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Explicit Discrete Solution for Some Optimization Problems and Estimations with Respect to the Exact Solution

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Abstract

We consider two steady-state heat conduction systems called, S and S_α, in a multidimensional bounded domain D for the Poisson equation with source energy g. In one system, we impose mixed boundary conditions (temperature b on the boundary Γ_1, heat flux q on Γ_2 and an adiabatic condition on Γ_3). In the other system, the condition on Γ_1 is replaced by a convective heat flux condition with coefficient α. For each of these systems, we consider three associated optimization problems (P_i) and $(P_{i\alpha})$, $i = 1,2,3$, where the variable is the source energy g, the heat flux q and the environmental temperature b, respectively. In the particular case where D is a rectangle, the explicit continuous optimization variables and the corresponding state of the systems are known. In the present work, by using a finite difference scheme, we obtain the discrete systems (S^h) and (S^h_α) and discrete optimization problems (P^h_i) and $(P^h_{i\alpha})$, $i = 1,2,3$, where h is the space step in the discretization. Explicit discrete solutions are found, and convergence and estimation errors results are proved when h goes to zero and when α goes to infinity. Moreover, some numerical simulations are provided in order to test theoretical results. Finally, we note that the use of a three-point finite-difference approximation for the Neumann or Robin boundary condition at the boundary improves the global order of convergence from $O(h)$ to $O(h^2)$.

Keywords: optimal control; finite difference method; explicit discrete solutions; estimation error

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1. Introduction

We consider a multidimensional bounded domain $\Omega \subset \mathbb{R}^n$ whose regular boundary Γ consists of three disjoint portions Γ_i with $meas(\Gamma_i) > 0$, for $i = 1,2,3$. We define two stationary heat conduction problems (S) and (S_α) with mixed boundary conditions which are given by (1) and (2), and by (1) and (3), respectively:

$$-\Delta u = g \quad in \quad \Omega, \tag{1}$$

$$u\Big|_{\Gamma_1} = b, \qquad -\frac{\partial u}{\partial n}\Big|_{\Gamma_2} = q, \qquad -\frac{\partial u}{\partial n}\Big|_{\Gamma_3} = 0, \tag{2}$$

$$-\frac{\partial u}{\partial n}\Big|_{\Gamma_1} = \alpha\,(u-b), \qquad -\frac{\partial u}{\partial n}\Big|_{\Gamma_2} = q \qquad -\frac{\partial u}{\partial n}\Big|_{\Gamma_3} = 0, \tag{3}$$

where g is the internal energy of the system in Ω, $b > 0$ the environmental temperature on Γ_1, q is the heat flux on Γ_2 and, $\alpha > 0$ is the convective heat coefficient on Γ_1. We assume that $g \in H = L^2(\Omega)$, $q \in Q = L^2(\Gamma_2)$ and $b \in B = H^{1/2}(\Gamma_1)$. These problems correspond to stationary Stefan problems [1,2]. Notice that mixed boundary conditions play an important role in several applications, e.g., heat conduction and electric potential problems [3].

The variational formulation of the elliptic problems (S) and (S_α), corresponding to (1), (2) and (1), (3), respectively, can be found in [2,4,5]. In general, solutions of mixed elliptic boundary value problems are not very regular [6], but there are cases in which they are regular [7–9]. Other theoretical optimization problems on the subject have been studied in [10,11].

We define the optimization problems (P_i) and $(P_{i\alpha})$ $i = 1, 2, 3$, associated to the systems (S) and (S_α), respectively (see [4,12–15]).

The distributed optimization problems (P_1) and $(P_{1\alpha})$ on the constant internal energy g are formulated as:

$$\text{find} \quad g_{op} \in \mathbb{R} \quad \text{such that} \quad J_1(g_{op}) = \min_{g\in\mathbb{R}} J_1(g) \tag{4}$$

$$\text{find} \quad g_{\alpha_{op}} \in \mathbb{R} \quad \text{such that} \quad J_{1\alpha}(g_{\alpha_{op}}) = \min_{g\in\mathbb{R}} J_{1\alpha}(g) \tag{5}$$

where $J_1 : \mathbb{R} \to \mathbb{R}_0^+$ and $J_{1\alpha} : \mathbb{R} \to \mathbb{R}_0^+$ are given by

$$J_1(g) = \frac{1}{2}\|u_g - z_d\|_H^2 + \frac{M_1}{2}\|g\|_H^2, \qquad J_{1\alpha}(g) = \frac{1}{2}\|u_{\alpha g} - z_d\|_H^2 + \frac{M_1}{2}\|g\|_H^2 \tag{6}$$

with $M_1 \in \mathbb{R}^+$ and $z_d \in \mathbb{R}$. For each $g \in \mathbb{R}$, u_g and $u_{\alpha g}$ denote the unique solutions to the systems (S) and (S_α), respectively, for given data $q \in \mathbb{R}$ and $b \in \mathbb{R}$. Here and throughout this section, $\|\cdot\|_H$ denotes the standard $L^2(\Omega)$ norm.

The boundary optimization problems (P_2) and $(P_{2\alpha})$ on the constant heat flux q on Γ_2 are defined as:

$$\text{find} \quad q_{op} \in \mathbb{R} \quad \text{such that} \quad J_2(q_{op}) = \min_{q\in\mathbb{R}} J_2(q) \tag{7}$$

$$\text{find} \quad q_{\alpha_{op}} \in \mathbb{R} \quad \text{such that} \quad J_{2\alpha}(q_{\alpha_{op}}) = \min_{q\in\mathbb{R}} J_{2\alpha}(q) \tag{8}$$

where $J_2 : \mathbb{R} \to \mathbb{R}_0^+$ and $J_{2\alpha} : \mathbb{R} \to \mathbb{R}_0^+$ are given by

$$J_2(q) = \frac{1}{2}\|u_q - z_d\|_H^2 + \frac{M_2}{2}\|q\|_Q^2, \qquad J_{2\alpha}(q) = \frac{1}{2}\|u_{\alpha q} - z_d\|_H^2 + \frac{M_2}{2}\|q\|_Q^2 \tag{9}$$

with $M_2 \in \mathbb{R}^+$ and $z_d \in \mathbb{R}$. For each $q \in \mathbb{R}$, we denote with u_q and $u_{\alpha q}$ the unique solutions to the systems (S) and (S_α) respectively, for data $g \in \mathbb{R}$ and $b \in \mathbb{R}$. Here and throughout this section, $\|\cdot\|_Q$ denotes the standard $L^2(\Gamma_2)$ norm.

The boundary optimization problems (P_3) and $(P_{3\alpha})$ on the constant temperature b in an external neighborhood of Γ_1 are set as

$$\text{find} \quad b_{op} \in \mathbb{R} \quad \text{such that} \quad J_3(b_{op}) = \min_{b\in\mathbb{R}} J_3(b) \tag{10}$$

$$\text{find} \quad b_{\alpha_{op}} \in \mathbb{R} \quad \text{such that} \quad J_{3\alpha}(b_{\alpha_{op}}) = \min_{b\in\mathbb{R}} J_{3\alpha}(b) \tag{11}$$

where $J_3 : \mathbb{R} \to \mathbb{R}_0^+$ and $J_{3\alpha} : \mathbb{R} \to \mathbb{R}_0^+$, given by

$$J_3(b) = \frac{1}{2}\|u_b - z_d\|_H^2 + \frac{M_3}{2}\|b\|_B^2, \qquad J_{3\alpha}(b) = \frac{1}{2}\|u_{\alpha b} - z_d\|_H^2 + \frac{M_3}{2}\|b\|_B^2 \tag{12}$$

with $M_3 \in \mathbb{R}^+$ and $z_d \in \mathbb{R}$. For every $b \in \mathbb{R}$, the functions u_b and $u_{\alpha b}$ are the unique solutions of systems (S) and (S_α) respectively, for data $g \in \mathbb{R}$ and $q \in \mathbb{R}$. Here and throughout this section, $\|\cdot\|_B$ denotes the standard norm in $B = H^{1/2}(\Gamma_1)$.

In [16], explicit solutions to the continuous systems (S) and (S_α) were derived, together with the associated optimization problems (P_i) and $(P_{i\alpha})$ for $i = 1, 2, 3$, in the particular case where the domain is a rectangle. These explicit solutions serve as a rigorous benchmark for assessing the accuracy and reliability of numerical methods.

The aim of this paper is three-fold: (i) to obtain explicit solutions to the systems (S) and (S_α) in a rectangular domain; (ii) to derive explicit discrete solutions for the optimization problems (P_i) and $(P_{i\alpha})$, $i = 1, 2, 3$, using finite difference methods; and (iii) to estimate the order of convergence of the discrete solutions by comparison with the exact explicit ones.

It is worth mentioning that there are several articles available in the literature that obtain explicit discrete solutions of some optimization problems [17,18]. For example, in [19], exact formulas are derived for the solution of an optimal boundary control problem governed by the one-dimensional heat equation where the control function measures the distance of the final state from the target. In [20] a finite element approximation is applied for some kind of parabolic optimal control problems with Neumann boundary conditions. Some numerical experiments are carried out setting a rectangular domain.

This paper is organized as follows: in Section 2 we obtain the discrete explicit solution to the systems (S) and (S_α) by the finite difference method. In Section 3, we obtain explicit discrete solutions to the discrete distributed optimization problems associated with (P_1) and $(P_{1\alpha})$, respectively, where the variable is the internal energy g. In Section 4, we define discrete boundary optimization problems where the variable is the heat flux q, associated with (P_2) and $(P_{2\alpha})$, respectively, obtaining the discrete explicit solutions. In the same manner, in Section 5, we derive explicit discrete solutions to the discrete boundary optimal control problems associated with (P_3) and $(P_{3\alpha})$, respectively, where the optimization variable is b. In all cases, when the step discretization goes to zero, convergence results are obtained by also estimating the order of convergence of the approximate solutions. In Section 6, we carry out some numerical simulations in order to illustrate the theoretical convergence results obtained in the previous sections. Finally, in Section 7, we analyze the order of convergence of the discrete systems associated with (S) and (S_α) by considering a modified approximation of the Neumann boundary condition on Γ_2, which leads to an improved convergence order.

The explicit continuous solutions of the systems and the associated optimal control problems in a rectangular domain are already available in the literature; in particular, they are given in [16].

The novelty of the present work can be summarized as follows: (i) the derivation of explicit discrete solutions for the state and the control variables; (ii) a rigorous analysis of the convergence of the discrete solutions, including the estimation of their orders of convergence; and (iii) an improved approximation of the boundary conditions in the discrete framework.

2. Discrete Systems for (S) and (S_α)

In this section we obtain the discrete explicit solutions to the systems (S) and (S_α) in a rectangular domain in the plane $\Omega = (0, x_0) \times (0, y_0)$ with $x_0 > 0$ and $y_0 > 0$. Its boundaries Γ_i for $i = 1, 2, 3$ are defined by:

$$\Gamma_1 = \{(0,y) : y \in (0,y_0]\}, \quad \Gamma_2 = \{(x_0,y) : y \in (0,y_0]\}$$

and

$$\Gamma_3 = \{(x,0) : x \in [0,x_0]\} \cup \{(x,y_0) : x \in [0,x_0]\}.$$

According to [16], the continuous solutions, in Ω, for the systems (S) and (S_α) defined by (1), (2) and (1), (3) are given by:

$$\begin{aligned} u(x,y) &= -\tfrac{1}{2}gx^2 + (gx_0 - q)x + b, \qquad \forall\ (x,y) \in \Omega \\ u_\alpha(x,y) &= -\tfrac{1}{2}gx^2 + (gx_0 - q)x + \tfrac{1}{\alpha}(g\,x_0 - q) + b, \qquad \forall\ (x,y) \in \Omega. \end{aligned} \tag{13}$$

As a consequence of the symmetry of domain Ω and the boundary conditions, the solutions u and u_α of systems (S) and (S_α) are independent of variable y, and therefore, we work with one-dimensional problems.

Given $n \in \mathbb{N}$, we define:

$$h = \tfrac{x_0}{n}; \quad x_i = (i-1)\,h, \text{ for } i = 1,\dots,n+1, \quad u_i^h \approx u(x_i,y) \text{ for } i = 2,\dots,n+1. \tag{14}$$

Here, n is the number of subintervals of $[0,x_0]$, h is the uniform mesh size, $u_1^h = b$, and u_i^h denotes the discrete approximation of the temperature at the node x_i, $i = 2,\dots,n+1$. Since the temperature is constant along the y-direction, u_i^h approximates $u(x_i,y)$ for any $(x_i,y) \in \Omega$.

We apply the classical finite-difference method to the system (S) described by Equations (1) and (2). Since the boundary condition on Γ_1 prescribes $u(0,y) = b$, we immediately obtain $u_1 = b$.

For the interior nodes, we use the classical centered second-order finite-difference approximation:

$$\frac{\partial^2 u}{\partial x^2}(x_i,y) \approx \frac{u(x_{i+1},y) - 2u(x_i,y) + u(x_{i-1},y)}{h^2}, \qquad i = 2,3,\dots,n, \tag{15}$$

and from the differential Equation (1), we impose that

$$-gh^2 = u_{i+1}^h - 2u_i^h + u_{i-1}^h, \qquad i = 2,3,\dots,n. \tag{16}$$

To incorporate the Neumann boundary condition on Γ_2, we use a backward finite difference for the first derivative:

$$\frac{\partial u}{\partial x}(x_{n+1},y) \approx \frac{u(x_{n+1},y) - u(x_n,y)}{h}, \tag{17}$$

which, using the boundary condition $\frac{\partial u}{\partial x}(x_{n+1},y) = q$, leads to assuming that

$$-qh = u_{n+1}^h - u_n^h. \tag{18}$$

Taking into account (16) and (18), the resulting discretization leads to the discrete linear system (S^h)

$$A\,v^h = T^h, \tag{19}$$

where $v^h = (u_i^h)_{i=2,\dots,n+1} \in \mathbb{R}^n$ denotes the vector of unknowns, A is the associated tridiagonal coefficient matrix:

$$A = \begin{pmatrix} -2 & 1 & 0 & \dots & \dots & & 0 \\ 1 & -2 & 1 & 0 & \dots & & 0 \\ 0 & 1 & -2 & 1 & 0 & \dots & 0 \\ \vdots & & & & & & \\ 0 & \dots & 0 & 1 & -2 & 1 & 0 \\ 0 & \dots & \dots & 0 & 1 & -2 & 1 \\ 0 & \dots & \dots & \dots & 0 & -1 & 1 \end{pmatrix}_{n \times n} \quad (20)$$

and $T^h \in \mathbb{R}^n$ is the vector of independent terms:

$$T^h = \Big(-g\,h^2 - b, -gh^2, \dots, -gh^2, -q\,h \Big)^t. \quad (21)$$

The square matrix A is invertible and its inverse matrix is given by

$$A^{-1} = \begin{pmatrix} -1 & -1 & -1 & \dots & \dots & -1 & 1 \\ -1 & -2 & -2 & \dots & \dots & -2 & 2 \\ -1 & -2 & -3 & \dots & \dots & -3 & 3 \\ \vdots & \vdots & \vdots & & & \vdots & \vdots \\ -1 & -2 & -3 & \dots & & -(n-1) & n-1 \\ -1 & -2 & -3 & \dots & & -(n-1) & n \end{pmatrix}_{n \times n}.$$

Then, the linear system (S^h) has a unique solution:

$$u_i^h = b + h^2 g\Big((i-1)\,n - \tfrac{i(i-1)}{2}\Big) - (i-1)hq, \quad i = 2, \dots, n+1.$$

As $n = \frac{x_0}{h}$ and $u_1^h = b$, it follows that:

$$u_i^h = b + (i-1)\,h\,(g\,x_0 - q) - h^2 g\,\tfrac{i(i-1)}{2}, \quad i = 1, \dots, n+1.$$

Then, the continuous solution $u(x,y)$ of system (S) can be approximated by the piecewise linear interpolant $u^h(x,y)$ obtained from the nodal values computed by the finite difference scheme. More precisely, we define

$$u^h(x,y) = (gx_0 - q - h\,g\,i)x + h^2 g\Big(\tfrac{i(i-1)}{2}\Big) + b, \quad x \in [x_i,\, x_{i+1}],\ \ y \in [0, y_0], \quad (22)$$

with $i = 1, \dots, n$.

The following lemma shows that the discrete solution u^h provides a first-order accurate approximation of the exact solution u and its derivative with respect to x.

Lemma 1.

(a) *For every grid point (x_i, y) with $i = 1, \dots, n+1$, $y \in [0, y_0]$, the following comparison holds:*
  - (i) *if $g > 0$ then $u^h(x_i, y) \le u(x_i, y)$.*
  - (ii) *if $g < 0$, then $u^h(x_i, y) \ge u(x_i, y)$.*

(b) *The approximation error satisfies first-order estimates in the H-norm, namely,*

$$\|u - u^h\|_H \le C_1\,h, \qquad \text{and} \qquad \|\tfrac{\partial u}{\partial x} - \tfrac{\partial u^h}{\partial x}\|_H \le \widetilde{C_1}\,h,$$

where the constants C_1 and $\widetilde{C_1}$, which do not depend on h, are given by $C_1 = x_0\,|g|\,\sqrt{\frac{2}{15} x_0\,y_0}$ and $\widetilde{C_1} = |g|\sqrt{\frac{1}{3} x_0\,y_0}$.

Proof.

(a) From the functions u and u^h, given by (13) and (22), respectively, we have

$$u(x_i, y) - u^h(x_i, y) = \frac{g\,h^2\,(i-1)}{2}, \quad i = 1, \ldots, n+1.$$

(b) From the definition of the norm in space H and Formulas (13) and (22) for functions u and u^h, respectively, it follows that:

$$\begin{aligned} ||u - u^h||_H^2 &= y_0 \sum_{i=1}^{n} \int_{x_i}^{x_{i+1}} \big(u(x,y) - u^h(x,y)\big)^2 dx \\ &= \tfrac{1}{120} y_0\, h^5\, g^2\, n^3 \left(\tfrac{1}{n^2} + \tfrac{5}{n} + 10\right) \\ &\leq \tfrac{2}{15}\, y_0\, h^5\, g^2\, n^3 = \tfrac{2}{15}\, h^2 g^2\, x_0^3\, y_0 = C_1^2 h^2. \end{aligned}$$

The norm $||\frac{\partial u}{\partial x} - \frac{\partial u^h}{\partial x}||_H$ can be computed analogously. □

We next apply the classical finite-difference method to the system (S_α) defined by Equations (1) and (3). For each $n \in \mathbb{N}$, we set $h = \frac{x_0}{n}$ and denote by $u^h_{\alpha,i} \approx u_\alpha(x_i, y)$ the approximate value of u_α at (x_i, y) for $i = 1, \cdots, n+1$.

The Robin boundary condition on Γ_1 is approximated by a classical forward finite-difference scheme, namely,

$$\frac{u_\alpha(x_2, y) - u_\alpha(x_1, y)}{h} \approx \frac{\partial u_\alpha}{\partial x}(x_1, y). \tag{23}$$

Taking into account that $\frac{\partial u_\alpha}{\partial x}(x_1, y) = \alpha\big(u_\alpha(x_1, y) - b\big)$, we impose that

$$\frac{u^h_{\alpha,2} - u^h_{\alpha,1}}{h} = \alpha\big(u^h_{\alpha,1} - b\big). \tag{24}$$

Moreover, at the interior nodes we use the approximation given in (16), while the Neumann boundary condition at x_{n+1} is discretized according to (17).

Then, we obtain the linear system (S^h_α):

$$A_\alpha\, v^h_\alpha = T^h_\alpha,$$

where the vector of unknowns $v^h_\alpha \in \mathbb{R}^{n+1}$ is given by $v^h_\alpha = (u^h_{\alpha,i})_{i=1,\ldots n+1}$, the tridiagonal coefficient matrix A_α of order $n+1$ is defined as:

$$A_\alpha = \begin{pmatrix} -(1+\alpha\, h) & 1 & 0 & \ldots & \ldots & & 0 \\ 1 & -2 & 1 & & & & \\ \vdots & \ldots & \ldots & & 1 & -2 & 1 \\ 0 & 0 & 0 & \ldots & 0 & -1 & 1 \end{pmatrix}_{(n+1)\times(n+1)} \tag{25}$$

and

$$T^h_\alpha = \Big(-\alpha\, b\, h, -gh^2, \ldots, -gh^2, -q\, h\Big)^t \in \mathbb{R}^{n+1}. \tag{26}$$

It can be seen that the square matrix A_α is invertible and its inverse matrix is given by

$$A_\alpha^{-1} = \frac{1}{\alpha h}\begin{pmatrix} -1 & -1 & -1 & \dots & \dots & -1 & 1 \\ -1 & -(1+\alpha h) & -(1+\alpha h) & \dots & \dots & -(1+\alpha h) & 1+\alpha h \\ -1 & -(1+\alpha h) & -(1+2\alpha h) & \dots & \dots & -(1+2\alpha h) & -(1+2\alpha h) \\ \vdots & \vdots & \vdots & & & \vdots & \vdots \\ -1 & -(1+\alpha h) & -(1+2\alpha h) & \dots & & -(1+(n-1)\alpha h) & 1+(n-1)\alpha h \\ -1 & -(1+\alpha h) & -(1+2\alpha h) & \dots & & -(1+(n-1)\alpha h) & 1+n\alpha h \end{pmatrix}_{(n+1)\times(n+1)}$$

Then, the linear system (S_α^h) has a unique solution:

$$u_{\alpha,i}^h = (b + \tfrac{g\,x_0 - q}{\alpha}) + (i-1)\,(g\,x_0 - q)\,h - \tfrac{g}{\alpha}\,h - g\,\tfrac{i(i-1)}{2}\,h^2, \quad i = 1,\dots,n+1.$$

As a consequence, the continuous solution $u_\alpha(x,y)$ of system (S_α) given by (13) can be approximated in $\overline{\Omega}$ by the discrete function $u_\alpha^h(x,y)$, defined as the piecewise linear interpolation of the nodal values obtained from the finite-difference system (S_α^h).

$$u_\alpha^h(x,y) = (g\,x_0 - q - h\,g\,i)x + b + \tfrac{g\,x_0 - q}{\alpha} - \tfrac{g\,h}{\alpha} + \tfrac{i^2 - i}{2}\,g\,h^2, \tag{27}$$

for $x \in [x_i,\, x_{i+1}], y \in [0, y_0], i = 1,\dots,n.$

Notice that $u_\alpha^h(x,y) \to u^h(x,y)$ when $\alpha \to \infty$ for every $(x,y) \in \overline{\Omega}$.

Lemma 2. *Let u_α be the solution of problem (S_α), where $\alpha > 0$ is the convective heat transfer coefficient appearing in the Robin boundary condition, and let u_α^h denote its piecewise linear discrete approximation defined in (27). Then, for each mesh size h, the following error estimates hold:*

$$\|u_\alpha - u_\alpha^h\|_H \le C_{1\alpha}\,h, \qquad \text{and} \qquad \|\tfrac{\partial u_\alpha}{\partial x} - \tfrac{\partial u_\alpha^h}{\partial x}\|_H \le \widetilde{C_1}\,h,$$

where $C_{1\alpha} = |g|x_0\sqrt{x_0\,y_0\left(\frac{2}{15} + \frac{2}{3}\frac{1}{\alpha x_0} + \frac{1}{\alpha^2 x_0^2}\right)}$ *and* $\widetilde{C_1} = |g|\sqrt{\frac{1}{3}x_0\,y_0}$ *are positive constants independent of h.*

Proof. Taking into account that u_α and u_α^h are given by (13) and (27), respectively, we have

$$\begin{aligned} ||u_\alpha - u_\alpha^h||_H^2 &= y_0 \sum_{i=1}^{n} \int_{x_i}^{x_{i+1}} \left(u_\alpha(x,y) - u_\alpha^h(x,y)\right)^2 dx \\ &= y_0\,g^2 \sum_{i=1}^{n} \frac{h^5 i^2}{4} - \frac{h^5 i}{6} + \frac{h^5}{20} - \frac{h^4}{3\alpha} + \frac{h^4 i}{\alpha} + \frac{h^3}{\alpha^2} \\ &= y_0\,g^2\left[h^5 n^3\left(\tfrac{1}{12} + \tfrac{1}{24n} + \tfrac{1}{120n^2}\right) + \tfrac{h^4 n^2}{\alpha}\left(\tfrac{1}{2} + \tfrac{1}{6n}\right) + \tfrac{h^3 n}{\alpha^2}\right] \\ &\le x_0\,y_0\,g^2\,h^2\left(\tfrac{2}{15}x_0^2 + \tfrac{2}{3}\tfrac{x_0}{\alpha} + \tfrac{1}{\alpha^2}\right) = C_{1\alpha}^2 h^2. \end{aligned}$$

In addition, the partial derivatives with respect to x of functions u_α and u_α^h are given by:

$$\tfrac{\partial u_\alpha}{\partial x}(x,y) = \tfrac{\partial u}{\partial x}(x,y) = -gx + gx_0 - q, \qquad \forall (x,y) \in \Omega$$

and

$$\tfrac{\partial u_\alpha^h}{\partial x}(x,y) = \tfrac{\partial u^h}{\partial x}(x,y) = g\,x_0 - q - h\,g\,i, \qquad x \in [x_i,\, x_{i+1}], \quad y \in [0, y_0].$$

Then, the bound for $\|\frac{\partial u_\alpha}{\partial x} - \frac{\partial u_\alpha^h}{\partial x}\|_H$ coincides with the bound for $\|\frac{\partial u}{\partial x} - \frac{\partial u^h}{\partial x}\|_H$, obtained in Lemma 1. □

Remark 1. *Notice that $C_{1\alpha} \to C_1$ when $\alpha \to \infty$, where C_1 is the constant appearing in Lemma 1. This shows that the error bound associated with the convective boundary condition converges to the one obtained for the Dirichlet problem as $\alpha \to \infty$.*

3. Distributed Optimization Problem with Variable g

In this section we obtain discrete optimal solutions to the continuous optimization problem (P_1) and $(P_{1\alpha})$ in the rectangular domain Ω for the case where the optimization variable is g.

3.1. Discrete Problem (P_1^h) Associated with (P_1)

Taking into account that b, g, q and the desired state z_d in (6) are constants, according to [16], the continuous quadratic functional cost for problem (P_1) is explicitly given by:

$$J_1(g) = \frac{1}{2} x_0^3\, y_0\, q^2 \Big\{ g^2 \frac{x_0^2}{q^2} \Big(\frac{2}{15} + \frac{M_1}{x_0^4} \Big) + g \frac{x_0}{q} \Big(-\frac{5}{12} + \frac{2}{3} \frac{(b-z_d)}{q x_0} \Big) + \frac{1}{3} - \frac{(b-z_d)}{q x_0} + \frac{(b-z_d)^2}{q^2 x_0^2} \Big\}. \quad (28)$$

Then, the solution to the distributed optimization problem (P_1) is g_{op} defined by:

$$g_{op} = \frac{q}{3\, x_0} \frac{\Big(\frac{5}{8} - \frac{(b-z_d)}{q\, x_0} \Big)}{\Big(\frac{2}{15} + \frac{M_1}{x_0^4} \Big)} \quad (29)$$

and the continuous optimization state when $g = g_{op}$ is

$$u_{g_{op}}(x, y) = -\frac{1}{2} g_{op}\, x^2 + (g_{op}\, x_0 - q) x + b. \quad (30)$$

We define the discrete distributed optimization problem (P_1^h) for the constant internal energy g as

$$\text{find} \quad g_{op}^h \in \mathbb{R} \quad \text{such that} \quad J_1^h(g_{op}^h) = \min_{g \in \mathbb{R}} J_1^h(g)$$

where the discrete cost function J_1^h is given by:

$$J_1^h(g) = \frac{1}{2} \| u_g^h - z_d \|_H^2 + \frac{1}{2} M_1 \| g \|_H^2.$$

Here, u_g^h denotes the discrete approximation corresponding to the internal energy g (see (22)), h is the discretization step defined in (14), z_d is the desired target, and $M_1 > 0$ is a regularization parameter.

Taking into account that the variable g is constant results in:

$$J_1^h(g) = \frac{1}{2} y_0 \Big\{ M_1\, g^2\, x_0 + \sum_{i=1}^{n} \int_{x_i}^{x_{i+1}} \Big(u_g^h(x, y) - z_d \Big)^2 dx \Big\} \quad (31)$$

and from algebraic work, it follows that

$$\begin{aligned} J_1^h(g) \quad &= \frac{1}{2} x_0^3\, y_0\, q^2 \Bigg\{ g^2 \frac{x_0^2}{q^2} \Big[\frac{2}{15} + \frac{M_1}{x_0^4} + \frac{1}{180} \frac{h^4}{x_0^4} + \frac{1}{24} \frac{h^3}{x_0^3} + \frac{1}{36} \frac{h^2}{x_0^2} - \frac{5}{24} \frac{h}{x_0} \Big] \\ &+ g \frac{x_0}{q} \Big[-\frac{5}{12} + \frac{2}{3} \frac{(b-z_d)}{q x_0} + \frac{h}{x_0} \Big(\frac{1}{3} + \frac{1}{12} \frac{h}{x_0} \Big) - \frac{h}{x_0} \frac{(b-z_d)}{q x_0} \Big(\frac{1}{2} + \frac{1}{6} \frac{h}{x_0} \Big) \Big] \\ &+ \frac{1}{3} - \frac{(b-z_d)}{q\, x_0} + \frac{(b-z_d)^2}{q^2\, x_0^2} \Bigg\}. \end{aligned} \quad (32)$$

Lemma 3. *For any given internal energy $g \in \mathbb{R}$, the following estimate holds for the discrete cost functional J_1^h:*

$$|J_1(g) - J_1^h(g)| \approx C_2\, h \quad (33)$$

where $C_2 = \frac{1}{2} x_0^3\, y_0\, g\, q \Big| -\frac{5}{24} \frac{g x_0}{q} + \frac{1}{3} - \frac{1}{2} \frac{(b-z_d)}{q x_0} \Big|$ is a constant independent of h.

Proof. From (28) and (32) we get

$$
\begin{aligned}
J_1^h(g) - J_1(g) &= \tfrac{1}{2}x_0^3\, y_0\, q^2 \Big\{ g^2 \tfrac{x_0^2}{q^2} \Big[\tfrac{1}{180}\tfrac{h^4}{x_0^4} + \tfrac{1}{24}\tfrac{h^3}{x_0^3} + \tfrac{1}{36}\tfrac{h^2}{x_0^2} - \tfrac{5}{24}\tfrac{h}{x_0} \Big] \\
&\quad + g\,\tfrac{h}{q}\Big[\tfrac{1}{3} + \tfrac{1}{12}\tfrac{h}{x_0} - \tfrac{(b-z_d)}{qx_0}\Big(\tfrac{1}{2} + \tfrac{1}{6}\tfrac{h}{x_0}\Big)\Big]\Big\} \\
&\approx \tfrac{1}{2}x_0^3\, y_0\, q^2 \Big[-\tfrac{5}{24}\tfrac{g^2 x_0}{q^2} + \tfrac{g}{q}\Big(\tfrac{1}{3} - \tfrac{1}{2}\tfrac{(b-z_d)}{qx_0}\Big)\Big] h \\
&= \tfrac{1}{2}x_0^3\, y_0\, g q \Big[-\tfrac{5}{24}\tfrac{g x_0}{q} + \tfrac{1}{3} - \tfrac{1}{2}\tfrac{(b-z_d)}{qx_0}\Big] h.
\end{aligned}
$$

Therefore, we obtain (33). □

From the optimality condition we obtain the following result:

Lemma 4.

(a) *The explicit expression for the optimal variable g_{op}^h is given by:*

$$
g_{op}^h = \frac{q}{3\,x_0} \frac{A_1 + \frac{h}{x_0}A_2 + \frac{h^2}{x_0^2}A_3}{A_4 + A_5(h)}, \tag{34}
$$

where

$$
\begin{aligned}
A_1 &= \tfrac{5}{8} - \tfrac{b-z_d}{qx_0}, & A_4 &= \tfrac{2}{15} + \tfrac{M_1}{x_0^4}, \\
A_2 &= \tfrac{3(b-z_d)}{4qx_0} - \tfrac{1}{2}, & A_5(h) &= \tfrac{h}{12\,x_0}\Big(\tfrac{h^3}{15\,x_0^3} + \tfrac{h^2}{2\,x_0^2} + \tfrac{h}{3\,x_0} - \tfrac{5}{2}\Big). \\
A_3 &= \tfrac{b-z_d}{4\,qx_0} - \tfrac{1}{8}, & &
\end{aligned} \tag{35}
$$

(b) *In addition, the following error estimates hold:*

$$
|g_{op} - g_{op}^h| \approx C_3\, h, \tag{36}
$$

$$
\Big| J_1(g_{op}) - J_1^h(g_{op}^h) \Big| \approx C_4\, h, \tag{37}
$$

where C_3 and C_4 do not depend on h.

Proof.

(a) It follows immediately from the expression of the derivative of J_1^h with respect to g.

(b) Rewriting g_{op} given by (29) as: $g_{op} = \frac{q}{3\,x_0}\frac{A_1}{A_4}$, it follows that:

$$
g_{op}^h - g_{op} = \frac{q}{3\,x_0} \frac{-A_1\,A_5(h) + \frac{h}{x_0}A_2\,A_4 + \frac{h^2}{x_0^2}A_3\,A_4}{A_4^2 + A_4\,A_5(h)} \approx \frac{q}{3\,x_0^2}\frac{A_2A_4 + \frac{5}{24}A_1}{A_4^2} h + o(h^2),
$$

and we obtain (36) with $C_3 = |C_3^*|$ where

$$
C_3^* = \frac{q}{3\,x_0^2}\frac{A_2\,A_4 + \frac{5}{24}A_1}{A_4^2}. \tag{38}
$$

From the expressions for $J_1(g)$ at $g = g_{op}$ and $J_1^h(g)$ at $g = g_{op}^h$, it follows that:

$$
\begin{aligned}
J_1(g_{op}) - J_1^h(g_{op}^h) &= \tfrac{1}{2}x_0^3\, y_0\, q^2 \Big[\tfrac{x_0^2}{q^2} A_4 (g_{op}^2 - (g_{op}^h)^2) - \tfrac{2}{3}\tfrac{x_0}{q} A_1 (g_{op} - g_{op}^h) \\
&\quad - (g_{op}^h)^2 \tfrac{x_0^2}{q^2} A_5(h) + \tfrac{2}{3} g_{op}^h \tfrac{h}{q}\Big(A_2 + \tfrac{h}{x_0}A_3\Big)\Big].
\end{aligned}
$$

By using (34) and (36) we get (37) where

$$
\begin{aligned}
C_4 &= \tfrac{1}{2} x_0^2 y_0 q^2 \left| -2A_4 C_3^* g_{op} \tfrac{x_0^3}{q^2} + \tfrac{5}{24} g_{op}^2 \tfrac{x_0^2}{q^2} + \tfrac{2}{3} A_1 C_3^* \tfrac{x_0^2}{q} + \tfrac{2}{3} A_2 g_{op} \tfrac{x_0}{q} \right| \\
&= \tfrac{1}{2} x_0^2 y_0 q^2 \left| \tfrac{A_1(5A_1+48A_2A_4)}{216A_4^2} \right|.
\end{aligned}
$$

□

Lemma 5. *Let us consider $u_{g_{op}}$ the solution of the system (S) given by (1) and (2) for $g = g_{op}$ and $u^h_{g^h_{op}}$ the discrete solution defined by (22) for $h > 0$ and for $g = g^h_{op}$, where g^h_{op} is the optimal value of the problem (P^h_1) given by (34). We have:*

$$
(a) \quad \|u_{g_{op}} - u^h_{g^h_{op}}\|_H \approx C_5\, h, \qquad\qquad (b) \quad \left\| \frac{\partial u_{g_{op}}}{\partial x} - \frac{\partial u^h_{g^h_{op}}}{\partial x} \right\|_H \approx C_6\, h
$$

where C_5 and C_6 are positive constants that are independent of parameter h.

Proof.

(a) From the definition of the norm in H, we obtain

$$
\begin{aligned}
||u_{g_{op}} - u^h_{g^h_{op}}||^2_H &= y_0 \sum_{i=1}^{n} \int_{x_i}^{x_{i+1}} \Big(u_{g_{op}}(x,y) - u^h_{g^h_{op}}(x,y) \Big)^2 dx \\
&= y_0 \sum_{i=1}^{n} \int_{x_i}^{x_{i+1}} \Big[-\tfrac{1}{2} g_{op} x^2 + x_0 x (g_{op} - g^h_{op}) + h g^h_{op} (ix - h \tfrac{i(i-1)}{2}) \Big]^2 dx \\
&= \tfrac{1}{120} x_0^3\, y_0\, g_{op}^2 \left[10 - 25\, \tfrac{x_0\, C_3^*}{g_{op}} + 16 \Big(\tfrac{x_0\, C_3^*}{g_{op}} \Big)^2 \right] h^2 + o(h^3).
\end{aligned}
$$

where C_3^* is given by (38). Therefore, it follows that

$$
\|u_{g_{op}} - u^h_{g^h_{op}}\|_H \approx C_5\, h,
$$

with

$$
C_5 = |g_{op}| \sqrt{\tfrac{1}{120} x_0^3\, y_0 \, \left[10 - 25\, \tfrac{x_0\, C_3^*}{g_{op}} + 16 \Big(\tfrac{x_0\, C_3^*}{g_{op}} \Big)^2 \right]}.
$$

(b) We have

$$
\begin{aligned}
&\left\| \frac{\partial u_{g_{op}}}{\partial x} - \frac{\partial u^h_{g^h_{op}}}{\partial x} \right\|^2_H \\
&= y_0 \sum_{i=1}^{n} \int_{x_i}^{x_{i+1}} \Big(-g_{op} x + h g^h_{op} i + x_0 (g_{op} - g^h_{op}) \Big)^2 dx \\
&= \tfrac{x_0 y_0}{6} \left[2 g_{op}^2 x_0^2 + g_{op}\, g^h_{op} (h^2 + 3 h x_0 - 4 x_0^2) + (g^h_{op})^2 (h^2 - 3 h x_0 + 2 x_0^2) \right] \\
&= \tfrac{x_0 y_0}{6} \left[2 x_0^2 (g_{op} - g^h_{op})^2 + 3h\, x_0\, g^h_{op}\, (g_{op} - g^h_{op}) + h^2 g^h_{op} (g_{op} + g^h_{op}) \right].
\end{aligned}
$$

Taking into account Lemma 4, we get

$$
\left\| \frac{\partial u_{g_{op}}}{\partial x} - \frac{\partial u^h_{g^h_{op}}}{\partial x} \right\|^2_H = \tfrac{x_0 y_0}{6}\, g_{op}^2 \left[2 + 3\, \tfrac{x_0 C_3^*}{g_{op}} + 2 \Big(\tfrac{x_0 C_3^*}{g_{op}} \Big)^2 \right] h^2 + o(h^3).
$$

Then,

$$
\left\| \frac{\partial u_{g_{op}}}{\partial x} - \frac{\partial u^h_{g^h_{op}}}{\partial x} \right\|_H \approx C_6 h,
$$

with
$$C_6 = |g_{op}| \sqrt{\frac{x_0 y_0}{6} \left[2 - 3\,\frac{x_0 C_3^*}{g_{op}} + 2 \left(\frac{x_0 C_3^*}{g_{op}} \right)^2 \right]},$$
where C_3^* is given by (38).

□

3.2. Discrete Problem $(P_{1\alpha}^h)$ Associated with $(P_{1\alpha})$

From [16], we know that the continuous quadratic functional cost in (6) for the optimization problem $(P_{1\alpha})$ is explicitly given by:

$$\begin{aligned} &J_{1\alpha}(g) = J_1(g) \\ &+ \frac{x_0^2 y_0 q^2}{2\alpha} \left\{ \frac{g^2 x_0^2}{q^2} \left(\frac{2}{3} + \frac{1}{\alpha x_0} \right) + \frac{g x_0}{q} \left(-\frac{5}{3} - \frac{2}{\alpha x_0} + \frac{2(b-z_d)}{q x_0} \right) + 1 + \frac{1}{\alpha x_0} - \frac{2(b-z_d)}{q x_0} \right\}, \end{aligned} \tag{39}$$

where J_1 is defined by (28). Moreover, the continuous optimal distributed variable denoted by $g_{\alpha_{op}}$ is given by

$$g_{\alpha_{op}} = \frac{q}{3\,x_0} \frac{\left(\frac{5}{8} - \frac{(b-z_d)}{q\,x_0} + \frac{5}{2\alpha x_0} + \frac{3}{\alpha^2 x_0^2} - \frac{3(b-z_d)}{\alpha q x_0^2} \right)}{\left(\frac{2}{15} + \frac{M_1}{x_0^4} + \frac{2}{3\alpha x_0} + \frac{1}{\alpha^2 x_0^2} \right)}. \tag{40}$$

The continuous associated state is established by:

$$u_{\alpha g_{\alpha_{op}}}(x,y) = -\frac{1}{2} g_{\alpha_{op}} x^2 + (g_{\alpha_{op}} x_0 - q)x + \frac{1}{\alpha}(g_{\alpha_{op}}\, x_0 - q) + b. \tag{41}$$

We define the discrete cost function as

$$J_{1\alpha}^h(g) = \frac{1}{2} \| u_{\alpha g}^h - z_d \|_H^2 + \frac{1}{2} M_1 \|g\|_H^2, \tag{42}$$

where function $u_{\alpha g}^h$, given in (27), denotes the discrete approximation corresponding to the internal energy g, $h > 0$ is the discretization step, z_d is the desired target, and $M_1 > 0$ is a constant parameter. We set the following discrete optimization problem $(P_{1\alpha}^h)$ on the constant internal energy g as

$$\text{find} \quad g_{\alpha_{op}}^h \in \mathbb{R} \quad \text{such that} \quad J_{1\alpha}^h(g_{\alpha_{op}}^h) = \min_{g \in \mathbb{R}} J_{1\alpha}^h(g).$$

The discrete cost function $J_{1\alpha}^h$ is explicitly given by

$$\begin{aligned} &J_{1\alpha}^h(g) = J_{1\alpha}(g) \\ &+ \frac{1}{2} x_0^3\, y_0\, g\, q\, h \left\{ \frac{g x_0}{q} \left[-\frac{5}{24} + \frac{1}{36}\frac{h}{x_0} + \frac{1}{24}\frac{h^2}{x_0^2} + \frac{1}{180}\frac{h^3}{x_0^3} + \frac{1}{\alpha x_0} \left(-\frac{7}{6} + \frac{1}{3}\frac{h}{x_0} + \frac{1}{6}\frac{h^2}{x_0^2} \right) \right. \right. \\ &\left. + \frac{1}{\alpha^2 x_0^2} \left(-2 + \frac{h}{x_0} \right) \right] + \frac{1}{3} + \frac{1}{12}\frac{h}{x_0} + \frac{1}{\alpha x_0} \left(\frac{3}{2} + \frac{1}{6}\frac{h}{x_0} \right) + \frac{2}{\alpha^2 x_0^2} \\ &\left. + \frac{(b-z_d)}{q x_0} \left(-\frac{1}{2} - \frac{1}{6}\frac{h}{x_0} - \frac{2}{\alpha x_0} \right) \right\}, \end{aligned} \tag{43}$$

where $J_{1\alpha}$ is given by (39).

Lemma 6. *For $g \in H,\ h > 0$ and $\alpha > 0$, the following estimate holds*

$$|J_{1\alpha}^h(g) - J_{1\alpha}(g)| \approx C_{2\alpha}\, h$$

where

$$C_{2\alpha} = \frac{1}{2}x_0^3\,y_0\,|g|\left|\frac{gx_0}{q}\Big(-\frac{5}{24}-\frac{7}{6}\frac{1}{\alpha x_0}-\frac{2}{\alpha^2 x_0^2}\Big)+\frac{1}{3}+\frac{3}{2}\frac{1}{\alpha x_0}+\frac{2}{\alpha^2 x_0^2}+\frac{(b-z_d)}{qx_0}\Big(-\frac{1}{2}-\frac{2}{\alpha x_0}\Big)\right|,$$

is a constant independent of h.

Proof. It follows immediately from expression (43). □

Remark 2. *$C_{2\alpha} \to C_2$ when $\alpha \to \infty$, where C_2 is given in Lemma 3.*

Lemma 7.

(a) *The explicit expression for the optimal control $g^h_{\alpha_{op}}$ is given by:*

$$g^h_{\alpha_{op}} = \frac{q}{3\,x_0}\frac{A_{1\alpha}+\frac{h}{x_0}\,A_{2\alpha}+\frac{h^2}{x_0^2}\,A_{3\alpha}}{A_{4\alpha}+A_{5\alpha}(h)}, \tag{44}$$

where

$$\begin{aligned}
A_{1\alpha} &= \frac{5}{8}-\frac{b-z_d}{qx_0}+\frac{1}{\alpha x_0}\left(\frac{5}{2}+\frac{3}{\alpha x_0}-\frac{3(b-z_d)}{qx_0}\right),\\
A_{2\alpha} &= \frac{3(b-z_d)}{4qx_0}-\frac{1}{2}+\frac{1}{\alpha x_0}\left(-\frac{9}{4}-\frac{3}{\alpha x_0}+\frac{3(b-z_d)}{qx_0}\right),\\
A_{3\alpha} &= \frac{b-z_d}{4\,qx_0}-\frac{1}{8}-\frac{1}{4\alpha x_0},\\
A_{4\alpha} &= \frac{2}{15}+\frac{M_1}{x_0^4}+\frac{1}{\alpha x_0}\left(\frac{2}{3}+\frac{1}{\alpha x_0}\right),\\
A_{5\alpha}(h) &= \frac{h}{12\,x_0}\left(\frac{h^3}{15\,x_0^3}+\frac{h^2}{2\,x_0^2}+\frac{h}{3\,x_0}-\frac{5}{2}\right)\\
&\quad+\frac{h}{\alpha x_0^2}\left(-\frac{7}{6}+\frac{h}{3x_0}+\frac{h^2}{6x_0^2}+\frac{1}{\alpha x_0}\left(-2+\frac{h}{x_0}\right)\right).
\end{aligned} \tag{45}$$

(b) *In addition, the following error estimates hold:*

$$|g_{\alpha_{op}} - g^h_{\alpha_{op}}| \approx C_{3\alpha}\,h, \tag{46}$$

$$\left|J_{1\alpha}(g_{\alpha_{op}}) - J^h_{1\alpha}(g^h_{\alpha_{op}})\right| \approx C_{4\alpha}\,h, \tag{47}$$

where $C_{3\alpha}$ and $C_{4\alpha}$ do not depend on h.

Proof.

(a) It follows immediately from the fact that

$$J^h_{1\alpha}(g) = \frac{1}{2}x_0^3y_0q^2\left[2g\frac{x_0^2}{q^2}\Big(A_{4\alpha}+A_{5\alpha}(h)\Big)-\frac{2}{3}\frac{x_0}{q}\Big(A_{1\alpha}+\frac{h}{x_0}A_{2\alpha}+\frac{h^2}{x_0^2}A_{3\alpha}\Big)\right].$$

(b) Notice that $g_{\alpha_{op}}$ given by (40) can be rewriten as $g_{\alpha_{op}} = \frac{q}{3x_0}\frac{A_{1\alpha}}{A_{4\alpha}}$. Then,

$$\begin{aligned}
g^h_{\alpha_{op}} - g_{\alpha_{op}} &= \frac{q}{3\,x_0}\frac{-A_{1\alpha}\,A_{5\alpha}(h)+\frac{h}{x_0}\,A_{2\alpha}\,A_{4\alpha}+\frac{h^2}{x_0^2}\,A_{3\alpha}\,A_{4\alpha}}{A_{4\alpha}^2+A_{4\alpha}\,A_{5\alpha}(h)}\\
&\approx \frac{q}{3\,x_0^2}\frac{A_{2\alpha}A_{4\alpha}+\frac{5}{24}A_{1\alpha}}{A_{4\alpha}^2}h+o(h^2).
\end{aligned}$$

and we obtain (46) with $C_{3\alpha} = |C^*_{3\alpha}|$, where

$$C^*_{3\alpha} = \frac{q}{3\,x_0^2}\,\frac{A_{2\alpha}\,A_{4\alpha} + \left(\frac{5}{24} + \frac{7}{6\alpha x_0}\right)A_{1\alpha}}{A_{4\alpha}^2}. \tag{48}$$

Following Lemma 4 we obtain Formula (47) with

$$\begin{aligned} C_{4\alpha} &= \tfrac{1}{2}x_0^2 y_0 q^2 \Big| -2A_{4\alpha}C^*_{3\alpha}g_{\alpha_{op}}\frac{x_0^3}{q^2} + \tfrac{5}{24}g^2_{\alpha_{op}}\frac{x_0^2}{q^2} \\ &\quad + \tfrac{7}{6}g^2_{\alpha_{op}}\frac{x_0}{\alpha q^2} + \tfrac{2}{3}A_{1\alpha}C^*_{3\alpha}\frac{x_0^2}{q} + \tfrac{2}{3}A_{2\alpha}g_{\alpha_{op}}\frac{x_0}{q}\Big|. \end{aligned}$$

□

Remark 3. *When $\alpha \to \infty$, we have $A_{i\alpha} \to A_i$, where A_i and $A_{i\alpha}$ are given by (35) and (45), respectively, for $i = 1, 2, \cdots, 5$. As an immediate consequence it follows that $C_{3\alpha} \to C_3$ and $C_{4\alpha} \to C_4$ when $\alpha \to \infty$, where C_3 and C_4 are defined in Lemma 4.*

Lemma 8. *Let us consider $u_{\alpha g_{\alpha_{op}}}$, the function given by (13) for $g = g_{\alpha_{op}}$ where $g_{\alpha_{op}}$ is the optimal variable of problem $(P_{1\alpha})$ given by (40), and $u^h_{\alpha g^h_{\alpha_{op}}}$, the function defined by (27) for $h > 0$ where $g = g^h_{\alpha_{op}}$ is the optimal control of $(P^h_{1\alpha})$ given by (44). We have:*

$$(a)\quad \|u_{\alpha g_{\alpha_{op}}} - u^h_{\alpha g^h_{\alpha_{op}}}\|_H \approx C_{5\alpha}\,h, \qquad (b)\quad \left\|\frac{\partial u_{\alpha g_{\alpha_{op}}}}{\partial x} - \frac{\partial u^h_{\alpha g^h_{\alpha_{op}}}}{\partial x}\right\|_H \approx C_{6\alpha}\,h,$$

where $C_{5\alpha}$ and $C_{6\alpha}$ are positive constants independent of parameter h.

Proof. Working algebraically we can obtain

$$\begin{aligned} C_{5\alpha} = |g_{\alpha_{op}}|\Bigg\{ &\frac{1}{120}x_0^3\,y_0\;\Bigg[10 - 25\frac{x_0\,C^*_{3\alpha}}{g_{op}} + 16\left(\frac{x_0\,C^*_{3\alpha}}{g_{op}}\right)^2 \\ &+ \frac{1}{\alpha x_0}\left(60 + \frac{120}{\alpha x_0} - 240\frac{C^*_{3\alpha}}{\alpha g_{\alpha_{op}}} + 120\frac{C^*_{3\alpha}x_0}{\alpha g^2_{\alpha_{op}}} - 140\frac{C^*_{3\alpha}x_0}{g_{\alpha_{op}}} + 80\frac{C^{*2}_{3\alpha}x_0^2}{g^2_{\alpha_{op}}}\right)\Bigg]\Bigg\}^{1/2}, \end{aligned}$$

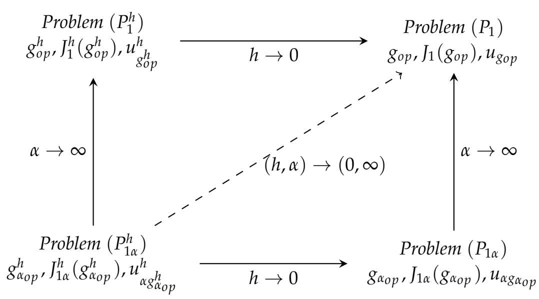
and

$$C_{6\alpha} = |g_{\alpha_{op}}|\sqrt{\frac{x_0 y_0}{6}\left[2 - 3\,\frac{x_0 C^*_{3\alpha}}{g_{\alpha_{op}}} + 2\left(\frac{x_0 C^*_{3\alpha}}{g_{\alpha_{op}}}\right)^2\right]},$$

where $C^*_{3\alpha}$ is given by (48). □

Remark 4. *$C_{5\alpha} \to C_5$ and $C_{6\alpha} \to C_6$ when $\alpha \to \infty$, where C_5 and C_6 are given in Lemma 5.*

Remark 5. *In [21] the double convergence when $(h, \alpha) \to (0, +\infty)$ of optimal control problem $(P^h_{1\alpha})$ was studied, obtaining a commutative diagram that relates the continuous and discrete optimal control problems (P_1), $(P_{1\alpha})$, (P^h_1) and $(P^h_{1\alpha})$ as in the following scheme:*



4. Boundary Optimization Problem with Variable q

4.1. Discrete Problem (P_2^h) Associated with (P_2)

Under the same considerations given in Section 3.1 and taking into account Formula (9), for a given $q \in Q$, we obtain the following quadratic cost function:

$$
\begin{aligned}
J_2(q) &= \tfrac{x_0 y_0}{2}\Big\{q^2 x_0^2\Big(\tfrac{1}{3}+\tfrac{M_2}{x_0^3}\Big)+q x_0\Big(-\tfrac{5}{12}g x_0^2-(b-z_d)\Big)\\
&+\tfrac{2}{15}g^2 x_0^4+(b-z_d)^2+\tfrac{2}{3}g x_0^2(b-z_d)\Big\}.
\end{aligned} \tag{49}
$$

Then, the boundary optimal control of problem (P_2), called q_{op}, and the associated continuous optimal state are given by:

$$
q_{op} = \frac{\frac{5}{12}g x_0^2 + (b-z_d)}{2x_0\big(\frac{1}{3}+\frac{M_2}{x_0^3}\big)}, \qquad u_{q_{op}}(x,y) = -\tfrac{1}{2}g\,x^2 + (g\,x_0 - q_{op})x + b. \tag{50}
$$

Associated with (P_2), we define the approximate discrete distributed optimal control problem (P_2^h) on the constant heat flux q as

$$
\text{find} \quad q_{op}^h \in \mathbb{R} \quad \text{such that} \quad J_2^h(q_{op}^h) = \min_{q\in\mathbb{R}} J_2^h(q)
$$

where the discrete cost function J_2^h is defined by

$$
J_2^h(q) = \tfrac{1}{2}\|u_q^h - z_d\|_H^2 + \tfrac{1}{2}M_2\|q\|_Q^2,
$$

where u_q^h, given in (22), denotes the discrete approximation for a fixed constant flux q, h is the spatial step, and z_d (the desired state). From the definition of the norm over Q, it results that:

$$
J_2^h(q) = \tfrac{1}{2}y_0\Big\{M_2\,q^2\,x_0 + \sum_{i=1}^{n}\int_{x_i}^{x_{i+1}}[u_q^h(x,y)-z_d]^2\,dx\Big\}
$$

and working algebraically, we get

$$
\begin{aligned}
J_2^h(q) = J_2(q) &+ \tfrac{x_0 y_0}{2}\Big\{h\,g\,x_0\Big[\tfrac{1}{3}q x_0 - \tfrac{5}{24}g x_0^2 - \tfrac{1}{2}(b-z_d)\Big]\\
&+h^2 g\Big[\tfrac{1}{36}x_0^2 g - \tfrac{1}{6}(b-z_d) + \tfrac{1}{12}q\,x_0\Big] + \tfrac{1}{24}h^3 g^2 x_0 + \tfrac{1}{180}h^4 g^2\Big\}.
\end{aligned} \tag{51}
$$

Lemma 9. *Given $q \in Q$ and $h > 0$, we have*

$$
|J_2^h(q) - J_2(q)| \approx C_7\,h, \tag{52}
$$

where $C_7 = \frac{1}{2}x_0^2\,y_0\,|g|\Big|\frac{1}{3}q x_0 - \frac{5}{24}g x_0^2 - \frac{1}{2}(b-z_d)\Big|$ is a constant independent of h.

Proof. It follows immediately from expression (51) for J_2^h. □

Lemma 10. *Let us consider $h > 0$.*

(a) The explicit expression for the optimal variable q_{op}^h is given by:

$$
q_{op}^h = q_{op} - B_1\,\frac{gh}{6} - B_2\,\frac{gh^2}{24\,x_0}, \tag{53}
$$

with $B_1 = B_2 = \Big(\frac{1}{3}+\frac{M_2}{x_0^3}\Big)^{-1}$.

(b) *The following error estimates hold:*

$$|q^h_{op} - q_{op}| \approx C_8\, h, \tag{54}$$

$$\left|J^h_2(q^h_{op}) - J_2(q_{op})\right| \approx C_9\, h, \tag{55}$$

where C_8 and C_9 are constants independent of h.

Proof.

(a) From the expression (51) for J^h_2, we have

$$\frac{d}{dq}J^h_2(q) = \tfrac{1}{2}x_0^2\, y_0\Big\{2\, x_0\, q\Big(\tfrac{1}{3} + \tfrac{M_2}{x_0^3}\Big) - \tfrac{5}{12}g\, x_0^2 - (b - z_d) + \tfrac{1}{3}\, g\, x_0\, h + \tfrac{1}{12}\, g\, h^2\Big\}.$$

Therefore,

$$q^h_{op} = \frac{\frac{5}{12}gx_0^2 + (b - z_d) - \frac{1}{12}gh^2 - \frac{1}{3}gx_0h}{2x_0\big(\frac{1}{3} + \frac{M_2}{x_0^3}\big)}.$$

Taking into account that q_{op} is given by (50), we obtain Formula (53).

(b) On one hand, expression (54) is a direct consequence of expression (53) where $C_8 = \left|\frac{g\,B_1}{6}\right|$.

On the other hand, taking into account Formula (51) for J^h_2, it follows that

$$\begin{aligned} J^h_2(q^h_{op}) - J_2(q_{op}) &= J_2(q^h_{op}) - J_2(q_{op}) \\ &+ \tfrac{x_0^2 y_0}{2} h\, g\Big[\tfrac{1}{3}q^h_{op}x_0 - \tfrac{5}{24}gx_0^2 - \tfrac{1}{2}(b - z_d)\Big] + o(h^2). \end{aligned} \tag{56}$$

From the definition of J_2 given by (49) and the explicit expression (53) for q^h_{op}, we get

$$\begin{aligned} J_2(q^h_{op}) - J_2(q_{op}) &= \tfrac{x_0^2 y_0}{2}\Big\{\tfrac{x_0}{B_1}\big((q^h_{op})^2 - q^2_{op}\big) \\ &+ (q^h_{op} - q_{op})\Big(-\tfrac{5}{12}gx_0^2 - (b - z_d)\Big)\Big\} \\ &= \tfrac{x_0^2 y_0}{2}(q^h_{op} - q_{op})\Big(\tfrac{x_0}{B_1}(q^h_{op} + q_{op}) - \tfrac{5}{12}gx_0^2 - (b - z_d)\Big) \\ &= \tfrac{x_0^2 y_0}{12}ghB_1\Big(-\tfrac{2x_0}{B_1}q_{op} + \tfrac{5}{12}gx_0^2 + b - z_d\Big) + o(h^2). \end{aligned} \tag{57}$$

Taking into account (56) and (57), we obtain estimate (55) with

$$C_9 = \frac{|g|x_0^2y_0}{2}\left|\tfrac{B_1}{6}\Big(-\tfrac{2x_0}{B_1}q_{op} + \tfrac{5}{12}gx_0^2 + b - z_d\Big) + \tfrac{1}{3}q_{op}x_0 - \tfrac{5}{24}gx_0^2 - \tfrac{1}{2}(b - z_d)\right|.$$

□

Lemma 11. *Consider $u_{q_{op}}$, the solution of* (1) *and* (2) *for $q = q_{op}$, and $u^h_{q^h_{op}}$, the discrete solution given by* (22) *for each $h > 0$ where $q = q^h_{op}$ is the optimal variable of the problem (P^h_2) given by* (53)*. Then, we have:*

$$\text{(a)}\quad \|u^h_{q^h_{op}} - u_{q_{op}}\|_H \approx C_{10}\, h, \qquad \text{(b)}\quad \left\|\frac{\partial u^h_{q^h_{op}}}{\partial x} - \frac{\partial u_{q_{op}}}{\partial x}\right\|_H \approx C_{11}\, h, \tag{58}$$

where C_{10} and C_{11} do not depend on the parameter h.

Proof.

(a) From the definition of u and u^h given by (13) and (22), respectively, it follows for $i = 1, \ldots, n$ that

$$\begin{aligned} u^h_{q^h_{op}}(x,y) - u_{q_{op}}(x,y) &= \tfrac{1}{2}\, g\, x^2 - (q^h_{op} - q_{op} + g\, i\, h)x + g\, \tfrac{i^2-i}{2}\, h^2, \\ &= \tfrac{1}{2}\, g\, x^2 + hgx\left(\tfrac{B_1}{6} - i\right) + h^2 g\left(\tfrac{B_1}{24}\tfrac{x}{x_0} + \tfrac{i(i-1)}{2}\right), \quad \forall x \in [x_i,\, x_{i+1}]. \end{aligned} \tag{59}$$

Then,

$$\begin{aligned} \|u^h_{q^h_{op}} - u_{q_{op}}\|^2_H &= y_0 \sum_{i=1}^{n} \int_{x_i}^{x_{i+1}} \left(u^h_{q^h_{op}}(x) - u_{q_{op}}(x)\right)^2 dx \\ &= x_0\, y_0\, g^2\Big[\tfrac{1}{108}(B_1-3)^2 h^2\, x_0^2 + \tfrac{1}{216} h^3\, x_0 (B_1-3)^2 \\ &\quad + \tfrac{1}{8640} h^4\, (72 - 30B_1 + 5B_1^2)\Big]. \end{aligned}$$

Therefore, we get Formula (58) (a) with $C_{10} = |g(B_1-3)|x_0\sqrt{\frac{x_0 y_0}{108}}$.

(b) In the same manner, we get

$$\begin{aligned} \left\|\frac{\partial u^h_{q^h_{op}}}{\partial x} - \frac{\partial u_{q_{op}}}{\partial x}\right\|^2_H &= y_0 \sum_{i=1}^{n} \int_{x_i}^{x_{i+1}} \left(g\, x - (q^h_{op} - q_{op}) - h\, g\, i\right)^2 dx \\ &= y_0\, g^2\left[\tfrac{1}{36}(12 - 6B_1 + B_1^2)h^2 x_0 + \tfrac{1}{72}(B_1-3)B_1 h^3 + \tfrac{1}{576} B_1^2 \tfrac{h^4}{x_0}\right]. \end{aligned}$$

Then, we get (58) (b) with

$$C_{11} = \frac{|g|}{6}\sqrt{x_0 y_0 (12 - 6B_1 + B_1^2)}.$$

□

4.2. Discrete Problem $(P^h_{2\alpha})$ Associated with $(P_{2\alpha})$

If we suppose that the desired state z_d is constant in (9), the quadratic cost function $J_{2\alpha}$ for optimal control problem $(P_{2\alpha})$ is explicitly given by:

$$\begin{aligned} J_{2\alpha}(q) = \frac{x_0 y_0}{2}\Big[& q^2 x_0^2 (D_{1\alpha} + \tfrac{M_2}{x_0^3}) + q x_0\left(D_{2\alpha} g x_0^2 + D_{3\alpha}(b - z_d)\right) \\ & + D_{4\alpha} g^2 x_0^4 + D_{5\alpha}(b - z_d)^2 + D_{6\alpha} g x_0^2 (b - z_d)\Big], \end{aligned} \tag{60}$$

where

$$\begin{aligned} &D_{1\alpha} = \tfrac{1}{3} + \tfrac{1}{\alpha x_0} + \tfrac{1}{\alpha^2 x_0^2}, && D_{2\alpha} = -\tfrac{5}{12} - \tfrac{5}{3\alpha x_0} - \tfrac{2}{\alpha^2 x_0^2}, \\ &D_{3\alpha} = -1 - \tfrac{2}{\alpha x_0}, && D_{4\alpha} = \tfrac{2}{15} + \tfrac{2}{3\alpha x_0} + \tfrac{1}{\alpha^2 x_0^2}, \\ &D_{5\alpha} = 1, && D_{6\alpha} = \tfrac{2}{3} + \tfrac{2}{\alpha x_0}. \end{aligned} \tag{61}$$

Then, the continuous boundary optimization control, called $q_{\alpha_{op}}$, and the associated state are:

$$\begin{aligned} q_{\alpha_{op}} &= -\frac{D_{2\alpha} g x_0^2 + D_{3\alpha}(b - z_d)}{2x_0\left(D_{1\alpha} + \frac{M_2}{x_0^3}\right)}, \\ u_{\alpha q_{\alpha_{op}}}(x,y) &= -\tfrac{1}{2} g\, x^2 + (g\, x_0 - q_{\alpha_{op}})x + \tfrac{1}{\alpha}(g\, x_0 - q_{\alpha_{op}}) + b. \end{aligned} \tag{62}$$

Remark 6. *Notice that* $J_{2\alpha}(q) \to J_2(q)$ *for all* $q \in Q$ *and* $q_{\alpha_{op}} \to q_{op}$ *when* $\alpha \to \infty$.

Define the discrete cost function as:

$$J^h_{2\alpha}(q) = \tfrac{1}{2}\|u^h_{\alpha q} - z_d\|^2_H + \tfrac{1}{2}M_2\|q\|^2_Q, \tag{63}$$

where $u^h_{\alpha q}$ is the solution of (S^h_α) given in (27) when q is fixed. We set the following discrete optimization problem $(P^h_{2\alpha})$ on the constant heat flux q as

$$\text{find} \quad q^h_{\alpha_{op}} \in \mathbb{R} \quad \text{such that} \quad J^h_{2\alpha}(q^h_{\alpha_{op}}) = \min_{q\in\mathbb{R}} J^h_{2\alpha}(q).$$

Working algebraically, the cost function $J^h_{2\alpha}$ can be written explicitly as:

$$\begin{aligned} J^h_{2\alpha}(q) &= J_{2\alpha}(q) \\ &+\tfrac{1}{2}x_0\, y_0\, g\, h\Big\{ \tfrac{q\, x_0^2}{3} - \tfrac{5\, g x_0^3}{24} - \tfrac{(b-z_d)\, x_0}{2} + \tfrac{3x_0 q}{2\,\alpha} - \tfrac{7g\, x_0^2}{6\alpha} - \tfrac{2(b-z_d)}{\alpha} + \tfrac{2(q-g\, x_0)}{\alpha^2} \\ &+h\Big(\tfrac{x_0^2\, g}{36} - \tfrac{(b-z_d)}{6} + \tfrac{q\, x_0}{12} + \tfrac{2g\, x_0+q}{6\,\alpha} + \tfrac{g}{\alpha^2}\Big) \\ &+h^2\, g\Big(\tfrac{x_0}{24} + \tfrac{1}{6\alpha}\Big) + \tfrac{1}{180} g\, h^3\Big\}. \end{aligned} \tag{64}$$

Lemma 12. *For each $q \in Q$ and $h > 0$, we have:*

$$|J^h_{2\alpha}(q) - J_{2\alpha}(q)| \approx C_{7\alpha}\, h, \tag{65}$$

with

$$C_{7\alpha} = \tfrac{1}{2}x_0^2\, y_0 \left| g\left(\tfrac{q\, x_0}{3} - \tfrac{5\, g x_0^2}{24} - \tfrac{(b-z_d)}{2} + \tfrac{3q}{2\,\alpha} - \tfrac{7g\, x_0}{6\alpha} - \tfrac{2(b-z_d)}{x_0\,\alpha} + \tfrac{2(q-g\, x_0)}{x_0\,\alpha^2}\right)\right|,$$

a constant independent of h.

Proof. It follows immediately from expression (64). □

Lemma 13. *Let us consider $h > 0$.*

(a) *The explicit expression for optimal control $q^h_{\alpha_{op}}$ is given by:*

$$q^h_{\alpha_{op}} = q_{\alpha_{op}}, - B_{1\alpha}\frac{gh}{6} - B_{2\alpha}\frac{gh^2}{24x_0} \tag{66}$$

with

$$B_{1\alpha} = \left(1 + \tfrac{9}{2\alpha x_0} + \tfrac{6}{\alpha^2 x_0^2}\right)\left(D_{1\alpha} + \tfrac{M_2}{x_0^3}\right)^{-1},$$

$$B_{2\alpha} = \left(1 - \tfrac{2}{\alpha x_0}\right)\left(D_{1\alpha} + \tfrac{M_2}{x_0^3}\right)^{-1}.$$

(b) *The following error estimates hold:*

$$|q^h_{\alpha_{op}} - q_{\alpha_{op}}| \approx C_{8\alpha}\, h, \tag{67}$$

$$\left|J^h_{2\alpha}(q^h_{\alpha_{op}}) - J_{2\alpha}(q_{\alpha_{op}})\right| \approx C_{9\alpha}\, h, \tag{68}$$

where $C_{8\alpha}$ and $C_{9\alpha}$ do not depend on h.

Proof.

(a) From the derivative of the control function $J^h_{2\alpha}$ given by

$$\frac{d}{dq}J^h_{2\alpha}(q) = \frac{d}{dq}J_{2\alpha}(q) + \frac{1}{2}x_0\,y_0\,g\,h\Big\{\frac{x_0^2}{3} + \frac{3x_0}{2\,\alpha} + \frac{2}{\alpha^2} + h\Big(\frac{x_0}{12} + \frac{1}{6\,\alpha}\Big)\Big\}$$
$$= \frac{x_0 y_0}{2}\Big\{2qx_0^2\Big(D_{1\alpha} + \frac{M_2}{x_0^3}\Big) + x_0\Big(D_{2\alpha}gx_0^2 + D_{3\alpha}(b - zd)\Big)$$
$$+gh\Big(\frac{x_0^2}{3} + \frac{3x_0}{2\,\alpha} + \frac{2}{\alpha^2} + h\Big(\frac{x_0}{12} + \frac{1}{6\,\alpha}\Big)\Big)\Big\},$$

it follows that

$$q^h_{\alpha_{op}} = \frac{-gh\Big(\frac{x_0^2}{3}+\frac{3x_0}{2\,\alpha}+\frac{2}{\alpha^2}\Big)-gh^2\Big(\frac{x_0}{12}+\frac{1}{6\,\alpha}\Big)-x_0\Big(D_{2\alpha}gx_0^2+D_{3\alpha}(b-zd)\Big)}{2x_0^2\Big(D_{1\alpha}+\frac{M_2}{x_0^3}\Big)}.$$

Working algebraically, we get Formula (66).

(b) Estimate (67) follows straightforwardly from (66) with

$$C_{8\alpha} = \left|\frac{gB_{1\alpha}}{6}\right|.$$

From Formula (64) we obtain that

$$J^h_{2\alpha}(q^h_{\alpha_{op}}) - J_{2\alpha}(q_{\alpha_{op}}) = J_{2\alpha}(q^h_{\alpha_{op}}) - J_{2\alpha}(q_{\alpha_{op}})$$
$$+\frac{1}{2}x_0\,y_0\,g\,h\Big(\frac{q^h_{\alpha_{op}}\,x_0^2}{3} - \frac{5\,gx_0^3}{24} - \frac{(b-z_d)\,x_0}{2} + \frac{3x_0q^h_{\alpha_{op}}}{2\,\alpha} \tag{69}$$
$$-\frac{7g\,x_0^2}{6\alpha} - \frac{2(b-z_d)}{\alpha} + \frac{2(q^h_{\alpha_{op}}-g\,x_0)}{\alpha^2}\Big) + o(h^2).$$

Moreover, taking into account the explicit expression for $J^h_{2\alpha}$ given by (64) and Formula (66), it follows that

$$J^h_{2\alpha}(q^h_{\alpha_{op}}) - J_{2\alpha}(q_{\alpha_{op}}) = \frac{x_0^2y_0}{2}(q^h_{\alpha_{op}} - q_{\alpha_{op}})\Big[(q^h_{\alpha_{op}} + q_{\alpha_{op}})x_0\Big(D_{1\alpha} + \frac{M_2}{x_0^3}\Big)$$
$$+D_{2\alpha}gx_0^2 + D_{3\alpha}(b - z_d)\Big] \tag{70}$$
$$= -\frac{x_0^2y_0}{12}gB_{1\alpha}\Big(2q_1x_0\Big(D_{1\alpha} + \frac{M_2}{x_0^3}\Big) + D_{2\alpha}gx_0^2 + D_{3\alpha}(b - z_d)\Big) + o(h^2).$$

Combining (69) and (70) we get estimate (68) with

$$C_{9\alpha} = \frac{x_0^2y_0}{2}|g|\,\Big| -\frac{B_{1\alpha}}{6}\Big(2q_{\alpha_{op}}x_0\Big(D_{1\alpha} + \frac{M_2}{x_0^3}\Big) + D_{2\alpha}gx_0^2 + D_{3\alpha}(b - z_d)\Big)$$
$$+\frac{q_{\alpha_{op}}\,x_0}{3} - \frac{5\,gx_0^2}{24} - \frac{(b-z_d)}{2} + \frac{3q_{\alpha_{op}}}{2\,\alpha} - \frac{7g\,x_0}{6\alpha} - \frac{2(b-z_d)}{\alpha x_0} + \frac{2(q_{\alpha_{op}}-g\,x_0)}{\alpha^2x_0}\Big|.$$

□

Lemma 14. *Let us consider $u_{q_{\alpha_{op}}}$ the solution of (1) and (3) for $q = q_{\alpha_{op}}$ and $u^h_{\alpha q^h_{\alpha_{op}}}$ the discrete solution given in (27) for $h > 0$ and $q = q^h_{\alpha_{op}}$. Then, we have:*

$$\begin{aligned} &(a) \quad \|u^h_{\alpha q^h_{\alpha op}} - u_{\alpha q_{\alpha op}}\|_H \approx C_{10\,\alpha}\, h, \\ &(b) \quad \left\| \frac{\partial u^h_{\alpha q^h_{\alpha op}}}{\partial x} - \frac{\partial u_{\alpha q_{\alpha op}}}{\partial x} \right\|_H \approx C_{11\,\alpha}\, h, \end{aligned} \tag{71}$$

Proof. Similarly to what was done in Lemma 12, we obtain

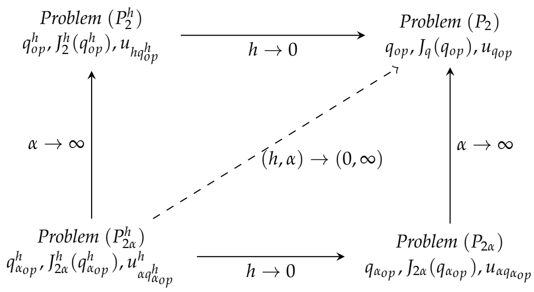
$$C_{10\,\alpha} = |g| x_0$$
$$\sqrt{\tfrac{x_0 y_0}{108}} \sqrt{B^2_{1\alpha}\left(\tfrac{3}{\alpha^2 x_0^2} + \tfrac{3}{\alpha x_0} + 1\right) + 9\left(\tfrac{12}{\alpha^2 x_0^2} + \tfrac{6}{\alpha x_0} + 1\right) - 3B_{1\alpha}\left(\tfrac{12}{\alpha^2 x_0^2} + \tfrac{9}{\alpha x_0} + 2\right)}$$

$$C_{11\,\alpha} = \frac{|g|}{6} \sqrt{x_0 y_0 (12 - 6B_{1\alpha} + B^2_{1\alpha})}.$$

□

Remark 7. *The constants verify that $C_{i\alpha} \to C_i$, when $\alpha \to \infty$, for each $i = 7, \ldots, 11$.*

Remark 8. *The double convergence when $(h, \alpha) \to (0, +\infty)$ of the optimal control of problem $(P^h_{2\alpha})$ holds. The relationship among optimal control problems (P_2), $(P_{2\alpha})$, (P^h_2) and $(P^h_{2\alpha})$ is given by the following diagram:*



5. Boundary Optimization Problem with Variable *b*

5.1. Discrete Problem (P^h_3) Associated with (P_3)

In this section we consider the boundary optimal control problem (P_3) given by (10). Taking into account expression (12), for a given constant b, we get

$$\begin{aligned} J_3(b) = \tfrac{x_0 y_0}{2}\Big\{ b^2\Big(1 + \tfrac{M_3}{x_0}\Big) + b\Big(\tfrac{2 g x_0^2}{3} - q\, x_0 - 2\, z_d\Big) \\ + \Big[\tfrac{2g^2 x_0^4}{15} - \tfrac{5 g q x_0^3}{12} + \tfrac{x_0^2}{3}(q^2 - 2g z_d) + z_d(z_d + q x_0)\Big]\Big\}. \end{aligned} \tag{72}$$

Then, the boundary optimal variable of problem (P_3), called b_{op}, and the associated continuous optimal state, are given respectively by:

$$b_{op} = \frac{-\frac{g x_0^2}{3} + \frac{q x_0}{2} + z_d}{1 + \frac{M}{x_0}}, \qquad u_{b_{op}} = -\tfrac{1}{2} g\, x^2 + (g\, x_0 - q)x + b_{op}. \tag{73}$$

We define the discrete optimal control problem (P^h_3) on the constant temperature b as

$$\text{find} \quad b^h_{op} \in \mathbb{R} \quad \text{such that} \quad J^h_3(b^h_{op}) = \min_{b \in \mathbb{R}} J^h_3(b)$$

where the discrete cost function $J_3^h(b)$ is defined as:

$$J_3^h(b) = \frac{1}{2}\|u_b^h - z_d\|_H^2 + \frac{1}{2}M_3\|b\|_B^2,$$

where u_b^h is given in (22) for a fixed constant b, h is the spatial step, and z_d (the desired state) is constant.

Notice that the cost function J_3^h can be explicitly written as:

$$\begin{aligned} J_3^h(b) = J_3(b) + \tfrac{x_0 y_0 g}{2}\Big\{ - bh\Big(\tfrac{x_0}{2} + \tfrac{h}{6}\Big) + hx_0\big(\tfrac{1}{3}qx_0 - \tfrac{5}{24}gx_0^2 + \tfrac{z_d}{2}\big) \\ + \tfrac{1}{6}h^2\Big(\tfrac{qx_0}{2} + \tfrac{1}{6}gx_0^2 + z_d\Big) + \tfrac{1}{24}gh^3x_0 + \tfrac{1}{180}gh^4\Big\}. \end{aligned} \tag{74}$$

Lemma 15. *Let $b \in R$ and $h > 0$; we have:*

$$|J_3^h(b) - J_3(b)| \approx C_{12}\, h, \tag{75}$$

where

$$C_{12} = \tfrac{1}{2}x_0^2\, y_0\, |g|\Big| - \tfrac{b}{2} + \tfrac{qx_0}{3} - \tfrac{5gx_0^2}{24} + \tfrac{z_d}{2}\Big|,$$

does not depend on h.

Proof. It follows from expression (74) for J_3^h. □

Lemma 16. *Let us consider $h > 0$.*

(a) *The explicit expression for the optimal variable b_{op}^h is given by:*

$$b_{op}^h = b_{op} + E_1 g x_0\, h\Big(1 + \tfrac{h}{3x_0}\Big), \qquad E_1 = \frac{1}{4\Big(1 + \frac{M_3}{x_0}\Big)}. \tag{76}$$

(b) *The following error estimates hold:*

$$|b_{op}^h - b_{op}| \approx C_{13}\, h, \tag{77}$$

$$\Big|J_3^h(b_{op}^h) - J_3(b_{op})\Big| \approx C_{14}\, h, \tag{78}$$

where C_{13} and C_{14} do not depend on h.

Proof.

(a) According to (74) we have

$$\frac{d}{db}J_3^h(b) = \frac{d}{db}J_3(b) - \tfrac{x_0 y_0}{2} gbh\Big(\tfrac{x_0}{2} + \tfrac{h}{6}\Big). \tag{79}$$

Then, Formula (76) for b_{op}^h follows immediately.

(b) Estimate (77) is a direct consequence of (76) with $C_{13} = |E_1 g x_0|$.
Moreover, taking into account Formulas (72) and (74) for J_3 and J_3^h and Formulas (73) and (76) for b_{op} and b_{op}^h, respectively, it follows that

$$\begin{aligned} &J_3^h(b_{op}^h) - J_3(b_{op}) \\ &= J_3(b_{op}^h) - J_3(b_{op}) + \tfrac{x_0^2 y_0 g}{2}\Big\{ - \tfrac{b_{op}^h}{2} + \tfrac{1}{3}qx_0 - \tfrac{5}{24}gx_0^2 + \tfrac{z_d}{2}\Big\}h + o(h^2). \end{aligned}$$

In addition, the expression $J_3(b_{op}^h) - J_3(b_{op})$ can be rewritten as

$$J_3(b_{op}^h) - J_3(b_{op}) = \frac{x_0^2 y_0 g}{2} E_1 \left(2b_{op}\left(1 + \frac{M_3}{x_0}\right) + \frac{2}{3} g x_0^2 - q x_0 - 2z_d\right) h + o(h^2).$$

Therefore, it follows that estimate (16) is given by

$$C_{14} = \frac{x_0^2 y_0 g}{2} \left| E_1 \left(2b_{op}\left(1 + \frac{M_3}{x_0}\right) + \frac{2}{3} g x_0^2 - q x_0 - 2z_d\right) - \frac{b_{op}}{2} + \frac{q x_0}{3} - \frac{5 g x_0^2}{24} + \frac{z_d}{2} \right|.$$

□

Lemma 17. *Let us consider $u_{b_{op}}$, the solution of (1), (3) for $b = b_{op}$, and $u^h_{b^h_{op}}$, the discrete solution given in (27) for $h > 0$ and $b = b^h_{op}$. Then, we have:*

$$\text{(a)} \quad \|u^h_{b^h_{op}} - u_{b_{op}}\|_H \approx C_{15}\, h, \qquad \text{(b)} \quad \left\| \frac{\partial u^h_{b^h_{op}}}{\partial x} - \frac{\partial u_{b_{op}}}{\partial x} \right\|_H \approx C_{16}\, h, \tag{80}$$

where C_{15} and C_{16} are constants that do not depend on h.

Proof. Working algebraically, we obtain

$$\|u^h_{b^h_{op}} - u_{b_{op}}\|^2_H = \frac{x_0^3 y_0 g^2}{2} \left(2E_1^2 - E_1 + \frac{1}{6}\right) h^2 + o(h^3).$$

Then, we obtain estimate (a) with

$$C_{15} = x_0 |g| \sqrt{\frac{x_0 y_0}{2}} \sqrt{2E_1^2 - E_1 + \frac{1}{6}}.$$

In a similar manner, we get that estimate (b) holds with

$$C_{16} = |g| \sqrt{\frac{x_0 y_0}{2}}.$$

□

5.2. Discrete Problem $(P^h_{3\alpha})$ Associated with $(P_{3\alpha})$

From [16], we know that the continuous quadratic functional cost in (6) for the optimization problem $(P_{3\alpha})$ is explicitly given by:

$$J_{3\alpha}(b) = J_3(b) + \frac{x_0 y_0}{2} \left\{ \frac{1}{3\alpha}(q - g x_0)(-6b + 3q x_0 - 2g x_0^2 + 6z_d) + \frac{1}{\alpha^2}(q - g x_0)^2 \right\} \tag{81}$$

where J_3 is defined by (72). Moreover, the continuous optimal boundary control $b_{\alpha_{op}}$ is given by

$$b_{\alpha_{op}} = b_{op} - \frac{g x_0 - q}{\alpha\left(1 + \frac{M_3}{x_0}\right)}. \tag{82}$$

The continuous associated state is established by:

$$u_{b_{\alpha_{op}}}(x, y) = -\frac{1}{2} g\, x^2 + (g\, x_0 - q)x + \frac{1}{\alpha}(g\, x_0 - q) + b_{\alpha_{op}}. \tag{83}$$

Define the discrete cost function as:

$$J^h_{3\alpha}(b) = \frac{1}{2}\|u^h_{\alpha b} - z_d\|^2_H + \frac{1}{2} M_3 \|b\|^2_B \tag{84}$$

where $u^h_{\alpha b}$ is the solution of (S^h_α) given in (27) for a fixed b. We set the following discrete optimization problem $(P^h_{3\alpha})$ as

$$\text{find} \quad b^h_{\alpha_{op}} \in \mathbb{R} \quad \text{such that} \quad J^h_{3\alpha}(b^h_{\alpha_{op}}) = \min_{b\in\mathbb{R}} J^h_{3\alpha}(b).$$

Working algebraically leads us to write $J^h_{3\alpha}$ as follows:

$$\begin{aligned} J^h_{3\alpha}(b) = J_{3\alpha}(b) + \tfrac{1}{2}x_0\, y_0\, g\, h\Big\{ -b\Big(\tfrac{x_0}{2} + \tfrac{2}{\alpha} + \tfrac{h}{6}\Big) + g\Big(\tfrac{-5x_0^3}{24} - \tfrac{7x_0^2}{6\alpha} - \tfrac{2x_0}{\alpha^2}\Big) \\ +q\Big(\tfrac{x_0^2}{3} + \tfrac{3x_0}{2\alpha} + \tfrac{2}{\alpha^2}\Big) + z_d\Big(\tfrac{x_0}{2} + \tfrac{2}{\alpha}\Big) \\ +h\Big[g\Big(\tfrac{x_0^2}{36} + \tfrac{x_0}{3\alpha} + \tfrac{1}{\alpha^2}\Big) + q\Big(\tfrac{x_0}{12} + \tfrac{1}{6\alpha}\Big) + \tfrac{z_d}{6}\Big] + h^2 g\Big(\tfrac{x_0}{24} + \tfrac{1}{6\alpha}\Big) + h^3\tfrac{g}{180}\Big\}. \end{aligned} \tag{85}$$

Lemma 18. *For $b \in B$ and $h > 0$, we have*

$$|J^h_{3\alpha}(b) - J_{3\alpha}(b)| \approx C_{12\alpha}\, h, \tag{86}$$

with

$$C_{12\alpha} = \tfrac{x_0\, y_0}{2}|g|\Big| - b\Big(\tfrac{x_0}{2} + \tfrac{2}{\alpha} + \tfrac{h}{6}\Big) + g\Big(\tfrac{-5x_0^3}{24} - \tfrac{7x_0^2}{6\alpha} - \tfrac{2x_0}{\alpha^2}\Big) + q\Big(\tfrac{x_0^2}{3} + \tfrac{3x_0}{2\alpha} + \tfrac{2}{\alpha^2}\Big) + z_d\Big(\tfrac{x_0}{2} + \tfrac{2}{\alpha}\Big)\Big|.$$

Proof. It arises immediately from (85). □

Lemma 19. *Let us consider $h > 0$.*

(a) *The explicit expression for optimal control $b^h_{\alpha_{op}}$ is given by:*

$$b^h_{\alpha_{op}} = b_{\alpha_{op}} + E_1 g x_0\, h\Big(1 + \tfrac{4}{\alpha x_0} + \tfrac{h}{3x_0}\Big), \tag{87}$$

where E_1 is given in (76).

(b) *The following error estimates hold:*

$$|b^h_{\alpha_{op}} - b_{\alpha_{op}}| \approx C_{13\alpha}\, h, \tag{88}$$

$$\Big|J^h_{3\alpha}(b^h_{\alpha_{op}}) - J_{3\alpha}(b_{\alpha_{op}})\Big| \approx C_{14\alpha}\, h, \tag{89}$$

where $C_{13\alpha}$ and $C_{14\alpha}$ do not depend on h.

Proof.

(a) It follows from expression $J^h_{3\alpha}$ given by (85).

(b) The estimate in (88) is obtained immediately from item (a) with

$$C_{13\alpha} = |E_1||g|x_0\Big|1 + \tfrac{4}{\alpha x_0}\Big|.$$

Taking into account (85) and (81) yields

$$J^h_{3\alpha}(b^h_{\alpha_{op}}) - J_{3\alpha}(b_{\alpha_{op}}) = J_{3\alpha}(b^h_{\alpha_{op}}) - J_{3\alpha}(b_{\alpha_{op}}) + F_{1\alpha}h + o(h^2).$$

with

$$F_{1\alpha} = \tfrac{1}{2}x_0^2\, y_0\, g\Big(-(b_{\alpha_{op}} - z_d)\Big(\tfrac{1}{2} + \tfrac{2}{\alpha x_0}\Big) + g x_0^2\Big(\tfrac{-5}{24} - \tfrac{7}{6\alpha x_0} - \tfrac{2}{\alpha^2 x_0^2}\Big) + q x_0\Big(\tfrac{1}{3} + \tfrac{3}{2\alpha x_0} + \tfrac{2}{\alpha^2 x_0^2}\Big)\Big).$$

In addition, from the definition of $J^h_{3\alpha}$ and $b^h_{\alpha_{op}}$, we have

$$J^h_{3\alpha}(b^h_{\alpha_{op}}) - J_{3\alpha}(b_{\alpha_{op}}) = J_3(b^h_{\alpha_{op}}) - J_3(b_{\alpha_{op}}) + F_{2\alpha}h + o(h^2)$$

with

$$F_{2\alpha} = \frac{x_0^2 y_0 g}{\alpha} E_1(-q + g x_0)\left(1 + \frac{4}{\alpha x_0}\right).$$

Finally, according to Formula (72) for J_3, we get

$$J_3(b^h_{\alpha_{op}}) - J_3(b_{\alpha_{op}}) = F_{3\alpha}h + o(h^2),$$

with

$$F_{3\alpha} = \frac{x_0^2 y_0 g}{2} E_1\left(1 + \frac{4}{\alpha x_0}\right)\left(2b_{\alpha_{op}}\left(1 + \frac{M_3}{x_0}\right) + \frac{2}{3} g x_0^2 - q x_0 - 2z_d\right).$$

Therefore, estimate (19) holds for

$$C_{14\alpha} = \left|F_{1\alpha} + F_{2\alpha} + F_{3\alpha}\right|.$$

□

Lemma 20. *Let us consider $u_{b_{\alpha_{op}}}$, the solution of (1) and (3) for $b = b_{\alpha_{op}}$, and $u^h_{\alpha b^h_{\alpha_{op}}}$, the discrete solution given in (27) for $h > 0$ and $b = b^h_{\alpha_{op}}$. Then, we have:*

$$\text{(a)} \quad \|u^h_{\alpha b^h_{\alpha_{op}}} - u_{\alpha b_{\alpha_{op}}}\|_H \approx C_{15\,\alpha}\, h, \quad \text{(b)} \quad \left\|\frac{\partial u^h_{\alpha b^h_{\alpha_{op}}}}{\partial x} - \frac{\partial u_{\alpha b_{\alpha_{op}}}}{\partial x}\right\|_H \approx C_{16\,\alpha}\, h, \tag{90}$$

Proof. Similarly to what was done in Lemma 12, we obtain

$$C_{15\,\alpha} = x_0|g|\sqrt{\frac{x_0 y_0}{2}}\mathcal{A},$$

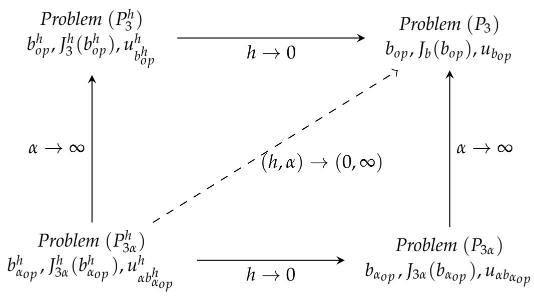
$$\mathcal{A} = \sqrt{E_1^2\left(2 + \frac{16}{\alpha x_0} + \frac{32}{\alpha^2 x_0^2}\right) + E_1\left(-1 - \frac{8}{\alpha x_0} - \frac{16}{\alpha^2 x_0^2}\right) + \frac{1}{6} + \frac{1}{\alpha x_0} + \frac{2}{\alpha^2 x_0^2}},$$

$$C_{16\,\alpha} = C_{16}.$$

□

Remark 9. *The constants obtained in the estimates of the previous lemmas verify that $C_{i\alpha} \to C_i$ when $\alpha \to \infty$ for $i = 12, \cdots, 16$.*

Remark 10. *The double convergence when $(h, \alpha) \to (0, +\infty)$ of the optimal control of problem $(P^h_{3\alpha})$ holds. The relationship among the optimal control of problems (P_3), $(P_{3\alpha})$, (P^h_3) and $(P^h_{3\alpha})$ is given by the following diagram:*



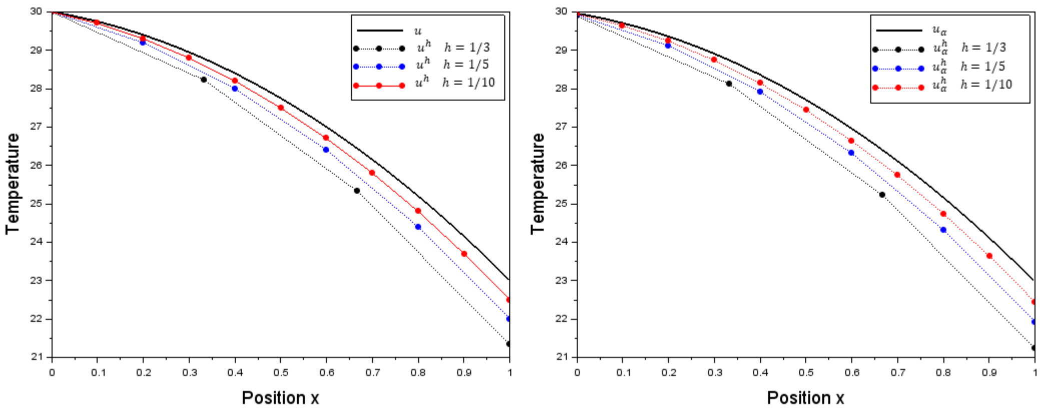
6. Numerical Results

We carried out some numerical simulations in order to illustrate the theoretical results obtained in the previous sections for the optimal control problems (P_i^h) and $(P_{i\alpha}^h)$ for $i = 1, 2, 3$.

Throughout this section we consider the domain $\Omega = [0, 1] \times [0, 1]$, i.e, $x_0 = y_0 = 1$.

Before analyzing the optimal control problems we illustrate the behavior of the continuous state of the systems (S) and (S_α) and the discrete state of the systems (S_h) and (S_α^h).

In Figure 1a we plotted the state of system u given by (13) and the approximate discrete function u^h defined by (22) against the position x for $h = 1/3, 1/5, 1/10$. As we saw in Lemma 1 for each fixed x, the functions $u^h(x)$ increase and get closer to the limit $u(x)$ as h decreases. In a similar manner, in Figure 1b, for $\alpha = 50$, we obtained system u_α given by (13) and the approximate discrete function u_α^h defined by (27) against the position x for $h = 1/3, 1/5, 1/10$. Notice that as h decreases, the functions $\{u_\alpha^h\}$ increase and get closer to the limit u_α as it was proved in Lemma 2.



(a) u and u^h for different values of h (b) u_α and u_α^h for different values of h with $\alpha = 50$

Figure 1. State of systems (S), (S^h), (S_α) and (S_α^h) using $q = 12$, $b = 30$, $z_d = 40$ and $g = 10$.

In addition in order to visualize the double convergence of $u_\alpha^h \to u$ when $(h, \alpha) \to (0, \infty)$, in Figure 2 we plotted u and u_α^h for $(h, \alpha) = (1/3, 10), (1/5, 50)$ and $(1/10, 500)$.

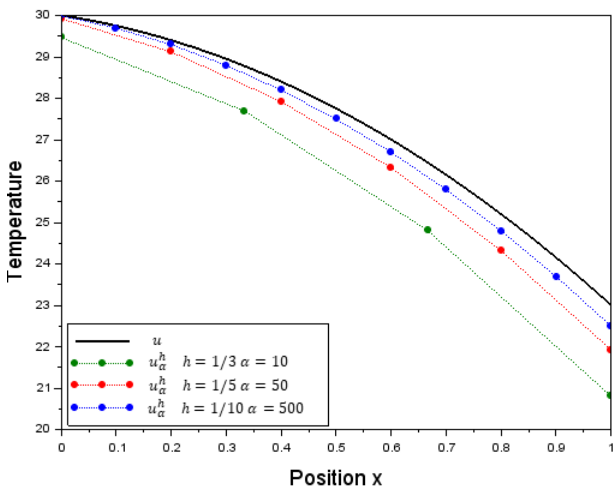


Figure 2. Plot of u and u_α^h against $n = 1/h$ for different values of (h, α).

Table 1 illustrates that the L^2 errors exhibit a linear rate of convergence. Indeed, each refinement step in which the mesh size h is divided by two produces an error that is approximately halved, confirming the expected first-order behavior.

Table 1. L^2 errors for $u - u^h$ and $u_\alpha - u_\alpha^h$ for different values of α.

h	$\|u - u^h\|_{L^2}$		$\|u_\alpha - u_\alpha^h\|_{L^2}$	
		$\alpha = 50$	$\alpha = 100$	$\alpha = 200$
0.250000	7.675914×10^{-1}	8.120374×10^{-1}	7.897314×10^{-1}	7.786397×10^{-1}
0.125000	3.722243×10^{-1}	3.942740×10^{-1}	3.832039×10^{-1}	3.777023×10^{-1}
0.062500	1.832549×10^{-1}	1.942324×10^{-1}	1.887200×10^{-1}	1.859813×10^{-1}
0.031250	9.091783×10^{-2}	9.639423×10^{-2}	9.364392×10^{-2}	9.227771×10^{-2}
0.015625	4.528211×10^{-2}	4.801716×10^{-2}	4.664351×10^{-2}	4.596121×10^{-2}

6.1. Control Variable g

In this subsection we obtain some computational examples for the optimal distributed control problems (P_1), (P_1^h), $(P_{1\alpha})$ and $(P_{1\alpha}^h)$. For each plot, we set $q = 12, b = 30, z_d = 40$ and $M_1 = 1$.

In Figure 3 we plotted the continuous quadratic cost function J_1 given by (28) and the discrete cost function J_1^h obtained in (32) against g for $h = 1/10, 1/50$ and $1/100$. Notice that as h decreases, the function $J_1^h = J_1^h(g)$ also decreases to the limit function $J_1 = J_1(g)$ in agreement with Lemma 3. In a similar manner in Figure 4, for $\alpha = 50$, we obtain the continuous function $J_{1\alpha}$ and the discrete functions $J_{1\alpha}^h$ for $h = 1/10, 1/50$ and $1/100$ observing the convergence of $J_{1\alpha}^h \to J_{1\alpha}$ as h decreases to zero. Moreover, Figure 5 shows the double convergence of $J_{1\alpha}^h \to J_1$ when $(h, \alpha) \to (0, \infty)$. We illustrate how $J_{1\alpha}^h$ gets closer to J_1 as the value of h decreases and the value of α increases.

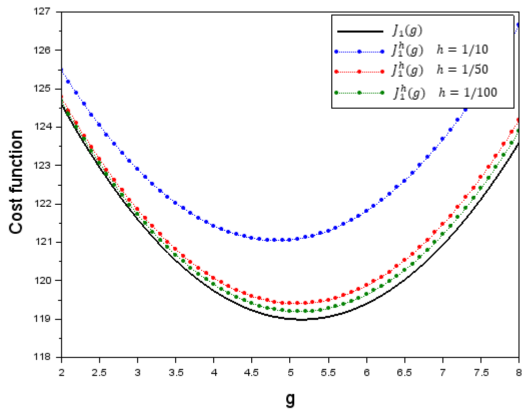


Figure 3. Plot of J_1 and J_1^h against g.

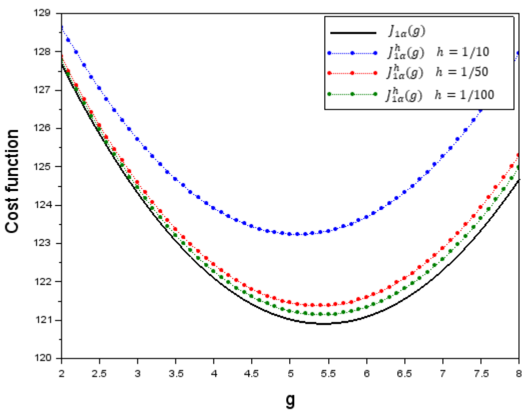


Figure 4. Plot of $J_{1\alpha}$ and $J_{1\alpha}^h$ for $\alpha = 50$ against g.

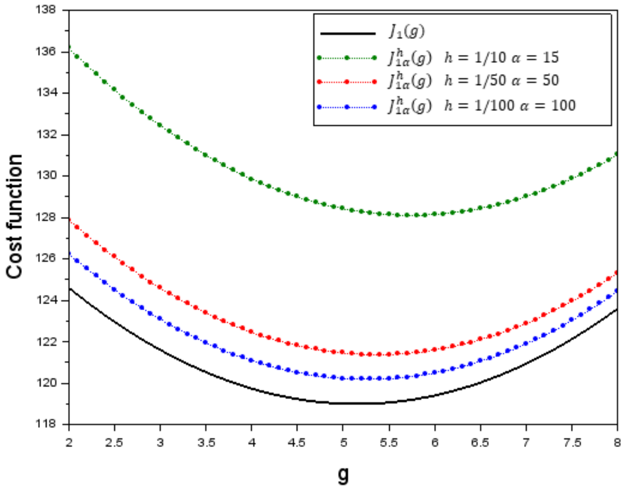


Figure 5. Plot of J_1 and $J^h_{1\alpha}$ against g.

In Figure 6 we plotted the continuous optimal control g_{op} for problem (P_1) given by (29) and optimal control $g_{\alpha_{op}}$ given by (40) for $\alpha = 15, 50, 100$. Notice that as α increases, $g_{\alpha_{op}}$ decreases to the limit g_{op}. In addition, we set different values of n between $n = 10$ and $n = 100$. Recalling that $h = \frac{x_0}{n} = \frac{1}{n}$, for each h, we obtained the optimal discrete control g^h_{op} to problem (P^h_1) defined by (4) and the optimal discrete control $g^h_{\alpha_{op}}$ to problem $(P^h_{1\alpha})$ given by (40) for $\alpha = 15, 50, 100$. For each α fixed, we observe the discrete solution $g^h_{\alpha_{op}} \to g_{\alpha_{op}}$ when $h \to 0$, i.e., $n \to \infty$.

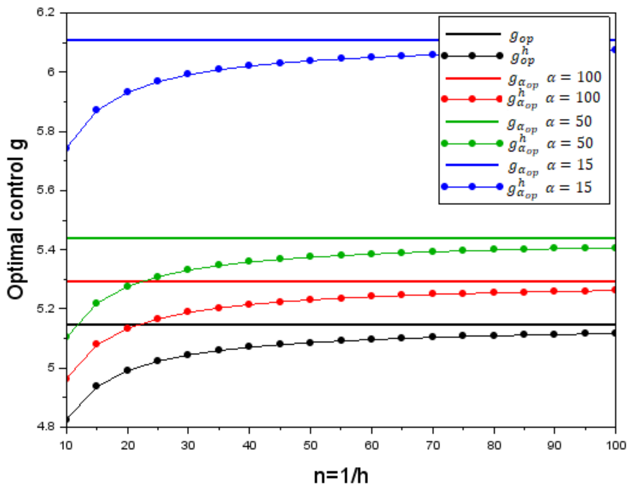


Figure 6. Plot of g_{op}, g^h_{op}, $g_{\alpha_{op}}$ and $g^h_{\alpha_{op}}$ against $n = 1/h$.

6.2. Control Variable q

In this subsection we ran some computational examples for the optimal boundary control problems (P_2), (P^h_2), $(P_{2\alpha})$ and $(P^h_{2\alpha})$. For each plot, we set $g = 10, b = 50, z_d = 40$ and $M_2 = 1$.

In Figure 7 we plotted the continuous quadratic cost function J_2 given by (49) and the discrete cost function J^h_2 obtained in (51) against q for $h = 1/10, 1/25$ and $1/50$. Observe that as h decreases, function $J^h_2 = J^h_2(q)$ also decreases to the limit function $J_2 = J_2(q)$. In a similar way, in Figure 8, for $\alpha = 100$, we obtained the continuous function $J_{2\alpha}$ and the discrete functions $J^h_{2\alpha}$ for $h = 1/10, 1/25$ and $1/50$. The convergences $J^h_2 \to J_2$ and $J^h_{2\alpha} \to J_{2\alpha}$ when $h \to 0$ are in agreement with Lemmas 9 and 12, respectively.

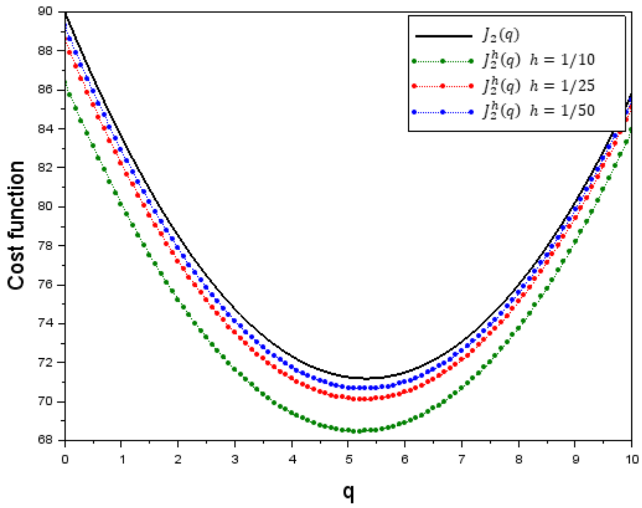


Figure 7. Plot of J_2 and J_2^h against q.

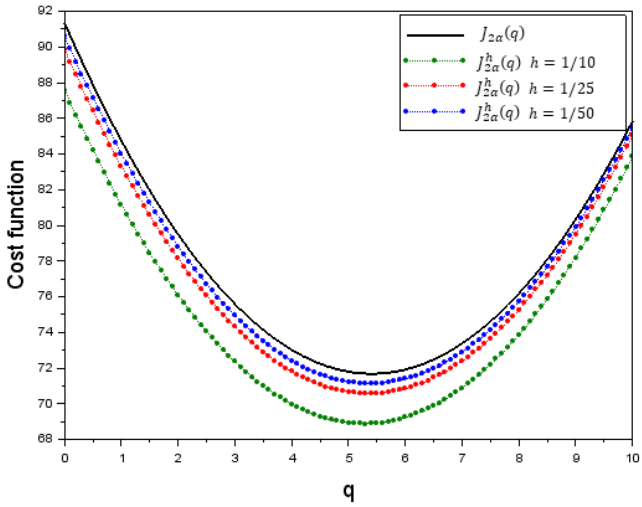


Figure 8. Plot of $J_{2\alpha}$ and $J_{2\alpha}^h$ for $\alpha = 100$ against q.

Moreover, Figure 9 shows the double convergence of $J_{2\alpha}^h \to J_2$ when $(h, \alpha) \to (0, \infty)$. We illustrate how $J_{2\alpha}^h$ gets closer to J_2 as the value of h decreases and the value of α increases.

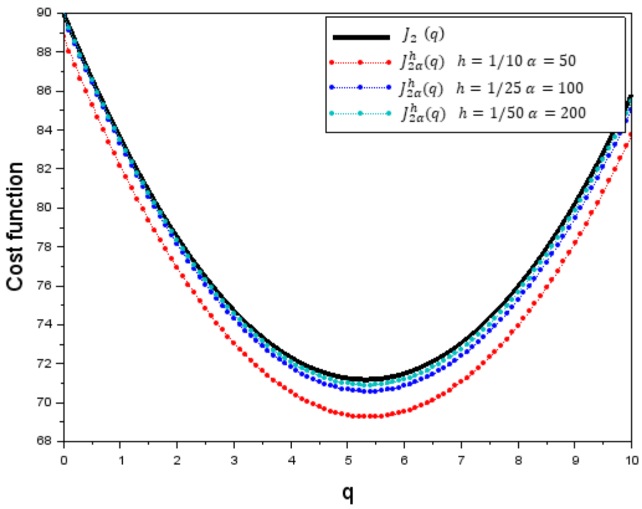


Figure 9. Plot of J_2 and $J_{2\alpha}^h$ against q.

In Figure 10 we plotted the continuous optimal control q_{op} for problem (P_2) given by (50) and optimal control $q_{\alpha_{op}}$ given by (62) for $\alpha = 50, 100, 200$. Notice that as α increases, $q_{\alpha_{op}}$ decreases to the limit q_{op}. In addition, we set different values of n between $n = 10$ and $n = 100$. Recalling that $h = \frac{x_0}{n} = \frac{1}{n}$, for each h, we obtained the optimal discrete control q^h_{op} to problem (P^h_2) defined by (53) and the optimal discrete control $q^h_{\alpha_{op}}$ to problem $(P^h_{2\alpha})$ given by (66) for $\alpha = 50, 100, 200$. For each α fixed, we observe the discrete solution $q^h_{\alpha_{op}} \to q_{\alpha_{op}}$ when $h \to 0$, i.e., $n \to \infty$.

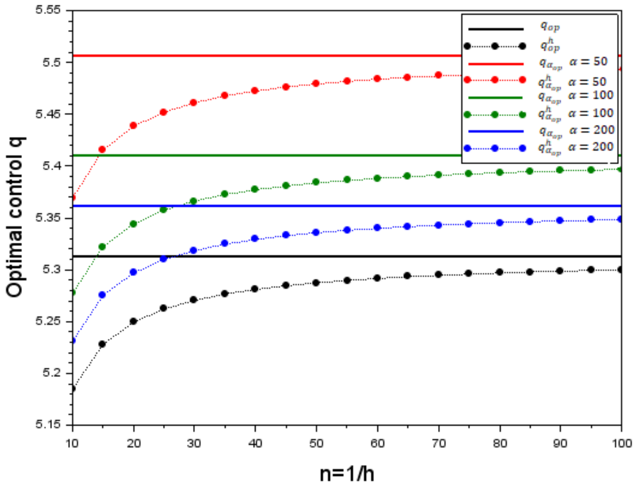


Figure 10. Plot of q_{op}, q^h_{op}, $q_{\alpha_{op}}$ and $q^h_{\alpha_{op}}$ against $n = 1/h$.

6.3. Control Variable b

In this section we obtain some computational examples for the optimal distributed control problems (P_3), (P^h_3), $(P_{3\alpha})$ and $(P^h_{3\alpha})$. For each plot, we set $q = 12, g = 10, z_d = 40$ and $M_3 = 1$.

In Figure 11 we plotted the continuous quadratic cost function J_3 given by (72) and the discrete cost function J^h_3 obtained in (74) against g for $h = 1/10$, $1/25$ and $1/100$. Notice that as h decreases, function $J^h_3 = J^h_3(b)$ also decreases to the limit function $J_3 = J_3(b)$ in agreement with Lemma 15. In a similar manner, in Figure 12, for $\alpha = 50$, we obtained the continuous function $J_{3\alpha}$ and the discrete functions $J^h_{3\alpha}$ for $h = 1/10, 1/25$ and $1/100$. Observe the convergence of $J^h_{3\alpha} \to J_{3\alpha}$ as $h \to 0$. Moreover, Figure 13 shows the double convergence of $J^h_{3\alpha} \to J_3$ when $(h, \alpha) \to (0, \infty)$. We illustrate how $J^h_{3\alpha}$ gets closer to J_3 as the value of h decreases and the value of α increases.

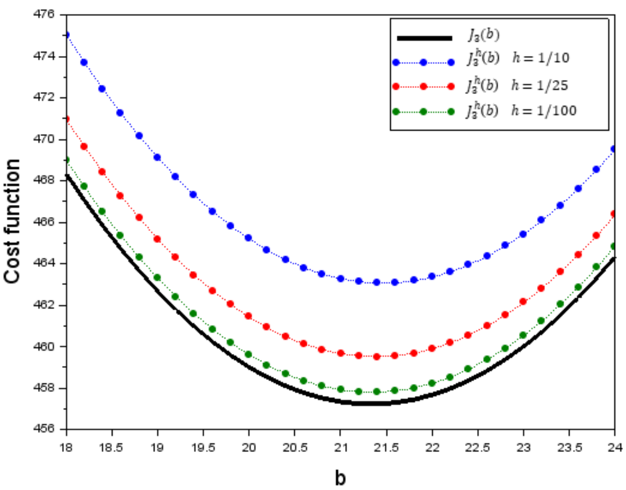


Figure 11. Plot of J_3 and J^h_3 against b.

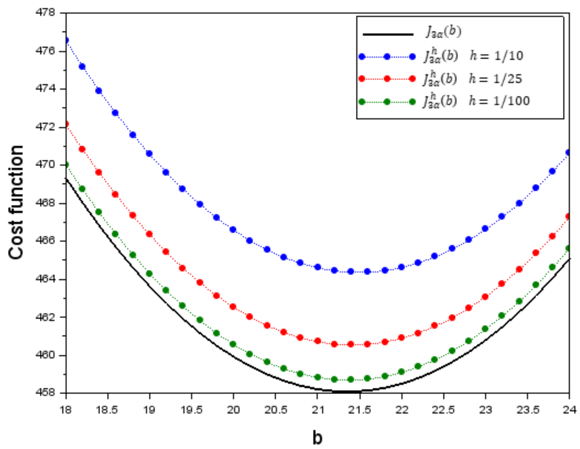


Figure 12. Plot of $J_{3\alpha}$ and $J^h_{3\alpha}$ for $\alpha = 100$ against b.

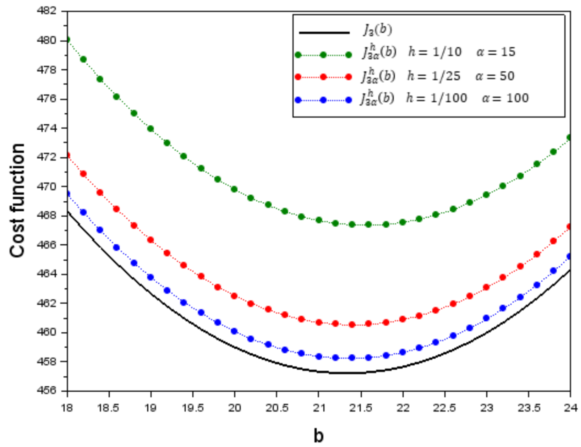


Figure 13. Plot of J_3 and $J_{3\alpha}$ against b.

In Figure 14 we plotted the continuous optimal control b_{op} for problem (P_3) given by (73) and optimal control $b_{\alpha_{op}}$ given by (82) for $\alpha = 15, 50, 100$. Notice that as α increases, $b_{\alpha_{op}}$ decreases to the limit b_{op}. In addition, we set different values of n between $n = 10$ and $n = 100$. Recalling that $h = \frac{x_0}{n} = \frac{1}{n}$, for each h, we obtained the optimal discrete control b^h_{op} to problem (P^h_3) defined by (76) and the optimal discrete control $b^h_{\alpha_{op}}$ to problem $(P^h_{3\alpha})$ given by (87) for $\alpha = 15, 50, 100$. For each α fixed, we observe the discrete solution $b^h_{\alpha_{op}}$ decreases to $b_{\alpha_{op}}$ when $h \to 0$.

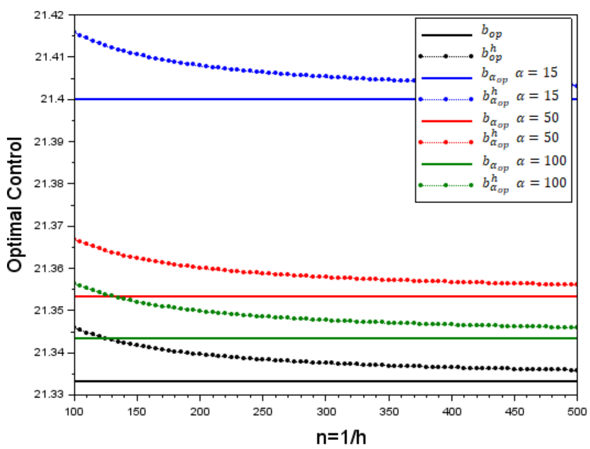


Figure 14. Plot of b_{op}, b^h_{op}, $b_{\alpha_{op}}$ and $b^h_{\alpha_{op}}$ against $n = 1/h$.

7. Improvement of the Order of Convergence

In this section, we introduce alternative discrete solutions $\widetilde{u}^h$ and $\widetilde{u}^h_\alpha$ associated with systems (S) and (S_α), respectively, and analyze the order of convergence of $\widetilde{u}^h$ to u and of $\widetilde{u}^h_\alpha$ to u_α as $h \to 0^+$. The Neumann boundary condition on Γ_2 is approximated by a three-point backward finite-difference scheme. Moreover, for the discrete solution $\widetilde{u}^h_\alpha$, the Robin boundary condition on Γ_1 is approximated by a three-point forward finite-difference scheme. These higher-order boundary approximations lead to an improved order of accuracy.

We consider the system (S) defined by Equations (1) and (2). From this system, we define the discrete problem $(\widetilde{S}^h)$, where for a fixed $h > 0$, $\widetilde{u}^h_i$ approximates $u(x_i, y)$, for $i = 1, \cdots, n+1$. Notice that from the Dirichlet condition on Γ_1, it follows immediately that $\widetilde{u}^h_1 = b$.

For the interior nodes, we employ the classical centered second-order finite-difference approximation given in (15), which leads to the discrete system (16) for $\widetilde{u}^h_i$, $i = 2, ..., n$.

For the Neumann boundary condition on Γ_2, we use the three-point backward approximation

$$\frac{\partial u}{\partial x}(x_{n+1}, y) \approx \frac{3u(x_{n+1}, y) - 4u(x_n, y) + u(x_{n-1}, y)}{2h}. \tag{91}$$

Thus, the discrete Neumann condition can be written as

$$-2qh = 3\widetilde{u}^h_{n+1} - 4\widetilde{u}^h_n + \widetilde{u}^h_{n-1}. \tag{92}$$

In addition, from (16) for $i = n$, we obtain

$$-gh^2 = \widetilde{u}^h_{n+1} - 2\widetilde{u}^h_n + \widetilde{u}^h_{n-1}. \tag{93}$$

Subtracting the two previous equations, it follows that

$$-\widetilde{u}^h_n + \widetilde{u}^h_{n+1} = \frac{gh^2}{2} - qh. \tag{94}$$

Therefore, the system given by (16) together with (94) can be written as

$$Aw^h = \widetilde{T}^h \tag{95}$$

where $w^h = (\widetilde{u}^h_i)_{i=2,\dots,n+1} \in \mathbb{R}^n$ is the vector of unknowns, A is the matrix given by (20) and $\widetilde{T}^h \in \mathbb{R}^n$ is the vector of independent terms:

$$\widetilde{T}^h = \Big(-g\,h^2 - b, -gh^2, \dots, -gh^2, -q\,h + \tfrac{gh^2}{2} \Big)^t. \tag{96}$$

Notice that system (95) differs from (19) in the last component of the vector of independent terms. Solving the linear system gives

$$\widetilde{u}^h_i = b + (i-1)h(gx_0 - q) - \frac{gh^2}{2}(i-1)^2. \tag{97}$$

Taking into account that for $i = 1, \cdots, n$

$$\widetilde{m}_i = \frac{\widetilde{u}^h_{i+1} - \widetilde{u}^h_i}{x_{i+1} - x_i} = gx_0 - q - (2i-1)\frac{gh}{2}, \tag{98}$$

and

$$\widetilde{h}_i = \widetilde{u}_i^h - \widetilde{m}_i x_i = b + \frac{gh^2}{2} i(i-1), \tag{99}$$

the linear approximation is given by $\widetilde{u}^h(x,y) = \widetilde{m}_i x + \widetilde{h}_i$, i.e.,

$$\widetilde{u}^h(x,y) = \left(gx_0 - q - (2i-1)\frac{gh}{2}\right)x + \frac{gh^2}{2} i(i-1) + b, \qquad x \in [x_i, x_{i+1}], \quad i = 1, \cdots, n \tag{100}$$

In the following lemma, we give some bounds for the approximate function $\widetilde{u}^h$:

Lemma 21. *The following bounds hold:*

$$\|u - \widetilde{u}^h\|_H \le D_1 h^2, \qquad \text{and} \qquad \|\tfrac{\partial u}{\partial x} - \tfrac{\partial \widetilde{u}^h}{\partial x}\|_H \le \widetilde{D}_1\, h,$$

where $D_1 = \sqrt{\frac{x_0 y_0}{120}}\, g$ *and* $\widetilde{D}_1 = \sqrt{\frac{x_0 y_0}{12}}\, g$.

Proof. From the definition of the norm in space H and using the expressions (13) and (100) for functions u and $\widetilde{u}^h$, respectively, it follows that

$$\begin{aligned} \|u - \widetilde{u}^h\|_H^2 &= \int_0^{y_0}\int_0^{x_0} \left(u(x,y) - \widetilde{u}^h(x,y)\right)^2 dx\, dy \\ &= y_0 \sum_{i=1}^n \int_{x_i}^{x_{i+1}} E_i^2(x)\, dx, \end{aligned} \tag{101}$$

where

$$E_i(x) = u(x,y) - \widetilde{u}^h(x,y), \quad x \in [x_i, x_{i+1}],\ y \in [0, y_0].$$

Note that, within each subinterval, $E_i(x)$ depends only on x and the index i, but not on y, since both u and $\widetilde{u}^h$ are constant along the y-direction.

A direct computation yields

$$E_i(x) = \frac{g}{2}\Big(-x^2 + (2i-1)hx - h^2 i(i-1)\Big) = -\frac{g}{2}(x - ih)(x - (i-1)h). \tag{102}$$

Then,

$$\begin{aligned} \int_{x_i}^{x_{i+1}} E_i^2(x)\, dx &= \frac{g^2}{4}\left[\frac{(x-ih)^5}{5} + \frac{h}{2}(x-ih)^4 + \frac{h^2}{3}(x-ih)^3\right]_{x_i}^{x_{i+1}} \\ &= \frac{g^2}{4}\left(\frac{h^5}{5} - \frac{h^5}{2} + \frac{h^5}{3}\right) = \frac{g^2}{120} h^5. \end{aligned} \tag{103}$$

As a consequence, from (101), it follows that

$$\|u - \widetilde{u}^h\|_H^2 = y_0 \frac{g^2 h^5}{120} n = \frac{x_0 y_0 g^2}{120} h^4,$$

and then

$$\|u - \widetilde{u}^h\|_H = \sqrt{\frac{x_0 y_0}{120}}\, g h^2.$$

In addition,

$$\begin{aligned} \|\tfrac{\partial u}{\partial x} - \tfrac{\partial \widetilde{u}^h}{\partial x}\|_H^2 &= \int_0^{y_0}\int_0^{x_0} \left(\tfrac{\partial u}{\partial x}(x,y) - \tfrac{\partial \widetilde{u}^h}{\partial x}(x,y)\right)^2 dx\, dy \\ &= y_0 \sum_{i=1}^n \int_{x_i}^{x_{i+1}} F_i^2(x)\, dx, \end{aligned} \tag{104}$$

where

$$F_i(x) = \frac{\partial u}{\partial x}(x,y) - \frac{\partial \widetilde{u}^h}{\partial x}(x,y) = -g\left(x - \frac{(2i-1)h}{2}\right),$$

for $x \in [x_i, x_{i+1}]$. Then,

$$\begin{aligned}\int_{x_i}^{x_{i+1}} F_i^2(x)\,dx &= \int_{x_i}^{x_{i+1}} g^2\left(x - \frac{(2i-1)h}{2}\right)^2 dx \\ &= g^2\left[\frac{1}{3}\left(x - \frac{(2i-1)h}{2}\right)^3\right]_{x_i}^{x_{i+1}} \\ &= g^2\,\frac{1}{3}\left(\frac{h^3}{8} + \frac{h^3}{8}\right) = \frac{g^2}{12}\,h^3.\end{aligned}$$

Therefore, from (104), we have

$$\left\|\frac{\partial u}{\partial x} - \frac{\partial \widetilde{u}^h}{\partial x}\right\|_H^2 = y_0\,\frac{g^2}{12}\,nh^3 = \frac{x_0 y_0 g^2}{12}\,h^2,$$

and finally

$$\left\|\frac{\partial u}{\partial x} - \frac{\partial \widetilde{u}^h}{\partial x}\right\|_H = \sqrt{\frac{x_0 y_0}{12}}\,g\,h.$$

□

Remark 11. *We emphasize that by improving the approximation of the Neumann boundary condition on Γ_2, the convergence order of the error $\|u - \widetilde{u}^h\|_H$ is increased to second order, namely, $O(h^2)$. The improvement is entirely due to the modification in the last component of vectors T^h and T_α^h in systems (S) and (S_α), respectively, where a term of order h^2 appears. This enhancement leads to a more accurate numerical approximation while remaining fully consistent with the theoretical convergence results established in [10,22].*

Remark 12. *The linear system* (95) *obtained by using the three-point backward finite-difference approximation for the Neumann boundary condition on Γ_2 can be equivalently interpreted by introducing a ghost point x_{n+2} outside the computational domain and assuming that the discrete differential equation holds at the boundary node x_{n+1}. Indeed, assuming that the equation is satisfied at $\widetilde{u}_{n+1}^h$, we have*

$$-gh^2 = \widetilde{u}_{n+2}^h - 2\widetilde{u}_{n+1}^h + \widetilde{u}_n^h,$$

while the Neumann boundary condition is approximated by

$$\frac{\widetilde{u}_{n+2}^h - \widetilde{u}_n^h}{2h} = -q.$$

Eliminating the ghost value $\widetilde{u}_{n+2}$ from these two expressions yields

$$-\widetilde{u}_n^h + \widetilde{u}_{n+1}^h = -qh + \frac{gh^2}{2},$$

which coincides with the boundary equation obtained in (94)*. Hence, the three-point backward finite-difference approximation of the Neumann condition is consistent with the ghost-point formulation and leads to the same discrete system.*

Analogously to the analysis of system (S), we propose a new discrete approximation $\widetilde{u}_\alpha^h$ for system (S_α) and study the order of convergence of $\widetilde{u}_\alpha^h$ to u_α as $h \to 0^+$. The associated discrete system $(\widetilde{S}_{h\alpha})$ employs a three-point backward finite-difference approximation for the Neumann boundary condition on Γ_2 and a three-point forward finite-difference approximation for the Robin boundary condition on Γ_1, leading to improved accuracy.

We consider system (S_α) defined by Equations (1) and (3) and define $\widetilde{u}^h_{\alpha,i} \approx u_\alpha(x_i, y)$.

For the interior nodes, $i = 2, \dots, n$, we employ the classical centered second-order finite-difference approximation given in (15):

$$\widetilde{u}^h_{\alpha,i+1} - 2\widetilde{u}^h_{\alpha,i} + \widetilde{u}^h_{\alpha,i-1} = -gh^2. \tag{105}$$

For the Robin boundary at Γ_1, we use the three-point forward approximation:

$$\frac{-3\widetilde{u}^h_{\alpha,1} + 4\widetilde{u}^h_{\alpha,2} - \widetilde{u}^h_{\alpha,3}}{2h} = \alpha(\widetilde{u}_{\alpha,1} - b). \tag{106}$$

Combining this expression with the interior equation at $i = 2$ yields the simplified discrete condition

$$-(1+\alpha h)\widetilde{u}^h_{\alpha,1} + \widetilde{u}^h_{\alpha,2} = -\alpha h b - \frac{gh^2}{2}. \tag{107}$$

For the Neumann boundary at Γ_2 we use the three-point backward approximation:

$$3\widetilde{u}^h_{\alpha,n+1} - 4\widetilde{u}^h_{\alpha,n} + \widetilde{u}^h_{\alpha,n-1} = -2qh. \tag{108}$$

Combining with the interior equation for $i = n$ gives

$$-\widetilde{u}^h_{\alpha,n} + \widetilde{u}^h_{\alpha,n+1} = -qh + \frac{gh^2}{2}. \tag{109}$$

The system given by (105), (107) and (109) can be rewritten as

$$A_\alpha w^h_\alpha = \widetilde{T}^h_\alpha \tag{110}$$

where $w^h_\alpha = (\widetilde{u}^h_{\alpha,i})_{i=1,\dots,n+1} \in \mathbb{R}^{n+1}$ is the vector of unknowns, A_α is the matrix given by (25) and $\widetilde{T}^h_\alpha \in \mathbb{R}^{n+1}$ is the vector of independent terms:

$$\widetilde{T}^h_\alpha = \Big(-\alpha\, b\, h - \tfrac{gh^2}{2}, -gh^2, \dots, -gh^2, -q\, h + \tfrac{gh^2}{2} \Big)^t \in \mathbb{R}^{n+1}. \tag{111}$$

It should be noted that only the first and last components of $\widetilde{T}^h_\alpha$ differ from those in T_α given by (26).

The solution of system (110) is given by

$$\widetilde{u}^h_{\alpha,i} = b + \frac{1}{\alpha}\Big(gx_0 - q\Big) + (i-1)h\,(gx_0 - q) - \frac{g}{2}\big((i-1)h\big)^2, \quad i = 1, \dots, n+1. \tag{112}$$

We define the linear interpolation on each subinterval $[x_i, x_{i+1}]$ by

$$\widetilde{u}^h_\alpha(x,y) = \widetilde{m}_{\alpha,i}x + \widetilde{h}_{\alpha,i}, \quad x \in [x_i, x_{i+1}],\ y \in [0, y_0], \tag{113}$$

where

$$\widetilde{m}_{\alpha,i} = gx_0 - q - gh\left(i - \frac{1}{2}\right), \quad i = 1, \dots, n, \tag{114}$$

$$\widetilde{h}_{\alpha,i} = b + \frac{1}{\alpha}(gx_0 - q) + \frac{gh^2}{2}(i-1)i, \quad i = 1, \dots, n. \tag{115}$$

From the previous expressions, we derive the following lemma.

Lemma 22. *The following bounds hold:*

$$\|u_\alpha - \widetilde{u}_\alpha^h\|_H \leq D_2 h^2, \qquad \|\tfrac{\partial u_\alpha}{\partial x} - \tfrac{\partial \widetilde{u}_\alpha^h}{\partial x}\|_H \leq \widetilde{D}_2 h,$$

where $D_2 = \sqrt{\frac{x_0 y_0}{120}} g$ *and* $\widetilde{D}_2 = \sqrt{\frac{x_0 y_0}{12}} g$.

Proof. By the definition of the H-norm, and using the expression for u_α in (13) as well as the definition of $\widetilde{u}_\alpha^h$ in (113), it follows that

$$\|u_\alpha - \widetilde{u}_\alpha^h\|_H^2 = y_0 \sum_{i=1}^{n} \int_{x_i}^{x_{i+1}} E_{\alpha,i}^2(x)\, dx, \tag{116}$$

where

$$E_{\alpha,i}(x) = u_\alpha(x,y) - \widetilde{u}_\alpha^h(x,y)$$

$$E_{\alpha,i}(x) = -\frac{g}{2}x^2 + \frac{g}{2}(2i-1)hx - \frac{gh^2}{2}(i^2 - i), \quad x \in [x_i, x_{i+1}] \tag{117}$$

We can notice that $E_{\alpha,i}(x) = E_i(x)$ where $E_i(x)$ is given by (102). Therefore, from (103), it follows immediately that

$$\|u_\alpha - \widetilde{u}_\alpha^h\|_H^2 \leq \frac{x_0 y_0 g^2}{120} h^4.$$

In addition,

$$\begin{aligned} \|\tfrac{\partial u}{\partial x} - \tfrac{\partial \widetilde{u}^h}{\partial x}\|_H^2 &= \int_0^{y_0}\int_0^{x_0} \left(\tfrac{\partial u}{\partial x}(x,y) - \tfrac{\partial \widetilde{u}^h}{\partial x}(x,y)\right)^2 dx\, dy \\ &= y_0 \sum_{i=1}^{n} \int_{x_i}^{x_{i+1}} g^2\left(x - \frac{(2i-1)h}{2}\right)^2 dx \\ &= y_0 \frac{g^2}{12} h^3\, n = \frac{x_0 y_0 g^2}{12} h^2. \end{aligned} \tag{118}$$

□

8. Conclusions

Applying the finite difference method, we derived discrete systems (S^h) and (S_α^h) and discrete optimization problems (P_i^h) and $(P_{i\alpha}^h)$, $i = 1, 2, 3$, where $\alpha > 0$ is a parameter that represents the heat transfer coefficient on a portion of the boundary of the domain. Explicit discrete solutions were obtained, and convergence results as discretization step $h \to 0$ and parameter $\alpha \to \infty$ were proved. Error estimations were also obtained as a function of step h. Some numerical computations were provided in order to illustrate the theoretical results.

The obtained results showed that the proposed numerical approach provided first-order accurate approximations for both state systems (S) and (S_α) and associated optimal control problems (P_i) and $(P_{i\alpha})$, $i = 1, 2, 3$, and that the discrete solutions converged to the corresponding continuous ones as discretization step $h \to 0$.

Finally, for systems (S) and (S_α), an alternative discretization of the Neumann boundary condition on Γ_2 and of the Robin boundary condition on Γ_1 for (S_α) was considered. By modifying the approximation of these boundary conditions, the order of convergence of the numerical solution was improved, leading to a more accurate approximation.

A main limitation of the present work is that the analysis is restricted to rectangular domains, which allows the derivation of explicit solutions and simplifies the numerical implementation. As a future development, the proposed methodology is expected to be extended to more general domains, including polar and spherical coordinate systems.

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