{g\,\overline{q}}), \overline{\overline{q}} - \overline{q}\right)_{\mathcal{Q}} = 0.$$

So, by the Cauchy-Schwarz inequality and the trace theorem, we have

$$||\overline{\overline{q}} - \overline{q}||_{\mathcal{Q}} \leq \frac{||\gamma_0||}{M} ||p_{\overline{\overline{g}}\,\overline{\overline{q}}} - p_{g\,\overline{q}}||_{L^2(V)}.$$

Now, if we prove that

$$||p_{\overline{\overline{g}}\,\overline{\overline{q}}} - p_{g\,\overline{q}}||_{L^2(V)} \leq \frac{1}{\lambda_0} ||u_{\overline{\overline{g}}\,\overline{\overline{q}}} - u_{g\,\overline{q}}||_{\mathcal{H}}$$

the estimation (27) holds. In fact, by the variational equality for the adjoint state given in Tarzia, Bollo and Gariboldi (2020), see (5), for $g = \overline{\overline{g}}$ and $q = \overline{\overline{q}}$, we have

$$-\langle \dot{p}_{\overline{\overline{g}}\,\overline{\overline{q}}}(t), v\rangle + a(p_{\overline{\overline{g}}\,\overline{\overline{q}}}(t), v) = (u_{\overline{\overline{g}}\,\overline{\overline{q}}}(t) - z_d(t), v)_H, \quad \forall v \in V_0,$$

and for fixed $g \in \mathcal{H}$ and $q = \overline{q}$

$$-\langle \dot{p}_{g\,\overline{q}}(t), v\rangle + a(p_{g\,\overline{q}}(t), v) = (u_{g\,\overline{q}}(t) - z_d(t), v)_H, \quad \forall v \in V_0.$$

Subtracting these equations, we obtain

$$-\langle \dot{p}_{\overline{\overline{g}}\,\overline{\overline{q}}}(t) - \dot{p}_{g\,\overline{q}}(t), v\rangle + a(p_{\overline{\overline{g}}\,\overline{\overline{q}}}(t) - p_{g\,\overline{q}}(t), v) = (u_{\overline{\overline{g}}\,\overline{\overline{q}}}(t) - u_{g\,\overline{q}}(t), v)_H, \quad \forall v \in V_0.$$

By replacing $v = p_{\overline{\overline{g}}\,\overline{\overline{q}}}(t) - p_{g\,\overline{q}}(t) \in V_0$ and using the fact that

$$2\langle \dot{p}_{\overline{\overline{g}}\,\overline{\overline{q}}}(t) - \dot{p}_{g\,\overline{q}}(t), p_{\overline{\overline{g}}\,\overline{\overline{q}}}(t) - p_{g\,\overline{q}}(t)\rangle = \frac{d}{dt} ||p_{\overline{\overline{g}}\,\overline{\overline{q}}}(t) - p_{g\,\overline{q}}(t)||_H^2,$$

we obtain

$$\begin{aligned} &-\frac{1}{2}\frac{d}{dt}||p_{\overline{\overline{g}}\,\overline{\overline{q}}}(t) - p_{g\,\overline{q}}(t)||_H^2 + \lambda_0 ||\nabla\left(p_{\overline{\overline{g}}\,\overline{\overline{q}}}(t) - p_{g\,\overline{q}}(t)\right)||_H^2 \\ &\leq ||u_{\overline{\overline{g}}\,\overline{\overline{q}}}(t) - u_{g\,\overline{q}}(t)||_H ||p_{\overline{\overline{g}}\,\overline{\overline{q}}}(t) - p_{g\,\overline{q}}(t)||_H. \end{aligned}$$

and by using Young inequality for $\epsilon = \lambda_0$, we have

$$\begin{aligned} &-\frac{1}{2}\frac{d}{dt}||p_{\overline{\overline{g}}\,\overline{\overline{q}}}(t) - p_{g\,\overline{q}}(t)||_H^2 + \lambda_0 ||p_{\overline{\overline{g}}\,\overline{\overline{q}}}(t) - p_{g\,\overline{q}}(t)||_V^2 \\ &\leq \frac{1}{2\lambda_0}||u_{\overline{\overline{g}}\,\overline{\overline{q}}}(t) - u_{g\,\overline{q}}(t)||_H^2 + \frac{\lambda_0}{2}||p_{\overline{\overline{g}}\,\overline{\overline{q}}}(t) - p_{g\,\overline{q}}(t)||_V^2. \end{aligned}$$

Then

$$-\frac{d}{dt}||p_{\overline{\overline{g}}\,\overline{\overline{q}}}(t) - p_{g\,\overline{q}}(t)||_H^2 + \lambda_0 ||p_{\overline{\overline{g}}\,\overline{\overline{q}}}(t) - p_{g\,\overline{q}}(t)||_V^2 \leq \frac{1}{\lambda_0}||u_{\overline{\overline{g}}\,\overline{\overline{q}}}(t) - u_{g\,\overline{q}}(t)||_H^2.$$

By integrating between 0 and $T$, and using that $p_{\overline{\overline{g}}\,\overline{\overline{q}}}(T) = p_{g\,\overline{q}}(T) = 0$, we deduce

$$||p_{\overline{\overline{g}}\,\overline{\overline{q}}}(0) - p_{g\,\overline{q}}(0)||_H^2 + \lambda_0 ||p_{\overline{\overline{g}}\,\overline{\overline{q}}} - p_{g\,\overline{q}}||_{L^2(V)}^2 \leq \frac{1}{\lambda_0}||u_{\overline{\overline{g}}\,\overline{\overline{q}}} - u_{g\,\overline{q}}||_{\mathcal{H}}^2$$

i.e.

$$||p_{\overline{\overline{g}}\,\overline{\overline{q}}} - p_{g\,\overline{q}}||_{L^2(V)} \leq \frac{1}{\lambda_0}||u_{\overline{\overline{g}}\,\overline{\overline{q}}} - u_{g\,\overline{q}}||_{\mathcal{H}}.$$

and then (27) holds. □

Now, if we consider the distributed optimal control problem

$$\text{find} \quad \overline{g} \in \mathcal{H} \quad \text{such that} \quad J_2(\overline{g}) = \min_{g \in \mathcal{H}} J_2(g) \qquad \text{for fixed} \quad q \in \mathcal{Q}, \tag{30}$$

where $J_2$ is the cost functional given in menaldi and tarzia (2007) plus the constant $\frac{M}{2}||q||^2_{\mathcal{Q}}$, that is, $J_2 : \mathcal{H} \to \mathbb{R}^+_0$ is given by

$$J_2(g) = \frac{1}{2}||u_g - z_d||^2_{\mathcal{H}} + \frac{M_1}{2}||g||^2_{\mathcal{H}} + \frac{M}{2}||q||^2_{\mathcal{Q}} \qquad (\text{fixed} \quad q \in \mathcal{Q}),$$

where $u_g$ is the unique solution of the problem (3) for fixed $q$, we can prove the following corollary.

Corollary 1 *If $(\overline{\overline{gq}}) \in \mathcal{H} \times \mathcal{Q}$ is the unique solution to the simultaneous optimal control problem studied in Tarzia, Bollo and Gariboldi (2020), see (5), $\overline{g}$ is the unique solution to the distributed optimal control problem (30) for fixed $q$ ($q = \overline{\overline{q}}$), and $\overline{q}$ is the unique solution to the problem (26) for fixed $g$ ($g = \overline{\overline{g}}$), then $\overline{g} = \overline{\overline{g}}$ and $\overline{q} = \overline{\overline{q}}$.*

Proof If we take $h = \overline{\overline{g}} - \overline{g}$ in the optimality condition for the problem given in Menaldi and Tarzia (2007), we have

$$\left(M_1\overline{g} + p_{\overline{g}\,\overline{\overline{q}}}\overline{\overline{g}} - \overline{g}\right)_{\mathcal{H}} = 0. \tag{31}$$

On the other hand, if we consider $h = \overline{\overline{g}} - \overline{g}$ and $\eta = 0$ in the optimality condition for the simultaneous optimal control problem studied in Tarzia, Bollo and Gariboldi (2020), we obtain

$$\left(M_1\overline{\overline{g}} + p_{\overline{\overline{g}}\,\overline{\overline{q}}}\overline{\overline{g}} - \overline{g}\right)_{\mathcal{H}}. \tag{32}$$

Subtracting (31) and (32), we deduce that

$$\left(M_1\overline{g} + p_{\overline{g}\,\overline{q}} - M_1\overline{\overline{g}} - p_{\overline{\overline{g}}\,\overline{\overline{q}}}\overline{\overline{g}} - \overline{g}\right)_{\mathcal{H}} = 0,$$

and therefore

$$\left(p_{\overline{g}\,\overline{q}} - p_{\overline{\overline{g}}\,\overline{\overline{q}}}\overline{\overline{g}} - \overline{g}\right)_{\mathcal{H}} - M_1\left(\overline{\overline{g}} - \overline{g}, \overline{\overline{g}} - \overline{g}\right)_{\mathcal{H}} = 0.$$

Next, by using the fact that

$$\left(p_{\overline{g}\,\overline{q}} - p_{\overline{\overline{g}}\,\overline{\overline{q}}}\overline{\overline{g}} - \overline{g}\right)_{\mathcal{H}} = -||u_{\overline{g}\,\overline{q}} - u_{\overline{\overline{g}}\,\overline{\overline{q}}}||^2_{\mathcal{H}}$$

we have

$$-||u_{\overline{g}\,\overline{q}} - u_{\overline{\overline{g}}\,\overline{\overline{q}}}||^2_{\mathcal{H}} = M_1||\overline{\overline{g}} - \overline{g}||^2_{\mathcal{H}}.$$

Hence, we deduce that $||\overline{\overline{g}} - \overline{g}||^2_{\mathcal{H}} = 0$ and therefore $\overline{g} = \overline{\overline{g}}$.
In a similar way we prove that $\overline{q} = \overline{\overline{q}}$. □

### 4.2. Estimations with respect to the problem $\mathbf{S}_\alpha$

For each $\alpha > 0$, we consider the following optimal control problem

$$\text{find} \quad \overline{q}_\alpha \in \mathcal{Q} \quad \text{such that} \quad J_{1\alpha}(\overline{q}_\alpha) = \min_{q\in\mathcal{Q}} J_{1\alpha}(q), \tag{33}$$

where $J_{1\alpha} : \mathcal{Q} \to \mathbb{R}_0^+$ is given by

$$J_{1\alpha}(q) = \frac{1}{2}||u_{\alpha q} - z_d||^2_{\mathcal{H}} + \frac{M_1}{2}||g||^2_{\mathcal{H}} + \frac{M}{2}||q||^2_{\mathcal{Q}} \qquad (\text{fixed} \quad g \in \mathcal{H}),$$

that is, $J_{1\alpha}$ is the functional (6) plus the constant $\frac{M_1}{2}||g||^2_{\mathcal{H}}$, and $u_{\alpha q}$ is the unique solution of the problem (4) for fixed $g$.

REMARK 2 *The functional $J^+_\alpha$ defined in Tarzia, Bollo and Gariboldi (2020), see (8), and the functional $J_{1\alpha}$, previously defined, satisfy the following estimate*

$$J^+_\alpha(\overline{\overline{g}}_\alpha, \overline{\overline{q}}_\alpha) \leq J_{1\alpha}(\overline{q}_\alpha), \quad \forall g \in \mathcal{H}.$$

Estimation between the solution of the Neumann boundary optimal control problem (33) with the second component of the solution to the simultaneous optimal control problem studied in Tarzia, Bollo and Gariboldi (2020), is given in the following theorem whose prove is omitted.

THEOREM 3 *If $(\overline{\overline{g}}_\alpha, \overline{\overline{q}}_\alpha) \in \mathcal{H} \times \mathcal{Q}$ is the unique solution of the simultaneous optimal control problem from Tarzia, Bollo and Gariboldi (2020), see (6), and $\overline{q}_\alpha$ is the unique solutions to the optimal control problem (33), then we have:*

$$||\overline{q}_\alpha - \overline{\overline{q}}_\alpha||_{\mathcal{Q}} \leq \frac{||\gamma_0||}{\lambda_\alpha M}||u_{\alpha \overline{\overline{g}}_\alpha \overline{\overline{q}}_\alpha} - u_{\alpha\, g\, \overline{q}_\alpha}||_{\mathcal{H}} \quad \forall g \in \mathcal{H}$$

*with $\lambda_\alpha = \lambda_1 \min\{1, \alpha\}$ and $\lambda_1$ the coerciveness constant of the bilinear form $a_1$.*

If we consider the following distributed optimal control problem, for each $\alpha > 0$

$$\text{find} \quad \overline{g}_\alpha \in \mathcal{H} \quad \text{such that} \quad J_{2\alpha}(\overline{g}_\alpha) = \min_{g\in\mathcal{H}} J_{2\alpha}(g), \tag{34}$$

where $J_{2\alpha} : \mathcal{H} \to \mathbb{R}_0^+$ is given by

$$J_{2\alpha}(g) = \frac{1}{2}||u_{\alpha g} - z_d||^2_{\mathcal{H}} + \frac{M_1}{2}||g||^2_{\mathcal{H}} + \frac{M}{2}||q||^2_{\mathcal{Q}} \qquad (\text{fixed} \quad q \in \mathcal{Q}),$$

that is, $J_{2\alpha}$ is the functional studied in Menaldi and Tarzia (2007) plus the constant $\frac{M}{2}||q||^2_{\mathcal{Q}}$, and $u_{\alpha g}$ is the unique solution of the problem (4) for fixed $q$, we give the following corollary, whose proof is omitted.

COROLLARY 2 *If $(\overline{\overline{g}}_\alpha, \overline{\overline{q}}_\alpha) \in \mathcal{H} \times \mathcal{Q}$ is the unique solution of the simultaneous optimal control problem studied in Tarzia, Bollo and Gariboldi (2020), see (6), $\overline{g}_\alpha$ is the unique solution of the problem (34) for fixed $q$ ($q = \overline{\overline{q}}_\alpha$), and $\overline{q}_\alpha$ is the unique solution of the problem (33) for fixed $g$ ($g = \overline{\overline{g}}_\alpha$), then $\overline{g}_\alpha = \overline{\overline{g}}_\alpha$ and $\overline{q}_\alpha = \overline{\overline{q}}_\alpha$.*

## 5. Real Neumann boundary optimal control problems

In this section, we consider the non-stationary real-boundary optimal control problems (8) and (9) and the stationary real-boundary optimal control problems (15) and (16). We prove existence and uniqueness of the solutions to these optimal control problems and monotonicity results are also obtained.

### 5.1. Real Neumann boundary optimal control problem in relation to the parabolic system S

If we consider the real-boundary optimal control problem (8) and we denote by $u_{bqg}$ the unique solution of the variational equality (3) for the data $b$, $q$ and $g$, and we take $q = \lambda q_0$ for fixed $q_0 \in \mathcal{Q}$ ($q_0 \neq 0$) and $\lambda \in \mathbb{R}$, we can prove that

$$u_{bqg}(t) = u_{b\lambda g}(t) = u_b(t) + u_q(t) + u_g(t), \quad \forall x \in \Omega,$$

where $u_b$ is the unique solution to the parabolic variational equality

$$\left\{ \begin{array}{l} u - v_b \in L^2(V_0), \qquad u(0) = v_b \quad \text{and} \quad \dot{u} \in L^2(V_0') \\ \text{such that} \quad \langle \dot{u}(t), v\rangle + a(u(t), v) = 0, \quad \forall v \in V_0, \end{array} \right. \tag{35}$$

$u_q$ is the unique solution to the parabolic variational equality

$$\left\{ \begin{array}{l} u \in L^2(V_0), \qquad u(0) = 0 \quad \text{and} \quad \dot{u} \in L^2(V_0') \\ \text{such that} \quad \langle \dot{u}(t), v\rangle + a(u(t), v) = -\lambda(q_0(t), v)_Q, \quad \forall v \in V_0, \end{array} \right. \tag{36}$$

and $u_g$ is the unique solution to the parabolic variational equality

$$\left\{ \begin{array}{l} u \in L^2(V_0), \qquad u(0) = 0 \quad \text{and} \quad \dot{u} \in L^2(V_0') \\ \text{such that} \quad \langle \dot{u}(t), v\rangle + a(u(t), v) = (g(t), v)_H, \quad \forall v \in V_0. \end{array} \right. \tag{37}$$

We note that, by the linearity, we can prove that $u_q(t) = \lambda u_{q_0}(t)$, where $u_{q_0}$ is the unique solution of (3) for $q = q_0$ and $b = g = 0$.
Next, for each $T > 0$, the functional $H_T(\lambda)$ can be written as

$$H_T(\lambda) = \frac{1}{2} \int_0^T \int_\Omega (u_b(t) + \lambda u_{q_0}(t) + u_g(t) - z_d(t))^2 dx dt + \frac{M\lambda^2}{2} \int_0^T \int_{\Gamma_2} q_0^2(t) d\gamma dt$$

therefore, $H_T(\lambda) = \lambda^2 A(T) + \lambda B(T) + C(T)$, where

$$A(T) = \frac{M}{2} \int_0^T \int_{\Gamma_2} q_0^2(t) d\gamma dt + \frac{1}{2} \int_0^T \int_\Omega u_{q_0}^2(t) dx dt$$

$$B(T) = \int_0^T \int_\Omega u_{q_0}(t)(u_b(t) + u_g(t) - z_d(t)) dx dt$$

$$C(T) = \frac{1}{2} \int_0^T \int_\Omega (u_b(t) + u_g(t) - z_d(t))^2 dxdt.$$

Here, taking into account that

$$\begin{aligned}
&4A(T)C(T) \\
&= \left( M \int_0^T \int_{\Gamma_2} q_0^2(t) d\gamma dt + \int_0^T \int_\Omega u_{q_0}^2(t) dxdt \right) \left( \int_0^T \int_\Omega (u_b(t) + u_g(t) - z_d(t))^2 dxdt \right) \\
&> \left( \int_0^T \int_\Omega u_{q_0}^2(t) dxdt \right) \left( \int_0^T \int_\Omega (u_b(t) + u_g(t) - z_d(t))^2 dxdt \right) \\
&= ||u_{q_0}||_{\mathcal{H}}^2 ||u_b + u_g - z_d||_{\mathcal{H}}^2 \geq (u_{q_0} u_b + u_g - z_d)_{\mathcal{H}}^2 \\
&= \left( \int_0^T \int_\Omega u_{q_0}(t)(u_b(t) + u_g(t) - z_d(t)) dxdt \right)^2 = (B(T))^2
\end{aligned}$$

we deduce that $(B(T))^2 - 4A(T)C(T) < 0$ and since $A(T) > 0$, because $q_0 \neq 0$, then there exists a unique $\overline{\lambda}(T) \in \mathbb{R}$ for each $T > 0$, such that it satisfies the problem (8), whose solution is given by the following expression:

$$\overline{\lambda}(T) = -\frac{B(T)}{2A(T)} = -\frac{\int_0^T \int_\Omega u_{q_0}(t)(u_b(t) + u_g(t) - z_d(t)) dxdt}{M \int_0^T \int_{\Gamma_2} q_0^2(t) d\gamma dt + \int_0^T \int_\Omega u_{q_0}^2(t) dxdt}. \tag{38}$$

Therefore, we have proven the following property.

THEOREM 4 *For each $T > 0$, there exists a unique solution $\overline{\lambda}(T) \in \mathbb{R}$ to the optimization problem (8).*

Now, we will prove some monotonicity properties.

PROPOSITION 3 *Let $q_1 = \lambda_1 q_0$ and $q_2 = \lambda_2 q_0$ ($q_0 > 0$), with $\lambda_2 \leq \lambda_1$ and $g_1 \leq g_2$, then $u_{b\lambda_1 g_1} \leq u_{b\lambda_2 g_2}$ in $\Omega \times [0, T]$.*

PROOF We define $w = u_{b\lambda_1 g_1} - u_{b\lambda_2 g_2}$ and we take $v = -w^+(t) \in V$ for $u_{b\lambda_1 g_1}$ in (3) and $v = w^+(t) \in V$ for $u_{b\lambda_2 g_2}$ in (3) (for the regularity of $w^+$ see Kinderlehrer and Stampalchia,2000). Adding the variational equalities, we have

$$\begin{aligned}
&\langle \dot{u}_{b\lambda_2 g_2}(t) - \dot{u}_{b\lambda_1 g_1}(t), w^+(t) \rangle + a(u_{b\lambda_2 g_2}(t) - u_{b\lambda_1 g_1}(t), w^+(t)) \\
&\quad = \int_\Omega (g_2(t) - g_1(t)) w^+(t) dx + (\lambda_1 - \lambda_2) \int_{\Gamma_2} q_0(t) w^+(t) d\gamma.
\end{aligned}$$

Next,

$$-\langle \dot{w^+}(t), w^+(t)\rangle - a(w^+(t), w^+(t)) = \int_\Omega (g_2(t) - g_1(t))w^+(t)dx + (\lambda_1 - \lambda_2)\int_{\Gamma_2} q_0(t)w^+(t)d\gamma.$$

Now, using the fact that

$$\langle \dot{w^+}(t), w^+(t)\rangle = \frac{1}{2}\frac{d}{dt}||w^+(t)||_H^2$$

and integrating between 0 and $T$, we prove that

$$\frac{1}{2}\left(||w^+(T)||_H^2 - ||w^+(0)||_H^2\right) + \int_0^T ||w^+(t)||_{V_0}^2 dt \le \int_0^T\int_\Omega (g_1(t) - g_2(t))w^+(t)dxdt + (\lambda_2 - \lambda_1)\int_0^T\int_{\Gamma_2} q_0(t)w^+(t)d\gamma dt.$$

Since $w^+(0) = (u_{b\lambda_1 g_1} - u_{b\lambda_2 g_2})^+(0) = \max\{0, (u_{b\lambda_1 g_1} - u_{b\lambda_2 g_2})(0)\} = 0$,

$$\frac{1}{2}||w^+(T)||_H^2 + \int_0^T ||w^+(t)||_{V_0}^2 dt \le \int_0^T\int_\Omega (g_1(t) - g_2(t))w^+(t)dxdt + (\lambda_2 - \lambda_1)\int_0^T\int_{\Gamma_2} q_0(t)w^+(t)d\gamma dt \le 0,$$

by using the facts that $g_1 \le g_2$, $\lambda_2 \le \lambda_1$ and $q_0 > 0$, here $||w^+||^2_{L^2(0,T;V_0)} = 0$, then $w^+ \equiv 0$ in $\Omega \times [0,T]$ and therefore $w \le 0$ in $\Omega \times [0,T]$, that is, the thesis holds. □

Corollary 3 *If $q_1 = \lambda_1 q_0$ and $q_2 = \lambda_2 q_0$ ($q_0 > 0$), with $\lambda_2 \le \lambda_1$, then $u_{b\lambda_1 g} \le u_{b\lambda_2 g}$ in $\Omega \times [0,T]$.*

Proof This results from taking $g = g_1 = g_2$ in the proof of the Proposition 3. □

Remark 3 *The previous monotonicity properties are still true if we consider $\lambda_1 \le \lambda_2$ with $q_0 < 0$.*

### 5.2. Real Neumann boundary optimal control problems in relation to the parabolic $\mathbf{S}_\alpha$

If we consider the real Neumann boundary optimal control problem (9) and for each $\alpha > 0$ we denote by $u_{\alpha bqg}$ the unique solution of the variational equality (4) for data $b$, $q$ and $g$, and we take $q = \lambda q_0$ for fixed $q_0 \in Q$ ($q_0 \neq 0$) and $\lambda \in \mathbb{R}$, we can see that

$$u_{\alpha bqg}(t) = u_{\alpha b\lambda g}(t) = u_{\alpha b}(t) + u_{\alpha q}(t) + u_{\alpha g}(t), \quad \forall x \in \Omega,$$

where $u_{\alpha b}, u_{\alpha q}, u_{\alpha g}$ are the unique solutions of (4) for $q = g = 0$, $b = g = 0$ and $b = q = 0$, respectively.

We note that, by the linearity, we can see that $u_{\alpha q}(t) = \lambda u_{\alpha q_0}(t)$, where $u_{\alpha q_0}$ is the unique solution of (4) for $q = q_0$ and $b = g = 0$. Next, for each $T > 0$, the functional $H_{\alpha T}(\lambda)$ can be written down as

$$H_{\alpha T}(\lambda) = \frac{1}{2}\int_0^T\int_\Omega (u_{\alpha b}(t) + \lambda u_{\alpha q_0}(t) + u_{\alpha g}(t) - z_d(t))^2 dxdt$$
$$+ \frac{M\lambda^2}{2}\int_0^T\int_{\Gamma_2} q_0^2(t) d\gamma dt = \lambda^2 A_\alpha(T) + \lambda B_\alpha(T) + C_\alpha(T),$$

where

$$A_\alpha(T) = \frac{M}{2}\int_0^T\int_{\Gamma_2} q_0^2(t) d\gamma dt + \frac{1}{2}\int_0^T\int_\Omega u_{\alpha q_0}^2(t) dxdt$$

$$B_\alpha(T) = \int_0^T\int_\Omega u_{\alpha q_0}(t)(u_{\alpha b}(t) + u_{\alpha g}(t) - z_d(t)) dxdt$$

$$C_\alpha(T) = \frac{1}{2}\int_0^T\int_\Omega (u_{\alpha b}(t) + u_{\alpha g}(t) - z_d(t))^2 dxdt.$$

Now, taking into account that $A_\alpha(T) > 0$, in a similar way as in the previous subsection, we can prove that $(B_\alpha(T))^2 - 4A_\alpha(T)C_\alpha(T) < 0$ and therefore there exists a unique solution $\overline{\lambda}_\alpha(T) \in \mathbb{R}$, for each $\alpha > 0$ and for each $T > 0$, for the optimal control problem (9), which is given by the following expression:

$$\overline{\lambda}_\alpha(T) = -\frac{\int_0^T\int_\Omega u_{\alpha q_0}(t)(u_{\alpha b}(t) + u_{\alpha g}(t) - z_d(t)) dxdt}{M\int_0^T\int_{\Gamma_2} q_0^2(t) d\gamma dt + \int_0^T\int_\Omega u_{\alpha q_0}^2(t) dxdt}. \tag{39}$$

Therefore, we have proven the following property.

Theorem 5 *For each $\alpha > 0$ and $T > 0$, there exists a unique solution $\overline{\lambda}_\alpha(T) \in \mathbb{R}$ to the optimization problem (9).*

Now, in similar way to the previous subsection, we can prove the following monotonicity properties, whose proof is omitted.

Proposition 4 *For each $\alpha > 0$, if $q_1 = \lambda_1 q_0$ and $q_2 = \lambda_2 q_0$ ($q_0 > 0$), with $\lambda_2 \leq \lambda_1$, $g_1 \leq g_2$, $b_1 \leq b_2$ on $\Gamma_1$ and initial conditions $v_{b_1} \leq v_{b_2}$, then $u_{\alpha b_1 \lambda_1 g_1} \leq u_{\alpha b_2 \lambda_2 g_2}$ in $\Omega \times [0,T]$.*

### 5.3. Real Neumann boundary optimal control problem in relation to the elliptic system P

Here, we consider the stationary real Neumann boundary optimal control problem (15). If we denote by $u_{\infty bqg}$ the unique solution of the variational equality (13) for data $b$, $q$ and $g$. and if we consider $q = \lambda q_0^*$ for fixed $q_0^* \in Q$ ($q_0^* \neq 0$) and $\lambda \in \mathbb{R}$, we can see that

$$u_{\infty bqg} = u_{\infty b\lambda g} = u_{\infty b} + u_{\infty q} + u_{\infty g}, \quad \forall x \in \Omega,$$

where $u_{\infty b}, u_{\infty q}, u_{\infty g}$ are the unique solutions of the variational equality (13) for $q = g = 0$, $b = g = 0$ and $b = q = 0$, respectively.
Now, taking into account that $u_{\infty q} = \lambda u_{\infty q_0^*}$, where $u_{\infty q_0^*}$ is the solution of (13) for $q = q_0^*$ and $b = g = 0$. Next, the functional $H(\lambda)$ can be written down as

$$H(\lambda) = \frac{1}{2}\int_\Omega (u_{\infty b} + \lambda u_{\infty q_0^*} + u_{\infty g} - z_d)^2 dx + \frac{M\lambda^2}{2}\int_{\Gamma_2} (q_0^*)^2 d\gamma.$$

Therefore $H(\lambda) = \lambda^2 A + \lambda B + C$, where

$$A = \frac{M}{2}\int_{\Gamma_2} (q_0^*)^2 d\gamma + \frac{1}{2}\int_\Omega u_{\infty q_0^*}^2 dx, \quad B = \int_\Omega u_{\infty q_0^*}(u_{\infty b} + u_{\infty g} - z_d)dx$$

$$C = \frac{1}{2}\int_\Omega (u_{\infty b} + u_{\infty g} - z_d)^2 dx.$$

Here, since $B^2 - 4AC < 0$, there exists a unique $\overline{\lambda}_\infty \in \mathbb{R}$ such that solves the problem (15), that is

$$\overline{\lambda}_\infty = -\frac{B}{2A} = -\frac{\int_\Omega u_{\infty q_0^*}(u_{\infty b} + u_{\infty g} - z_d)dx}{M\int_{\Gamma_2} (q_0^*)^2 d\gamma + \int_\Omega u_{\infty q_0^*}^2 dx} \tag{40}$$

Therefore, we have proven the following theorem.

Theorem 6 *There exists a unique solution $\overline{\lambda}_\infty \in \mathbb{R}$ to the optimization problem (15).*

Now, we will give some monotonicity property, whose proof is omitted.

PROPOSITION 5 *Let $q_1 = \lambda_1 q_0^*$ and $q_2 = \lambda_2 q_0^*$ ($q_0^* > 0$), with $\lambda_2 \leq \lambda_1$ and $g_1 \leq g_2$, then $u_{\infty b \lambda_1 g_1} \leq u_{\infty b \lambda_2 g_2}$.*

### 5.4. Real Neumann boundary optimal control problems in relation to the elliptic system $\mathbf{P}_\alpha$

We consider the stationary real Neumann boundary optimal control problem (16) and we denote by $u_{\infty \alpha bqg}$ the unique solution to the variational equality (14) for the data $b$, $q$ and $g$. If we consider $q = \lambda q_0^*$ for fixed $q_0^* \in Q$ ($q_0^* \neq 0$) and $\lambda \in \mathbb{R}$, we can see that

$$u_{\infty \alpha bqg} = u_{\infty \alpha b \lambda g} = u_{\infty \alpha b} + u_{\infty \alpha q} + u_{\infty \alpha g}, \quad \forall x \in \Omega,$$

where $u_{\infty \alpha b}, u_{\infty \alpha q}, u_{\infty \alpha g}$ are the unique solutions of the variational equality (14) for $q = g = 0$, $b = g = 0$ and $b = q = 0$, respectively.
Now, taking into account the fact that $u_{\infty \alpha q} = \lambda u_{\infty \alpha q_0^*}$, where $u_{\infty \alpha q_0^*}$ is the solution of (14) for $q = q_0^*$ and $b = g = 0$, the functional $H_\alpha(\lambda)$ can be written down as

$$H_\alpha(\lambda) = \frac{1}{2} \int_\Omega (u_{\infty \alpha b} + \lambda u_{\infty \alpha q_0^*} + u_{\infty \alpha g} - z_d)^2 dx + \frac{M \lambda^2}{2} \int_{\Gamma_2} (q_0^*)^2 d\gamma = \lambda^2 A_\alpha + \lambda B_\alpha + C_\alpha,$$

where

$$A_\alpha = \frac{M}{2} \int_{\Gamma_2} (q_0^*)^2 d\gamma + \frac{1}{2} \int_\Omega u_{\infty \alpha q_0^*}^2 dx, \quad B_\alpha = \int_\Omega u_{\infty \alpha q_0^*} (u_{\infty \alpha b} + u_{\infty \alpha g} - z_d) dx$$

$$C_\alpha = \frac{1}{2} \int_\Omega (u_{\infty \alpha b} + u_{\infty \alpha g} - z_d)^2 dx.$$

Here, since $B_\alpha^2 - 4 A_\alpha C_\alpha < 0$, there exists a unique $\overline{\lambda}_\alpha \in \mathbb{R}$, which satisfies the optimization problem (16), that is

$$\overline{\lambda}_\alpha = -\frac{B_\alpha}{2A_\alpha} = -\frac{\int_\Omega u_{\infty \alpha q_0^*} (u_{\infty \alpha b} + u_{\infty \alpha g} - z_d) dx}{M \int_{\Gamma_2} (q_0^*)^2 d\gamma + \int_\Omega u_{\infty \alpha q_0^*}^2 dx}. \tag{41}$$

Therefore, we have proven the following theorem.

THEOREM 7 *For each $\alpha > 0$, there exists a unique solution $\overline{\lambda}_\alpha \in \mathbb{R}$ to the optimization problem (16).*

Now, we will give some monotonicity property, whose proof is omitted.

PROPOSITION 6 *For each $\alpha > 0$, if $q_1 = \lambda_1 q_0^*$ and $q_2 = \lambda_2 q_0^*$ ($q_0^* > 0$), with $\lambda_2 \leq \lambda_1$, $g_1 \leq g_2$, $b_1 \leq b_2$ on $\Gamma_1$, then $u_{\alpha b_1 \lambda_1 g_1} \leq u_{\alpha b_2 \lambda_2 g_2}$ in $\Omega$.*

## 6. Asymptotic behaviour of solutions when t $\to +\infty$

In this section, we study the convergence of the solutions to the problem (3) for fixed data $b \in H^{\frac{1}{2}}(\Gamma_1)$, $q \in \mathcal{Q}$ and $g \in \mathcal{H}$, to the solution to the problem (13) for the same $b \in H^{\frac{1}{2}}(\Gamma_1)$ and fixed $q \in Q$ and $g \in H$, when $t \to +\infty$. Here, for the sake of simplicity, we denote by $u_\infty$ the unique solution to the variational equality (13) for data $q_\infty \in Q$ and $g_\infty \in H$.

If we define

$$F_1(t) = e^{\lambda_0 t}||g(t) - g_\infty||_H^2, \qquad F_2(t) = e^{\lambda_0 t}||\gamma_0||^2||q(t) - q_\infty||_Q^2$$

with $g \in \mathcal{H}$, $q \in \mathcal{Q}$, $\lambda_0$ the coerciveness constant of the bilinear form $a$ and $\gamma_0$ the trace operator, we can prove the following theorem.

Theorem 8 *If $b \in H^{\frac{1}{2}}(\Gamma_1)$, $q \in \mathcal{Q}$, $g \in \mathcal{H}$, $F_1 \in L^1(0,\infty)$ and $F_2 \in L^1(0,\infty)$, then*

$$||u_{bqg}(t) - u_\infty||_H^2 \leq ||u_{bqg}(0) - u_\infty||_H^2 e^{-\lambda_0 t} + \frac{2e^{-\lambda_0 t}}{\lambda_0}\left(||F_1||_{L^1(0,\infty)} + ||F_2||_{L^1(0,\infty)}\right)$$

*and therefore*

$$\lim_{t\to+\infty} u_{bqg}(t) = u_\infty \quad \textit{in } H \textit{ strong (exponentially).}$$

Proof If we consider $w(t) = u_{bqg}(t) - u_\infty$, we have that $w(t) \in V_0$, $w(0) = v_b - u_\infty$ and $\dot{w} = \dot{u}_{bqg}$. Therefore, by taking $v = w(t)$ in the variational equalities (3) and (13) respectively, and subtracting then [????] we obtain

$$\langle \dot{w}(t), w(t)\rangle + a(w(t), w(t)) = \langle L(t) - L_\infty, w(t)\rangle,$$

that is

$$\frac{1}{2}\frac{d}{dt}||w(t)||_H^2 + \lambda_0||w(t)||_V^2 \leq (g(t) - g_\infty, w(t))_H - (q(t) - q_\infty, w(t))_Q.$$

Next,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}||w(t)||_H^2 + \lambda_0||w(t)||_V^2 &\leq ||g(t) - g_\infty||_H||w(t)||_V + ||q(t) - q_\infty||_Q||w(t)||_Q \\ &\leq \left(||g(t) - g_\infty||_H + ||\gamma_0||||q(t) - q_\infty||_Q\right)||w(t)||_V \\ &\leq \frac{\lambda_0}{2}||w(t)||_V^2 \\ &\quad + \frac{1}{2\lambda_0}\left(||g(t) - g_\infty||_H + ||\gamma_0||||q(t) - q_\infty||_Q\right)^2,\end{aligned}$$

where $\gamma_0$ is the trace operator. Then

$$\frac{1}{2}\frac{d}{dt}||w(t)||_H^2 + \frac{\lambda_0}{2}||w(t)||_H^2 \leq \frac{1}{\lambda_0}\left(||g(t) - g_\infty||_H^2 + ||\gamma_0||^2||q(t) - q_\infty||_Q^2\right).$$

Next, if we call $F(t) = ||g(t) - g_\infty||_H^2 + ||\gamma_0||^2||q(t) - q_\infty||_Q^2$, we have

$$\frac{d}{dt}\left(||w(t)||_H^2\right) e^{\lambda_0 t} + \lambda_0 ||w(t)||_H^2 e^{\lambda_0 t} \leq \frac{2}{\lambda_0} F(t) e^{\lambda_0 t}$$

or, equivalently,

$$\frac{d}{dt}\left(||w(t)||_H^2 e^{\lambda_0 t}\right) \leq \frac{2}{\lambda_0} F(t) e^{\lambda_0 t}.$$

Now, by integrating between 0 and $t$, we obtain

$$||w(t)||_H^2 e^{\lambda_0 t} - ||w(0)||_H^2 \leq \frac{2}{\lambda_0} \int_0^t F(\tau) e^{\lambda_0 \tau} d\tau$$

therefore

$$||w(t)||_H^2 \leq ||w(0)||_H^2 e^{-\lambda_0 t} + \frac{2e^{-\lambda_0 t}}{\lambda_0} \int_0^t F(\tau) e^{\lambda_0 \tau} d\tau$$

and the thesis holds. □

COROLLARY 4 *If $b \in H^{\frac{1}{2}}(\Gamma_1)$, $F_1 \in L^1(0,\infty)$ and $F_2 \in L^1(0,\infty)$, with $g(t) = g_\infty \in H$ and $q(t) = q_\infty \in Q$, then we have*

$$||u_{bqg}(t) - u_\infty||_H \leq ||u_{bqg}(0) - u_\infty||_H e^{-\frac{\lambda_0}{2} t}. \tag{42}$$

PROOF This results directly from Theorem 8. □

REMARK 4 *We note that if we consider the hypothesis: there exist $m \in (0, \lambda_0)$ and $C_1 = const. > 0$ such that*

$$\lim_{t \to +\infty} \frac{||F_1||_{L^1(0,t)} + ||F_2||_{L^1(0,t)}}{e^{mt}} \leq C_1$$

*then we can prove the asymptotic behaviour obtained in Theorem 8.*

COROLLARY 5 *a) If $b \in H^{\frac{1}{2}}(\Gamma_1)$, with $q(t) = q_\infty \in Q$ and there exist $m \in (0, \lambda_0)$ and $C_2 = const. > 0$ such that $\lim_{t \to +\infty} \frac{||F_1||_{L^1(0,t)}}{e^{mt}} \leq C_2$, then*

$$\lim_{t \to +\infty} ||u_{bqg}(t) - u_\infty||_H = 0.$$

*b) If $b \in H^{\frac{1}{2}}(\Gamma_1)$, with $g(t) = g_\infty \in H$ and there exist $m \in (0, \lambda_0)$ and $C_3 = const. > 0$ such that $\lim_{t \to +\infty} \frac{||F_2||_{L^1(0,t)}}{e^{mt}} \leq C_3$, then*

$$\lim_{t \to +\infty} ||u_{bqg}(t) - u_\infty||_Q = 0.$$

Remark 5 *An open problem is to study whether (38) is convergent to (40) and (39) is convergent to (41) (for each $\alpha > 0$), when $T \to +\infty$. We hope that these convergences do not happen. This assumption [????] is based on the fact that a function $g(t)$ $(\forall t > 0)$ can be strongly convergent in $H$ to a function $g_\infty$, but $g$ is not necessarily strongly convergent to the same function $g_\infty$ in $\mathcal{H}$, which is shown in the following counterexample.*

Example 1 *Let $\Omega = (0,1)$ be and $g(t) = g_\infty + e^{-t}$, then we have that*

$$\int_0^1 (g(t) - g_\infty)^2 dx = \int_0^1 (e^{-t})^2 dx = e^{-2t} \to 0, \quad \text{if } t \to +\infty,$$

*and*

$$\int_0^t \int_0^1 (g(\tau) - g_\infty)^2 dx d\tau = \int_0^t (e^{-\tau})^2 d\tau = \frac{1}{2}(1 - e^{-2t}) \to \frac{1}{2}, \quad \text{if } t \to +\infty.$$

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