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Exact and τ-method approximate solutions to a two-phase Stefan problem with Robin condition and exponential sources

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ABSTRACT

This work examines a two-phase Stefan problem in a semi-infinite domain with a convective (Robin-type) boundary condition at the fixed face and with exponential internal heat sources in terms of a similarity-type variable. Existence and uniqueness of the similarity-type solution are established with exactness, and asymptotic analysis of the system is carried out, showing convergence towards the associated two-phase Stefan problem with the imposed temperature condition at the fixed face in the limit when the heat transfer coefficient tends to infinity. A numerical example pertaining to melting paraffin is given to support the theoretical results. Moreover, the Tau-method based on shifted Chebyshev and Legendre polynomials are employed to construct approximate solutions that are then compared to the exact similarity-type solution that serves as benchmark for quantifying the accuracy of spectral approximations.

1. Introduction

Stefan problems belong to an interesting area because they originate from many important applications in physics, engineering, and industrial processes. They are basic to the comprehension of phase transition phenomena particularly in situations involving heat transfer and solidification or melting processes. The objective of Stefan problems is to represent the solid and liquid phases of a material that is undergoing a phase change and to locate the interface between these phases, or in other words, to find the position of the free boundary. In many practical situations, transient heat conduction problems with one or more phase changes are encountered. Applications of Stefan-type problems include solidification of binary alloys, [15,27,31,36], continuous casting of steel [5], and cell cryopreservation [18]. Moreover, numerous applications of phase-change processes can be found in books such as [4,17,19,22,23,25,37].

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In Stefan problems, which model phase-change processes such as melting, the boundary conditions play a fundamental role in determining the evolution of the temperature and the free boundary [30], [16], [20], [14], [38], [34], [32]. A particularly important condition from the physical point of view is the convective boundary condition, also known as a Robin-type condition [33]. This condition describes situations in which the heat flux at the boundary is proportional to the difference between the material temperature and the ambient temperature, thus representing a thermal exchange mechanism with the surrounding medium, i.e.,

$$k\frac{\partial \Phi}{\partial x}(0,t) = H(t)\left(\Phi(0,t) - B_{\infty}\right),$$

where Φ is the temperature of the material, k is the thermal conductivity, $H(t)$ characterizes the heat transfer at the fixed face, and $B_{\infty} > 0$ represents the ambient temperature at $x = 0$. The melting of a semi-infinite material under a convective boundary condition at the fixed face $x = 0$ is considered. It takes the form $H(t) = \frac{h_0}{\sqrt{t}}$, with $h_0 > 0$, as in [35]. Unlike conditions of the Dirichlet type (prescribed temperature) or Neumann type (prescribed flux), in many physical and engineering situations -e.g. concerning solidification of metals or cooling of some materials that are subject to a convective environment- the convective condition represents more realistic modeling.

Heat conduction problems are considered non-classical when the source term is linear or nonlinear, depending on the heat flux or temperature at the boundary of the domain, in accordance with the corresponding imposed boundary condition. Non-classical problems are motivated by the modeling of temperature regulation systems in isotropic media, where the source term describes a cooling or heating effect depending on different types of sources, which are related to the evolution of the unknown boundary condition at the domain boundary. The first article connecting the non-classical heat equation with a phase-change process (i.e., the Stefan problem) was [12], followed by several other works on the topic, such as [11,9,10,6].

Explicit or exact solutions yield the whole explicit knowledge of a phase-change process under consideration; however, it is by no means easy to obtain such solutions. Since Stefan problems are nonlinear, analytical solutions are possible for just a few instances. Thus, it becomes important to study different methods that can give approximate solutions. Another major thrust area in Stefan problems is developing numerical methods for getting approximate solutions. Herein, these may be reduced to ordinary differential equation problems using some similarity-type variable through which originally moving-boundary problem becomes defined on a single independent variable. This is reduction facilitates the application of approximate methods in solving ordinary differential equations obtained thereof. For getting precise approximations, spectral approaches in particular provide several benefits, including great precision and high efficiency. The basic notion of spectral methods is to express the approximate solution as a finite sum of orthogonal polynomials multiplied by undetermined coefficients. This simplifies the typical differential equation to a set of algebraic linear or nonlinear equations for the coefficients, which may then be resolved by means of standard numerical methods. In spectral approaches, the orthogonal polynomials -the basis functions- are smooth and mutually orthogonal. Different shapes have been used, including shifted Chebyshev and Legendre polynomials. Spectral techniques have been often used to get exact approximate solutions for ordinary, partial, and fractional differential equations. Among these methods, the Tau-method has been mostly used for getting approximate solutions of ordinary differential equations.

The main goal of this work is to find a similarity-type solution for a two-phase Stefan problem, which includes a convective boundary condition at the fixed face and an internal heat source that follows an exponential pattern based on the similarity-type variable (see [13,24,29]). The behavior of the problem is also explored as the heat transfer coefficient of the Robin condition approaches infinity. Moreover, approximate solutions are obtained using the Tau-method with shifted Chebyshev and shifted Legendre polynomials [2,1,8,3,21,28], and then compared with the exact similarity-type solution, which acts as the benchmark. This study also involves the development of a computer code in SciLab that computes an approximate solution to the two-phase Stefan problem by applying the Tau-method with shifted Chebyshev and Legendre polynomials for both phases.

The article is structured in a clear way. Section 2 introduces a one-dimensional two-phase Stefan problem that models the melting of a semi-infinite material in the region $x \geq 0$. This section also discusses a convective boundary condition at the fixed face $x = 0$ and incorporates an internal heat source. Section 3 addresses the existence and uniqueness of a solution under suitable assumptions on the problem's data. To provide a concrete illustration, a computational example is included to show the application of the model to paraffin melting. Section 4 examines the asymptotic behavior of the system and shows that the solution converges to the unique solution of the corresponding Stefan problem with a prescribed temperature at $x = 0$. Section 5 presents approximate solutions obtained through the Tau method with shifted Chebyshev and Legendre polynomials, comparing them with the exact solution to evaluate the associated errors of each method. Finally, Section 6 concludes the article.

2. Mathematical formulation

A one-dimensional two-phase Stefan problem is considered, describing the melting of a homogeneous semi-infinite material ($x \geq 0$), where a convective (Robin-type) boundary condition is prescribed at the fixed face $x = 0$. The objective is to determine the temperature distribution

$$\Phi(x,t) = \begin{cases} \Phi_2(x,t) & \text{if} \quad 0 \leq x \leq s(t), \quad t > 0, \\ \Phi_1(x,t) & \text{if} \quad s(t) < x, \quad\quad\quad t > 0, \end{cases} \tag{1}$$

and the free boundary $x = s(t)$, $t > 0$, which separates the solid and liquid phases.

Initially, at time $t = 0$, the material is in a solid state, as shown in Fig. 1. From a time $t > 0$ onward, the temperature distribution in each region is as shown in Fig. 2, where $x = s(t)$ is the free boundary separating both regions.

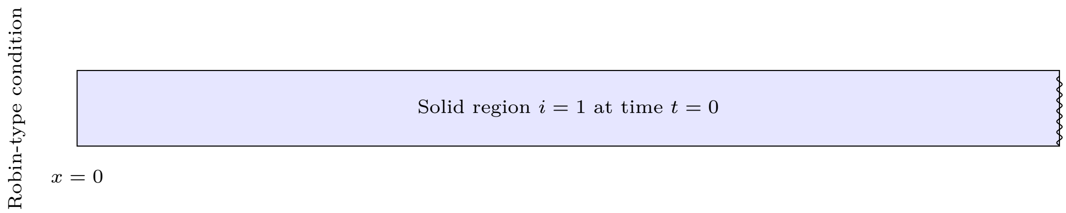

Fig. 1. Temperature profile in the solid region at time $t = 0$.

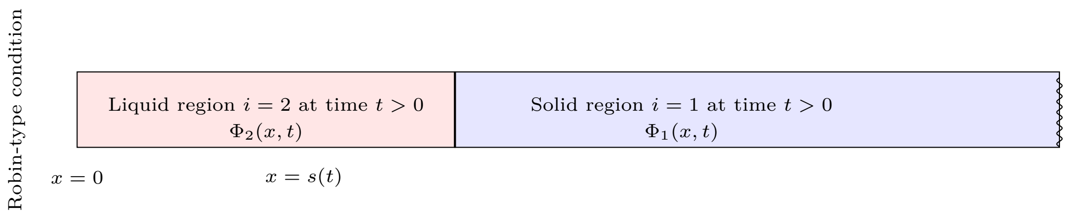

Fig. 2. Spatial distribution by regions: solid and liquid at time $t > 0$.

A mathematical description of the model is given by

$$\frac{\partial \Phi_2}{\partial t} = a_2^2 \frac{\partial^2 \Phi_2}{\partial x^2} + \frac{1}{\rho c_2} g_2(x,t), \qquad 0 < x < s(t),\ t > 0, \tag{2}$$

$$\frac{\partial \Phi_1}{\partial t} = a_1^2 \frac{\partial^2 \Phi_1}{\partial x^2} + \frac{1}{\rho c_1} g_1(x,t), \qquad x > s(t),\ t > 0, \tag{3}$$

for two internal heat sources given by:

$$g_i(x,t) = (-1)^{i+1} \tfrac{\rho \ell}{t} \exp\left\{ - \left(\tfrac{x}{2a_i\sqrt{t}} + d_i \right)^2 \right\}, \quad i = 1,2, \tag{4}$$

where the positive constants $a_i^2 = \frac{k_i}{\rho c_i}$ and c_i represent the thermal diffusivity and specific heat, respectively, for the phases $i = 1$ (solid region) and $i = 2$ (liquid region), with $d_i > 0$, $i = 1, 2$, $\ell > 0$ the latent heat per unit mass, and $\rho > 0$ the mass density common to both phases.

This type of heat source term is relevant due to the use of microwave energy, as indicated in [29], which was applied in [13,24].

The initial temperature and the temperature at $x \to +\infty$ are assumed to be constant

$$\Phi_1(x,0) = -C < 0, \qquad x > 0, \tag{5}$$

$$\Phi_1(+\infty,t) = -C < 0, \qquad t > 0. \tag{6}$$

At the fixed boundary $x = 0$, a convective boundary condition is imposed

$$k_2 \frac{\partial \Phi_2}{\partial x}(0,t) = \frac{h_0}{\sqrt{t}} \left(\Phi_2(0,t) - B_\infty \right), \qquad t > 0, \tag{7}$$

where $h_0 > 0$ is the coefficient characterizing the heat transfer at the fixed boundary $x = 0$, and $B_\infty > 0$ is the ambient temperature.

The interface condition for the phase change is derived from an energy balance at the free boundary $x = s(t)$ (Stefan condition)

$$k_1 \frac{\partial \Phi_1}{\partial x}(s(t),t) - k_2 \frac{\partial \Phi_2}{\partial x}(s(t),t) = \rho \ell \dot{s}(t), \qquad t > 0, \tag{8}$$

where the temperature at the interface is assumed to be constant:

$$\Phi_1(s(t),t) = \Phi_2(s(t),t) = 0, \qquad t > 0. \tag{9}$$

In addition, the position of the free boundary at the initial time is

$$s(0) = 0. \tag{10}$$

3. Existence and uniqueness of solution

In this section, the existence of a similarity-type solution to the Stefan problem (2)-(10) is proven. Under suitable assumptions on the initial data, uniqueness of the solution is also ensured.

By employing the domain immobilization method, solutions of the form

$$\Phi_i(x,t) = \zeta_i(\omega), \quad i = 1,2, \tag{11}$$

are considered, where the transformed spatial variable ω is defined by

$$\omega = \frac{x}{s(t)}. \tag{12}$$

Under this transformation, the condition (8) becomes

$$k_1\zeta_1'(1) - k_2\zeta_2'(1) = \rho\ell\, s(t)\, \dot{s}(t), \tag{13}$$

which implies that $s(t)\,\dot{s}(t)$ remains constant. As a consequence, the free boundary evolves according to

$$s(t) = 2a_2\lambda\sqrt{t}, \quad t > 0, \tag{14}$$

with $\lambda > 0$ denoting an unknown dimensionless parameter that characterizes the interface.

To proceed, we define

$$y_i(\eta) = \zeta_i\left(\frac{\eta}{\lambda}\right), \quad i = 1, 2, \tag{15}$$

where $\omega = \frac{\eta}{\lambda}$ represents the similarity variable.

This leads directly to the following theorem, which establishes the equivalence between the Stefan problem (2)–(10) and a system of two coupled ordinary differential equations:

Theorem 3.1. *The Stefan problem defined by* (2)-(10) *has a similarity-type solution* (Φ, s) *given by*

$$\Phi_2(x,t) = y_2\left(\tfrac{x}{2a_2\sqrt{t}}\right), \qquad 0 \le x \le s(t),\ t > 0, \tag{16}$$

$$\Phi_1(x,t) = y_1\left(\tfrac{x}{2a_2\sqrt{t}}\right), \qquad x > s(t),\ t > 0, \tag{17}$$

$$s(t) = 2a_2\lambda\sqrt{t}, \qquad t > 0, \tag{18}$$

if and only if the functions $y_1 = y_1(\eta)$ *is twice differentiable in* $(\lambda, +\infty)$, $y_2 = y_2(\eta)$ *is twice differentiable in* $(0, \lambda)$ *and the parameter* $\lambda > 0$, *satisfy the following ordinary differential problems:*

$$y_2'' + 2\eta\, y_2' = \tfrac{4}{\mathrm{Ste}_2}\exp\left(-(\eta + d_2)^2\right), \qquad 0 < \eta < \lambda, \tag{19}$$

$$y_2(\lambda) = 0, \tag{20}$$

$$y_2'(0) = 2\mathrm{Bi}\left(y_2(0) - B_\infty\right), \tag{21}$$

and

$$y_1'' + 2\tfrac{a_2^2}{a_1^2}\eta\, y_1' = -\tfrac{4a_2^2}{a_1^2}\tfrac{1}{\mathrm{Ste}_1}\exp\left(-\left(\tfrac{a_2}{a_1}\eta + d_1\right)^2\right), \qquad \eta > \lambda, \tag{22}$$

$$y_1(\lambda) = 0, \tag{23}$$

$$y_1(+\infty) = -C, \tag{24}$$

coupled through the following condition:

$$y_1'(\lambda) - \frac{k_2}{k_1}y_2'(\lambda) = 2\tfrac{k_2}{k_1}\tfrac{\lambda}{\mathrm{Ste}_2}, \tag{25}$$

where $\mathrm{Ste}_1 = \dfrac{c_1 C}{\ell}$ *and* $\mathrm{Ste}_2 = \dfrac{c_2 B_\infty}{\ell}$ *are the Stefan numbers, while* $\mathrm{Bi} = \dfrac{a_2 h_0}{k_2}$ *denotes the Biot number.*

The existence and uniqueness of the solution to ordinary differential problem (19)-(25) will be established below.

Theorem 3.2. *If*

$$\mathrm{Ste}_1 \geq \frac{\sqrt{\pi}}{d_1 \exp(d_1^2)} \left(1 - \exp(d_1^2)\,\mathrm{erfc}(d_1)\right), \tag{26}$$

and

$$\mathrm{Bi} > h_2 := \tfrac{1}{\mathrm{Ste}_2} \left(\mathrm{Ste}_1 - \tfrac{\sqrt{\pi}}{d_1 \exp(d_1^2)} \left(1 - \exp(d_1^2)\,\mathrm{erfc}(d_1)\right)\right), \tag{27}$$

then the problem (19)–(25) *admits a unique solution of the form*

$$y_2(\eta) = \varphi_2(\eta) + \Psi_{2,\mathrm{Bi}}(\eta), \qquad 0 \leq \eta \leq \lambda, \tag{28}$$

$$y_1(\eta) = \varphi_1(\eta) + \Psi_1(\eta), \qquad \eta > \lambda, \tag{29}$$

where λ *is the unique solution to the equation*

$$G_{\mathrm{Bi}}(z) = \frac{\mathrm{Ste}_2}{\sqrt{\pi}}, \quad z > 0, \tag{30}$$

with

$$\varphi_2(z) = -\tfrac{\sqrt{\pi} B_\infty}{d_2 \mathrm{Ste}_2} \left(\mathrm{erf}(z+d_2) - \mathrm{erf}(d_2) - \tfrac{\mathrm{erf}(z)}{\exp(d_2^2)}\right), \tag{31}$$

$$\Psi_{2,\mathrm{Bi}}(z) = \frac{B_\infty \sqrt{\pi}(\mathrm{erf}(\lambda) - \mathrm{erf}(z)) - \varphi_2(\lambda)\left(\frac{1}{\mathrm{Bi}} + \sqrt{\pi}\,\mathrm{erf}(z)\right)}{\frac{1}{\mathrm{Bi}} + \sqrt{\pi}\,\mathrm{erf}(\lambda)}, \tag{32}$$

$$\varphi_1(z) = \tfrac{C\sqrt{\pi}}{d_1 \mathrm{Ste}_1} \left[\frac{\mathrm{erf}\left(\frac{a_2 \lambda}{a_1}\right) - \mathrm{erf}\left(\frac{a_2 z}{a_1}\right)}{\exp\left(\frac{2 a_2 \lambda d_1}{a_1} + d_1^2\right)} + \mathrm{erf}\left(\tfrac{a_2 z}{a_1} + d_1\right) - \mathrm{erf}\left(\tfrac{a_2 \lambda}{a_1} + d_1\right) \right], \tag{33}$$

$$\Psi_1(z) = -\frac{C + \varphi_1(+\infty)}{\mathrm{erfc}\left(\frac{a_2\lambda}{a_1}\right)} \left(\mathrm{erf}\left(\tfrac{a_2 z}{a_1}\right) - \mathrm{erf}\left(\tfrac{a_2 \lambda}{a_1}\right)\right), \tag{34}$$

$$\varphi_1(+\infty) = \tfrac{\ell\sqrt{\pi}}{c_1 d_1} \left[\mathrm{erfc}\left(\tfrac{a_2\lambda}{a_1} + d_1\right) - \frac{\mathrm{erfc}\left(\frac{a_2\lambda}{a_1}\right)}{\exp\left(d_1^2 + \frac{2 a_2 \lambda d_1}{a_1}\right)}\right], \tag{35}$$

$$G_{\mathrm{Bi}}(z) = \frac{F_{0,\mathrm{Bi}}(z) h_1(z)}{Q\left(\frac{a_2}{a_1} z\right)} + \widetilde{F}_{0,\mathrm{Bi}}(z), \tag{36}$$

$$F_{0,\mathrm{Bi}}(z) = \left(\tfrac{1}{\sqrt{\pi}\mathrm{Bi}} + \mathrm{erf}(z)\right) z \exp\left(z^2\right), \tag{37}$$

$$h_1(z) = \mathrm{Ste}_1 + \tfrac{\sqrt{\pi}}{d_1} \left[\mathrm{erfc}\left(\tfrac{a_2}{a_1} z + d_1\right) - \frac{\mathrm{erfc}\left(\frac{a_2}{a_1} z\right)}{\exp\left(d_1^2 + \frac{2 a_2 d_1}{a_1} z\right)}\right], \tag{38}$$

$$Q(z) = \sqrt{\pi}\, z \exp\left(z^2\right) \mathrm{erfc}(z), \tag{39}$$

$$\widetilde{F}_{0,\mathrm{Bi}}(z) = F_{0,\mathrm{Bi}}(z) + \tfrac{\mathrm{erf}(z+d_2) - \mathrm{erf}(d_2)}{d_2} + \tfrac{1 - \exp(-2 d_2 z)}{d_2 \sqrt{\pi}\mathrm{Bi} \exp(d_2^2)} - \tfrac{\exp\left(-d_2^2 - 2 d_2 z\right) \mathrm{erf}(z)}{d_2}. \tag{40}$$

Proof. Let the function F_0 be defined by

$$F_0(z) = z\,\mathrm{erf}(z)\,\exp\left(z^2\right), \quad z > 0. \tag{41}$$

This function satisfies $F_0(0) = 0$, $F_0(+\infty) = +\infty$, and increases strictly.

The function $F_{0,\mathrm{Bi}}$ defined in (37), can be written as

$$F_{0,\mathrm{Bi}}(z) = \tfrac{1}{\sqrt{\pi}\mathrm{Bi}}\, z \exp(z^2) + F_0(z), \quad z > 0,$$

and satisfies $F_{0,\mathrm{Bi}}(0) = 0$, $F_{0,\mathrm{Bi}}(+\infty) = +\infty$, being strictly increasing.

On the other hand, taking into account hypothesis (26), it follows that the function h_1, defined in (38), satisfies

$$h_1(0) = \mathrm{Ste}_1 + \tfrac{\sqrt{\pi}}{d_1}\left[\mathrm{erfc}(d_1) - \tfrac{1}{\exp(d_1^2)}\right] \geq 0, \quad h_1(+\infty) = \mathrm{Ste}_1,$$

and

$$h_1'(z) = \frac{2a_2\sqrt{\pi}\,\mathrm{erfc}\left(\frac{a_2}{a_1}z\right)}{a_1 \exp\left(d_1^2 + \frac{2d_1 a_2}{a_1}z\right)} > 0, \quad z > 0,$$

so that the function h_1 is positive and strictly increasing for all $z > 0$.

Moreover, the function Q, defined in (39), is strictly increasing for all $z > 0$, and satisfies $Q(0) = 0$, $Q(+\infty) = 1$, and $Q'(0) = \sqrt{\pi}$.

The function G_{Bi}, defined by (36), can be rewritten as

$$G_{\mathrm{Bi}}(z) = W(z)\left(\tfrac{1}{\sqrt{\pi}\mathrm{Bi}} + \mathrm{erf}(z)\right) h_1(z) + \widetilde{F}_{0,\mathrm{Bi}}(z), \tag{42}$$

where

$$W(z) = \tfrac{z\exp(z^2)}{Q\left(\frac{a_2}{a_1}z\right)}, \quad z > 0. \tag{43}$$

Then,

$$G_{\mathrm{Bi}}'(z) = W'(z)\left(\tfrac{1}{\sqrt{\pi}\mathrm{Bi}} + \mathrm{erf}(z)\right) h_1(z) + W(z)\left[\left(\tfrac{1}{\sqrt{\pi}\mathrm{Bi}} + \mathrm{erf}(z)\right) h_1(z)\right]' + \widetilde{F}_{0,\mathrm{Bi}}'(z). \tag{44}$$

Since

$$W'(z) = \frac{2z^2\exp(z^2)Q\left(\frac{a_2}{a_1}z\right) + 2\left(\frac{a_2}{a_1}z\right)^2 \exp(z^2)\left(1 - Q\left(\frac{a_2}{a_1}z\right)\right)}{Q^2\left(\frac{a_2}{a_1}z\right)}, \tag{45}$$

it follows that $W'(z) > 0$ for all $z > 0$.

Moreover, the function $\widetilde{F}_{0,\mathrm{Bi}}$, defined by (40), satisfies the following properties:

$$\widetilde{F}_{0,\mathrm{Bi}}(0) = F_{0,\mathrm{Bi}}(0) = 0,$$
$$\widetilde{F}_{0,\mathrm{Bi}}(+\infty) = +\infty,$$

and its derivative satisfies

$$\widetilde{F}_{0,\mathrm{Bi}}'(z) = F_{0,\mathrm{Bi}}'(z) + 2\exp(-d_2^2 - 2d_2 z)\left(\tfrac{2}{\sqrt{\pi}\mathrm{Bi}} + \mathrm{erf}(z)\right) > 0,$$

which implies that $\widetilde{F}_{0,\mathrm{Bi}}$ is strictly increasing for all $z > 0$.

From the above, the function G_{Bi} is strictly increasing for all $z > 0$. Moreover,

$$G_{\mathrm{Bi}}(0) = \lim_{z\to 0} \frac{[F_{0,\mathrm{Bi}}(z)h_1(z)]'}{Q'\left(\frac{a_2}{a_1}z\right)\frac{a_2}{a_1}} + \widetilde{F}_{0,\mathrm{Bi}}(0) = \frac{a_1}{a_2\pi \mathrm{Bi}} h_1(0) \geq 0,$$

and

$$G_{\mathrm{Bi}}(+\infty) = +\infty.$$

Finally, by hypothesis (27), it holds that $G_{\mathrm{Bi}}(0) < \frac{\mathrm{Ste}_2}{\sqrt{\pi}}$, which guarantees the existence of a unique solution $\lambda > 0$ to equation (30). □

From Theorems 3.1 and 3.2, the following immediately follows:

Theorem 3.3. *If hypotheses* (26) *and* (27) *hold, then the Stefan problem defined by* (2)-(10) *has a unique similarity-type solution given by* (16)-(18)*, where* y_2 *and* y_1 *are determined by* (28) *and* (29)*, respectively, and* $\lambda > 0$ *is the unique solution to equation* (30)*.*

Remark 3.4. If hypothesis (26) is not satisfied, the existence of at least one solution to the Stefan problem (2)–(10) remains guaranteed.

Example 3.5. A computational example is provided to illustrate the applicability of the two-phase Stefan problem under a convective boundary condition imposed at the fixed boundary $x = 0$, as established in Theorem 3.3. The example focuses on the melting process of paraffin, assuming constant thermal properties, which are specified in Table 1.

Table 1
Thermal coefficients for the paraffin in the phases solid $i = 1$ and liquid $i = 2$.

Phase	Themal conductivity k_i (W/m·K)	Specific heat c_i (J/kg·K)	Latent heat ℓ (J/kg)	Density ρ (kg/m^3)
$i = 1$	$0, 2$	2200	200000	800
$i = 2$	$0, 4$	2200	200000	800

If the initial temperature is $-C = -250\,\mathrm{K}$ and the ambient temperature is $B_\infty = 300\,\mathrm{K}$, considering the data provided in Table 1 and the definitions of thermal diffusivity $a_i^2 = \frac{k_i}{\rho c_i}$ $(i = 1, 2)$ and the Stefan numbers Ste_1 and Ste_2 as defined in Theorem 3.2, the parameters listed in Table 2 are derived.

Table 2
Parameters deduced from Table 1.

Thermal diffusivity phase 1 a_1^2 (m^2/s)	Thermal diffusivity phase 2 a_2^2 (m^2/s)	Ste_1 –	Ste_2 –
1.13×10^{-7}	2.27×10^{-7}	2.75	3.3

Taking into account the thermal coefficients listed in Table 1 and the parameters derived in Table 2, for the case of the positive constants $d_1 = 0.1$ and $d_2 = 0.05$, it follows that $h_2 = 0.2827$. By selecting $Bi = 0.2863$, the unique solution of equation (30), which characterizes the free boundary of problem (2)–(10), is given by $\lambda = 0.0921568$.

Fig. 3 presents a color map of the temperature distribution given by (16) and (17), along with the free boundary $x = s(t)$ for this case.

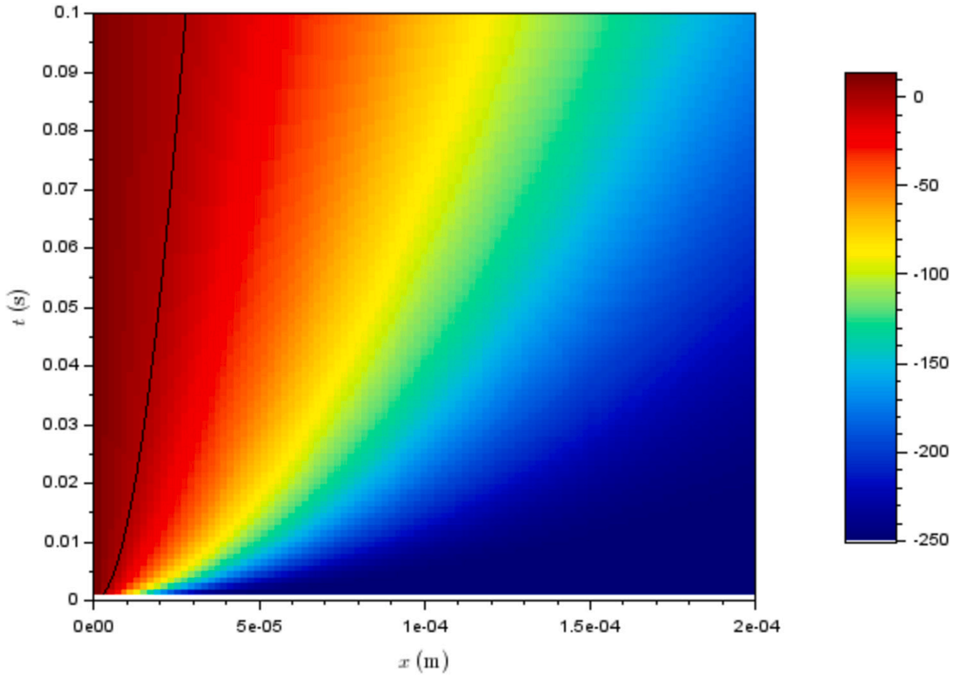

Fig. 3. Temperature color map and free boundary.

4. Asymptotic behavior

If the coefficient h_0, which characterizes the heat flux at the fixed boundary $x = 0$, tends to infinity, then the corresponding Biot number $\mathrm{Bi} = \frac{a_2 h_0}{k_2}$ also tends to infinity. In this limit, the Stefan problem with Robin-type boundary condition defined by (2)–(10) reduces to a Stefan problem with Dirichlet-type boundary condition at the fixed face. Consequently, the solution of the two-phase Stefan problem with a Robin condition converges to the solution of the Stefan problem with a Dirichlet condition at the fixed face $x = 0$.

In [13], the one-dimensional two-phase Stefan problem modeling the melting of a semi-infinite homogeneous material occupying the domain $x \geq 0$ is analyzed, where a temperature (Dirichlet-type) boundary condition is imposed at the fixed boundary $x = 0$. The problem consists of determining the temperature distribution

$$\theta(x,t) = \begin{cases} \theta_2(x,t), & 0 \leq x \leq w(t), \quad t > 0, \\ \theta_1(x,t), & w(t) < x, \quad t > 0, \end{cases}$$

and the free boundary $x = w(t)$, $t > 0$, which separates the two phases and satisfies:

$$\frac{\partial \theta_2}{\partial t} = a_2^2 \frac{\partial^2 \theta_2}{\partial x^2} + \frac{1}{\rho c_2} g_2(x,t), \qquad 0 < x < w(t), \quad t > 0, \tag{46}$$

$$\frac{\partial \theta_1}{\partial t} = a_1^2 \frac{\partial^2 \theta_1}{\partial x^2} + \frac{1}{\rho c_1} g_1(x,t), \qquad x > w(t), \quad t > 0, \tag{47}$$

$$\theta_1(x,0) = -C < 0, \qquad x > 0, \tag{48}$$

$$\theta_1(+\infty,t) = -C < 0, \qquad t > 0, \tag{49}$$

$$\theta_2(0,t) = B_\infty, \qquad t > 0, \tag{50}$$

$$k_1 \frac{\partial \theta_1}{\partial x}(w(t),t) - k_2 \frac{\partial \theta_2}{\partial x}(w(t),t) = \rho \ell \dot{w}(t), \qquad t > 0, \tag{51}$$

$$\theta_1(w(t),t) = \theta_2(w(t),t) = 0, \qquad t > 0, \tag{52}$$

$$w(0) = 0, \tag{53}$$

where the internal heat sources $g_i = g_i(x,t)$, $i = 1,2$, are given by (4). Existence and uniqueness of the solution to this problem are established in the following theorem:

Theorem 4.1. *If*

$$\mathrm{Ste}_1 \geq 2, \tag{54}$$

then the Stefan problem defined by (46)–(53) *admits a unique similarity-type solution given by*

$$\theta_2(x,t) = \varphi_2\left(\frac{x}{2a_2\sqrt{t}}\right) + \Psi_2\left(\frac{x}{2a_2\sqrt{t}}\right), \qquad 0 \leq x \leq w(t), \tag{55}$$

$$\theta_1(x,t) = \varphi_1\left(\frac{x}{2a_2\sqrt{t}}\right) + \Psi_1\left(\frac{x}{2a_2\sqrt{t}}\right), \qquad x > w(t), \tag{56}$$

$$w(t) = 2a_2\xi\sqrt{t}, \qquad t > 0, \tag{57}$$

where ξ *is the unique solution of the equation*

$$G(z) = \frac{\mathrm{Ste}_2}{\sqrt{\pi}}, \quad z > 0, \tag{58}$$

and the functions φ_2, φ_1, *and* Ψ_1 *are defined in* (31), (33), *and* (34), *respectively, while*

$$\Psi_2(z) = \frac{B_\infty\sqrt{\pi}\,(\mathrm{erf}(\xi) - \mathrm{erf}(z)) - \varphi_2(\xi)\sqrt{\pi}\,\mathrm{erf}(z)}{\sqrt{\pi}\,\mathrm{erf}(\xi)}, \quad z > 0, \tag{59}$$

$$G(z) = \frac{F_0(z)h_1(z)}{Q\left(\frac{a_2}{a_1}z\right)} + \widetilde{F}_0(z), \quad z > 0, \tag{60}$$

where the functions h_1, Q, *and* F_0 *are defined in* (38), (39), *and* (41), *respectively, and*

$$\widetilde{F}_0(z) = F_0(z) + \frac{\mathrm{erf}(z+d_2) - \mathrm{erf}(d_2)}{d_2} - \frac{\exp(-d_2^2 - 2d_2 z)\,\mathrm{erf}(z)}{d_2}, \quad z > 0. \tag{61}$$

To establish the convergence of the unique solution of the Stefan problem with a convective boundary condition defined by (2)–(10) to the unique solution of the Stefan problem with a temperature boundary condition given by (46)–(53) as $\mathrm{Bi} \to +\infty$, the following preliminary result must be proved:

Lemma 4.2. *If hypothesis* (54) *holds, then hypothesis* (26) *is also satisfied.*

Proof. Consider the function defined by

$$f(z) = \frac{\sqrt{\pi}}{z}\left(\exp(-z^2) - \mathrm{erfc}(z)\right), \quad z > 0. \tag{62}$$

Clearly, $f(0) = 2$, $f(+\infty) = 0$, and

$$f'(z) = \frac{\sqrt{\pi}}{z^2\exp(z^2)}\left(-2z^2 + \frac{2}{\sqrt{\pi}}z - 1 + \exp(z^2)\,\mathrm{erfc}(z)\right). \tag{63}$$

Defining

$$F(z) = -2z^2 + \frac{2}{\sqrt{\pi}} z - 1 + \exp(z^2)\,\mathrm{erfc}(z), \quad z > 0, \tag{64}$$

the function F satisfies:

$$F(0) = 0,$$
$$F(+\infty) = \lim_{z\to+\infty} F(z) = -\infty.$$

Moreover,

$$F'(z) = -\frac{2}{\sqrt{\pi}}\left(2\sqrt{\pi}z - Q(z)\right),$$

where Q, defined by (39), satisfies $Q(0) = 0$, $Q(+\infty) = 1$, is strictly increasing for all $z > 0$, and $Q'(0) = \sqrt{\pi}$. Therefore, $F'(z) < 0$ for all $z > 0$, then $F(z) < 0$ for all $z > 0$.

It follows that $f'(z) < 0$ for all $z > 0$, which implies

$$f(z) \leq 2, \quad z > 0,$$

and in particular, $f(d_1) \leq 2$.

Consequently, if (54) holds, then

$$\mathrm{Ste}_1 \geq 2 \geq f(d_1)$$

for all $d_1 > 0$, thereby verifying (26). □

Theorem 4.3. *If hypothesis (54) holds, then the similarity-type solution (Φ, s) of the Stefan problem defined by (2)-(10) converges to the similarity-type solution (θ, w) of the Stefan problem defined by (46)-(53) when* $\mathrm{Bi} \to +\infty$. *Convergence is understood in the following sense*

$$\lim_{\mathrm{Bi}\to+\infty} s(t) = w(t), \quad \forall t > 0,$$
$$\lim_{\mathrm{Bi}\to+\infty} \Phi_2(x,t) = \theta_2(x,t), \quad \forall t > 0,\ 0 \leq x \leq w(t),$$
$$\lim_{\mathrm{Bi}\to+\infty} \Phi_1(x,t) = \theta_1(x,t), \quad \forall t > 0,\ x > w(t).$$

Proof. Let G_{Bi} be the function defined by (36). If $\mathrm{Bi}_1 < \mathrm{Bi}_2$, then the following inequalities hold:

$$F_{0,\mathrm{Bi}_1}(z) > F_{0,\mathrm{Bi}_2}(z), \qquad \widetilde{F}_{0,\mathrm{Bi}_1}(z) > \widetilde{F}_{0,\mathrm{Bi}_2}(z),$$

for all $z > 0$, where $F_{0,\mathrm{Bi}}$ and $\widetilde{F}_{0,\mathrm{Bi}}$ are defined in (37) and (40), respectively. From this relation, it follows that G_{Bi} is a strictly decreasing function with respect to Bi.

Denote by λ_{Bi} the solution to equation (30) for each Bi. It can be observed that λ_{Bi} is strictly increasing with respect to Bi.

Since

$$F_{0,\mathrm{Bi}}(z) > F_0(z) \quad \text{and} \quad \widetilde{F}_{0,\mathrm{Bi}}(z) > \widetilde{F}_0(z),$$

for all $z > 0$, where F_0 and $\widetilde{F}_0$ are defined in (41) and (61), respectively, it follows that

$$G_{\mathrm{Bi}}(z) > G(z) \quad \text{for all} \quad z > 0.$$

Consequently, for every Bi, it holds that $\lambda_{\mathrm{Bi}} \leq \xi$, where ξ is the unique solution of (58).

The function λ_{Bi}, being strictly increasing and bounded above, possesses a limit. Denote this limit by

$$\lim_{\mathrm{Bi}\to+\infty} \lambda_{\mathrm{Bi}} = \lambda^*.$$

It will be shown that $\lambda^* = \xi$. For each Bi, from (30) it holds that

$$\frac{F_{0,\mathrm{Bi}}(\lambda_{\mathrm{Bi}})\, h_1(\lambda_{\mathrm{Bi}})}{Q\left(\frac{a_2}{a_1}\lambda_{\mathrm{Bi}}\right)} + \widetilde{F}_{0,\mathrm{Bi}}(\lambda_{\mathrm{Bi}}) = \frac{\mathrm{Ste}_2}{\sqrt{\pi}}.$$

Taking the limit as $\mathrm{Bi} \to +\infty$, one obtains

$$\frac{F_0(\lambda^*)\, h_1(\lambda^*)}{Q\left(\frac{a_2}{a_1}\lambda^*\right)} + \widetilde{F}_0(\lambda^*) = \frac{\mathrm{Ste}_2}{\sqrt{\pi}}.$$

This implies that λ^* satisfies equation (58). Since ξ is the unique solution to this equation, it follows that $\lambda^* = \xi$. $\square$

Example 4.1. With reference to Example 3.5, the unique solution of equation (58) that characterizes the free boundary of the Stefan problem with a Dirichlet-type boundary condition at the fixed boundary defined by (46)-(53), is given by $\xi = 0,5046270$.

Fig. 4 shows the convergence of the parameter λ_{Bi}, which characterizes the free boundary of the Stefan problem with a Robin-type condition at the fixed boundary (2)-(10), to the parameter ξ, which characterizes the free boundary of the Stefan problem with a Dirichlet-type condition at the fixed boundary (46)-(53), as $Bi \to +\infty$.

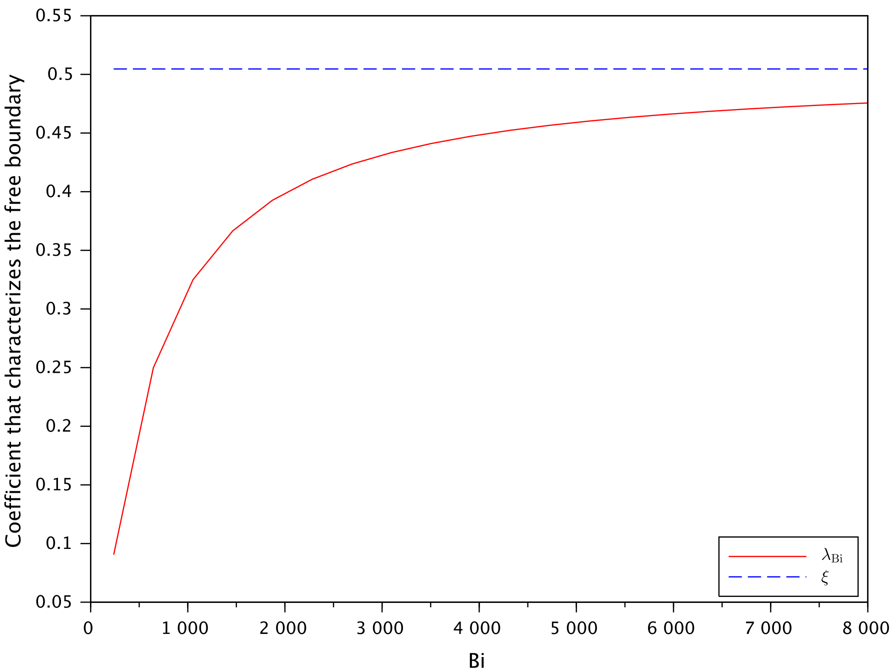

Fig. 4. Asymptotic behavior of λ_{Bi}.

5. τ-Method approximate solutions

In this section, numerical spectral methods based on the τ-method are applied to obtain an approximate solution to the two-phase Stefan problem defined by (2)-(10), using operational matrices of differentiation of shifted Chebyshev and Legendre polynomials.

5.1. Some preliminaries

Spectral methods are founded on finite-order expansions of smooth functions, in which a function $u = u(z)$ is approximated by truncating its infinite series representation. The approximation is expressed as a weighted sum of trial functions that constitute an orthogonal basis in an appropriate Hilbert space of continuous functions.

Any function $u = u(z) \in L^2[a,b]$, can be expanded according to the set of basis functions as follows

$$u(z) = \sum_{j=0}^{\infty} a_j\, \phi_j(z), \tag{65}$$

where $\{\phi_j(z)\}_{j\geq 0}$ are known global functions (e.g., Chebyshev or Legendre polynomials, or trigonometric functions) and a_k are coefficients to be determined.

In practice, spectral methods truncate this series at a finite order N, yielding the approximate solution

$$u(z) \approx \sum_{j=0}^{N} a_j\, \phi_j(z), \tag{66}$$

where N denotes the truncation order in the spectral method.

If the basis functions are shifted Chebyshev orthogonal polynomials [3,21], the Chebyshev spectral method is formed. In this case, the trial functions $\phi_j(z) = P_j^{[a,b]}(z)$, $z \in [a,b]$ for $j = 0, 1, 2, ...$ satisfying the following recurrence formula:

$$P_0^{[a,b]}(z) = 1, \tag{67}$$

$$P_1^{[a,b]}(z) = \frac{2}{b-a} z - \frac{b+a}{b-a}, \tag{68}$$

$$P_j^{[a,b]}(z) = P_j\left(\frac{2}{b-a} z - \frac{b+a}{b-a}\right), \qquad j = 2, 3, 4, \ldots, \tag{69}$$

where $P_j(z)$ with $j = 0, 1, 2, ...$ are defined on the interval $[-1, 1]$ by:

$$P_0(z) = 1, \tag{70}$$

$$P_1(z) = z, \tag{71}$$

$$P_j(z) = 2zP_{j-1}(z) - P_{j-2}(z), \quad j = 2, 3, 4, ... \tag{72}$$

The set $\left\{P_j^{[a,b]}(z)\right\}_{j\geq 0}$ is orthogonal with respect to the inner product with a weight function $w(z) = \frac{1}{\sqrt{1-\left(\frac{2z-(a+b)}{b-a}\right)^2}}$, $z \in [a,b]$, i.e.,

$$\left\langle P_m^{[a,b]},\, P_n^{[a,b]} \right\rangle = \int_a^b P_m^{[a,b]}(z)\, P_n^{[a,b]}(z)\, w(z)\, dz = \begin{cases} \frac{\pi(b-a)}{2} & \text{if} \quad m = n = 0 \\ \frac{\pi(b-a)}{4} & \text{if} \quad m = n \neq 0 \\ 0 & \text{if} \quad m \neq n \end{cases}, \tag{73}$$

where $\langle\cdot,\cdot\rangle$ denotes the usual inner product in $L^2[a,b]$.

Similarly, choosing shifted Legendre polynomials [1] as basis functions yields the Legendre spectral method. In this case, the trial functions $\phi_j(z) = L_j^{[a,b]}(z)$, $z \in [a,b]$, $j = 0,1,2,\ldots$, are defined by the recurrence relations:

$$L_0^{[a,b]}(z) = 1, \tag{74}$$

$$L_1^{[a,b]}(z) = \frac{2}{b-a}z - \frac{b+a}{b-a}, \tag{75}$$

$$L_j^{[a,b]}(z) = L_j\left(\frac{2}{b-a}z - \frac{b+a}{b-a}\right), \qquad j = 2,3,4,\ldots, \tag{76}$$

where $L_j(z)$ with $j = 0,1,2,\ldots$ are defined on the interval $[-1,1]$ by:

$$L_0(z) = 1, \tag{77}$$

$$L_1(z) = z, \tag{78}$$

$$L_j(z) = \tfrac{2j-1}{j}\, z\, L_{j-1}(z) - \tfrac{j-1}{j}\, L_{j-2}(z), \qquad j = 2,3,4,\ldots. \tag{79}$$

The set $\left\{L_j^{[a,b]}(z)\right\}_{j\geq 0}$ is orthogonal with respect to the inner product with a weight function $w(z) = 1$, $z \in [a,b]$, i.e.,

$$\left\langle L_m^{[a,b]},\, L_n^{[a,b]}\right\rangle = \int_a^b L_m^{[a,b]}(z)\, L_n^{[a,b]}(z)\, w(z)\, dz = \begin{cases} \frac{b-a}{2n+1} & \text{if} \quad m = n \\ 0 & \text{if} \quad m \neq n \end{cases}. \tag{80}$$

Spectral methods are one of the main techniques for the numerical solution of differential equations, based on series expansions of global functions. To solve the differential equation

$$\mathcal{L}[u](z) = v(z), \quad z \in [a,b], \tag{81}$$

where $\mathcal{L}$ is a given linear differential operator and $v(z)$ is a known function, the solution $u(z) \in L^2[a,b]$ is expanded in a basis of global functions as in (65).

The τ-method based on shifted Chebyshev or Legendre polynomials consists of substituting the truncated expansion (66) into the differential equation (81) and requiring that the residual

$$R_{N,u_N}(z) := \mathcal{L}[u_N](z) - v(z), \tag{82}$$

be orthogonal to all but the last M basis functions. The τ-method requires that the first $N-M$ coefficients of the residuals expansion in the Chebyshev orthogonal basis $\{P_j^{[a,b]}(z)\}_{j\geq 0}$ or the Legendre orthogonal basis $\{L_j^{[a,b]}(z)\}_{j\geq 0}$ vanish. Formally, this condition is expressed as

$$\int_a^b R_{N,u_N}(z)\, \phi_j(z)\, w(z)\, dz = 0, \qquad j = 0,1,\ldots,N-M, \tag{83}$$

and the M additional equations come from the boundary conditions, leading to a closed algebraic system for the coefficients a_j.

In the τ-method based on shifted Chebyshev polynomials, the series in (65) is typically truncated after the first $N+1$ terms, yielding the approximation

$$u(z) \approx u_N(z) = \sum_{j=0}^{N} a_j P_j^{[a,b]}(z) = \mathbf{A}\,\mathbf{P}^{[a,b]}(z), \quad z \in [a,b], \tag{84}$$

where $\mathbf{A} = [a_0\ a_1\ \cdots\ a_N]$ denotes a row vector of length $(N+1)$, and

$$\mathbf{P}^{[a,b]}(z) = \left[P_0^{[a,b]}(z)\ P_1^{[a,b]}(z)\ \cdots\ P_N^{[a,b]}(z)\right]^t, \quad z \in [a,b],$$

is a column vector of dimension $(N+1)$.

Following [28], it can be deduced that

$$\mathbf{P}^{[a,b]}(z) = \mathbf{T}\,\mathbf{W}_{a,b}\,\mathbf{Z}(z), \quad z \in [a,b], \tag{85}$$

where the $(N+1)$-vector $\mathbf{Z} = \mathbf{Z}(z)$ is defined by

$$\mathbf{Z}(z) = \begin{bmatrix} 1 & z & z^2 & \cdots & z^N \end{bmatrix}^t, \quad z \in [a,b], \tag{86}$$

and $\mathbf{T} = [t_{kj}]$ is the $(N+1) \times (N+1)$ matrix given by

$$\mathbf{T} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ -1 & t_{21} & 2 & 0 & 0 & 0 & \cdots & 0 \\ 0 & t_{31} & t_{32} & 2^2 & 0 & 0 & \cdots & 0 \\ 1 & t_{41} & t_{42} & t_{43} & 2^3 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ \cos\left(\frac{N\pi}{2}\right) & t_{N1} & t_{N2} & t_{N3} & t_{N4} & t_{N5} & \cdots & 2^{N-1} \end{bmatrix}, \tag{87}$$

with

$$t_{kj} = \begin{cases} \cos\left(\frac{k\pi}{2}\right) & \text{if} \quad k = 0, 1, ..., N,\ j = 0, \\ 2^{k-1} & \text{if} \quad k = j = 1, 2, ..., N, \\ \operatorname{sgn}(t_{k-1,\,j-1})\,(2|t_{k-1,\,j-1}| + |t_{k-2,\,j}|) & \text{if} \quad k = 2, 3, ..., N, \\ & \qquad j = 1, ..., k-1. \\ 0 & \qquad \text{otherwise,} \end{cases}$$

and

$$\mathbf{W}_{a,b} = [w_{kj}], \tag{88}$$

is the matrix $(N+1) \times (N+1)$ where

$$w_{kj} = \begin{cases} \binom{k}{j}\left(-\frac{b+a}{b-a}\right)^{k-j}\left(\frac{2}{b-a}\right)^{j} & \text{if} \quad k = 0, 1, ..., N,\ j = 0, 1, ..., k, \\ 0 & \text{if} \quad k < j. \end{cases}$$

Based on [26], the n-th order derivative of the vector $\mathbf{P}^{[a,b]}(z)$ defined in (85) can be expressed as

$$\left(\mathbf{P}^{[a,b]}\right)^{(n)}(z) = \mathbf{T}\,\mathbf{W}_{a,b}\,\mathbf{B}^n\mathbf{Z}(z), \quad z \in [a,b], \tag{89}$$

where

$$\mathbf{B} = [b_{kj}], \tag{90}$$

is the $(N+1) \times (N+1)$ matrix with entries

$$b_{kj} = \begin{cases} j+1, & \text{if } k = j+1, \quad j = 0, 1, \ldots, N, \\ 0, & \text{otherwise,} \end{cases}$$

and $n \in \mathbb{N}$ denotes the n-th power of the matrix $\mathbf{B}$. Consequently, from (84), the n-th order derivative of the function u_N is given by

$$u_N^{(n)}(z) = \mathbf{A}\,\mathbf{T}\,\mathbf{W}_{a,b}\,\mathbf{B}^n\mathbf{Z}(z), \quad z \in [a,b]. \tag{91}$$

In the τ-method based on shifted Legendre polynomials, the series in (65) is truncated after the first $N+1$ terms, giving the approximation

$$u(z) \approx u_N(z) = \sum_{j=0}^{N} a_j L_j^{[a,b]}(z) = \mathbf{A}\,\mathbf{L}^{[a,b]}(z), \quad z \in [a,b], \tag{92}$$

where $\mathbf{A} = [a_0\; a_1\; \cdots\; a_N]$ is a row vector of length $N+1$, and

$$\mathbf{L}^{[a,b]}(z) = \left[L_0^{[a,b]}(z)\; L_1^{[a,b]}(z)\; \cdots\; L_N^{[a,b]}(z)\right]^t, \quad z \in [a,b],$$

is a column vector of dimension $N+1$.

Based on [1], the n-th order derivative of the vector $\mathbf{L}^{[a,b]}(z)$ can be expressed as

$$\left(\mathbf{L}^{[a,b]}\right)^{(n)}(z) = \mathbf{D}_{a,b}^n\mathbf{L}^{[a,b]}(z), \quad z \in [a,b], \tag{93}$$

where

$$\mathbf{D}_{a,b} = [d_{kj}]^t, \tag{94}$$

is the $(N+1) \times (N+1)$ matrix with entries

$$d_{kj} = \begin{cases} \dfrac{2(2j+1)}{b-a}, & \text{if } k > j \text{ and } k-j \text{ is odd,} \\ 0, & \text{otherwise,} \end{cases} \qquad k, j = 0, 1, \ldots, N.$$

and $n \in \mathbb{N}$ denotes the n-th power of the matrix $\mathbf{D}_{a,b}$. Consequently, from (84), the n-th order derivative of the function u_N is given by

$$u_N^{(n)}(z) = \mathbf{A}\,\mathbf{D}_{a,b}^n\mathbf{L}^{[a,b]}(z), \quad z \in [a,b]. \tag{95}$$

5.2. τ-Method implementation

To obtain approximate solutions to the Stefan problem (2)-(10), and following [7], it is assumed that in the solid phase there exists a heat penetration depth $\widetilde{r}(t) > s(t)$, known as the thermal layer, beyond which no heat transfer occurs. This assumption implies that for every $x > \widetilde{r}(t)$, the slab remains at an equilibrium temperature, so that

$$\frac{\partial \Phi_1}{\partial x}(\widetilde{r}(t), t) = 0, \qquad \Phi_1(\widetilde{r}(t), t) = -C, \qquad t > 0. \tag{96}$$

Based on these assumptions, the goal is to find the approximate temperature distribution given by

$$\widetilde{\Phi}(x,t) = \begin{cases} \widetilde{\Phi}_2(x,t) & \text{if} \quad 0 \leq x \leq \widetilde{s}(t), \quad t>0, \\ \widetilde{\Phi}_1(x,t) & \text{if} \quad \widetilde{s}(t) < x < \widetilde{r}(t), \quad t>0, \end{cases} \tag{97}$$

the approximate free boundary $x = \widetilde{s}(t)$, $t > 0$, which separates the solid phase from the liquid phase and the thermal layer $x = \widetilde{r}(t)$, $t > 0$, satisfying the conditions of the modified two-phase Stefan problem stated as follows:

$$\frac{\partial \widetilde{\Phi}_2}{\partial t} = a_2^2 \frac{\partial^2 \widetilde{\Phi}_2}{\partial x^2} + \frac{1}{\rho c_2} g_2(x,t), \qquad 0 < x < \widetilde{s}(t),\ t>0, \tag{98}$$

$$\frac{\partial \widetilde{\Phi}_1}{\partial t} = a_1^2 \frac{\partial^2 \widetilde{\Phi}_1}{\partial x^2} + \frac{1}{\rho c_1} g_1(x,t), \qquad s(t) < x < \widetilde{r}(t),\ t>0, \tag{99}$$

$$\widetilde{\Phi}_1(x,0) = -C < 0, \qquad x>0, \tag{100}$$

$$\widetilde{\Phi}_1(\widetilde{r}(t),t) = -C < 0, \qquad t>0, \tag{101}$$

$$\frac{\partial \widetilde{\Phi}_1}{\partial x}(\widetilde{r}(t),t) = 0, \qquad t>0, \tag{102}$$

$$k_2 \frac{\partial \widetilde{\Phi}_2}{\partial x}(0,t) = \frac{h_0}{\sqrt{t}}\left(\widetilde{\Phi}_2(0,t) - B_\infty\right), \qquad t>0, \tag{103}$$

$$k_1 \frac{\partial \widetilde{\Phi}_1}{\partial x}(\widetilde{s}(t),t) - k_2 \frac{\partial \widetilde{\Phi}_2}{\partial x}(\widetilde{s}(t),t) = \rho \ell \dot{\widetilde{s}}(t), \qquad t>0, \tag{104}$$

$$\widetilde{\Phi}_1(\widetilde{s}(t),t) = \widetilde{\Phi}_2(\widetilde{s}(t),t) = 0, \qquad t>0, \tag{105}$$

$$\widetilde{s}(0) = 0, \tag{106}$$

$$\widetilde{r}(0) = 0. \tag{107}$$

Through the change of variables given by (12) and (15), the following direct result can be established:

Theorem 5.1. *The Stefan problem defined by (98)-(107) has a similarity-type solution $(\widetilde{\Phi},\ \widetilde{s},\ \widetilde{r})$ given by:*

$$\widetilde{\Phi}_2(x,t) = \widetilde{y}_2\left(\frac{x}{2a_2\widetilde{\lambda}\sqrt{t}}\right), \qquad 0 \leq x \leq \widetilde{s}(t),\ t>0, \tag{108}$$

$$\widetilde{\Phi}_1(x,t) = \widetilde{y}_1\left(\frac{x}{2a_2\widetilde{\lambda}\sqrt{t}}\right), \qquad \widetilde{s}(t) < x \leq \widetilde{r}(t),\ t>0, \tag{109}$$

$$\widetilde{s}(t) = 2a_2\widetilde{\lambda}\sqrt{t}, \qquad t>0, \tag{110}$$

$$\widetilde{r}(t) = 2a_2\widetilde{\mu}\sqrt{t}, \qquad t>0, \tag{111}$$

if and only if the functions $\widetilde{y}_1 = \widetilde{y}_1(\eta)$ is twice differentiable in $(\widetilde{\lambda}, \widetilde{\mu})$, $\widetilde{y}_2 = \widetilde{y}_2(\eta)$ is twice differentiable in $(0, \widetilde{\lambda})$, and the parameters $0 < \widetilde{\lambda} < \widetilde{\mu}$ satisfy the ordinary differential problem defined by:

$$\widetilde{y}_2'' + 2\eta\, \widetilde{y}_2' = \frac{4}{\mathrm{Ste}_2} \exp\left(-(\eta + d_2)^2\right), \qquad 0 < \eta < \widetilde{\lambda}, \tag{112}$$

$$\widetilde{y}_1'' + 2\frac{a_2^2}{a_1^2}\eta\, \widetilde{y}_1' = -4\frac{a_2^2}{a_1^2}\frac{1}{\mathrm{Ste}_1} \exp\left(-(\frac{a_2}{a_1}\eta + d_1)^2\right), \qquad \widetilde{\lambda} < \eta < \widetilde{\mu}, \tag{113}$$

$$\widetilde{y}_1(\widetilde{\mu}) = -C, \tag{114}$$

$$\widetilde{y}_1'(\widetilde{\mu}) = 0, \tag{115}$$

$$\widetilde{y}_2'(0) = 2\text{Bi}\left(\widetilde{y}_2(0) - B_\infty\right), \tag{116}$$

$$\widetilde{y}_1'(\widetilde{\lambda}) - \frac{k_2}{k_1}\widetilde{y}_2'(\widetilde{\lambda}) = 2\frac{k_2}{k_1}\frac{\widetilde{\lambda}}{\text{Ste}_2}, \tag{117}$$

$$\widetilde{y}_2(\widetilde{\lambda}) = 0, \tag{118}$$

$$\widetilde{y}_1(\widetilde{\lambda}) = 0. \tag{119}$$

5.2.1. τ-Method based on shifted Chebyshev polynomials

To obtain approximate solutions to the ordinary differential problem (112)-(119) on the bounded domain $[0, \widetilde{\mu}]$, the τ-method based on shifted Chebyshev polynomials (τ-MCP) is applied. Assuming that the unknown functions $\widetilde{y}_1 = \widetilde{y}_1(\eta)$ and $\widetilde{y}_2 = \widetilde{y}_2(\eta)$ are represented in terms of shifted Chebyshev polynomials as in (84):

$$\widetilde{y}_2(\eta) \approx \widetilde{y}_{2_N}(\eta) = \sum_{j=0}^{N} a_{2j} P_j^{[0,\widetilde{\lambda}_N]}(\eta) = \mathbf{A}_2 \mathbf{P}^{[0,\widetilde{\lambda}_N]}(\eta), \qquad \eta \in [0, \widetilde{\lambda}_N], \tag{120}$$

$$\widetilde{y}_1(\eta) \approx \widetilde{y}_{1_N}(\eta) = \sum_{j=0}^{N} a_{1j} P_j^{[\widetilde{\lambda}_N,\widetilde{\mu}_N]}(\eta) = \mathbf{A}_1 \mathbf{P}^{[\widetilde{\lambda}_N,\widetilde{\mu}_N]}(\eta), \qquad \eta \in [\widetilde{\lambda}_N, \widetilde{\mu}_N], \tag{121}$$

where $\mathbf{A}_2 = [a_{20}\ a_{21}\ ...\ a_{2N}]$ and $\mathbf{A}_1 = [a_{10}\ a_{11}\ ...\ a_{1N}]$ are row $(N+1)$-vectors to be determined, according to (85), we have that

$$\widetilde{y}_{2_N}(\eta) = \mathbf{A}_2 \mathbf{T}\ \mathbf{W}_{0,\widetilde{\lambda}_N} \mathbf{Z}(\eta) \qquad \eta \in [0, \widetilde{\lambda}_N], \tag{122}$$

$$\widetilde{y}_{1_N}(\eta) = \mathbf{A}_1 \mathbf{T}\ \mathbf{W}_{\widetilde{\lambda}_N,\widetilde{\mu}_N} \mathbf{Z}(\eta), \qquad \eta \in [\widetilde{\lambda}_N, \widetilde{\mu}_N], \tag{123}$$

where $\mathbf{T}$ is given by (87), $\mathbf{W}_{a,b}$ is defined by (88) and $\mathbf{Z}$ is given by (86).

Based on expressions (122) and (123), and the derivative approximation given in (95), the residual $R_{N,\,\widetilde{y}_{i_N}}(\eta)$, for $i = 1, 2$, is defined as in (82):

$$R_{N,\,\widetilde{y}_{2_N}}(\eta) = \mathbf{A}_2 \mathbf{T}\ \mathbf{W}_{0,\widetilde{\lambda}_N} \mathbf{B}\left(\mathbf{B} + 2\eta\mathbf{I}\right)\mathbf{Z}(\eta) - 4\tfrac{1}{\text{Ste}_2}\exp\left(-(\eta + d_2)^2\right), \tag{124}$$

$$R_{N,\,\widetilde{y}_{1_N}}(\eta) = \mathbf{A}_1 \mathbf{T}\ \mathbf{W}_{\widetilde{\lambda}_N,\widetilde{\mu}_N} \mathbf{B}\left(\mathbf{B} + 2\tfrac{a_2^2}{a_1^2}\eta\mathbf{I}\right)\mathbf{Z}(\eta) + 4\tfrac{k_2}{k_1}\tfrac{1}{\text{Ste}_2}\exp\left(-\left(\tfrac{a_2}{a_1}\eta + d_1\right)^2\right), \tag{125}$$

where $\mathbf{I}$ is the $(N+1) \times (N+1)$ identity matrix.

In accordance with the τ-method, minimizing the residuals so that the first $(N+1)$ terms of their spectral expansions vanish, according to the definition given in (83), leads to the following system of $2N-2$ nonlinear equations:

$$\left\langle R_{N,\,\widetilde{y}_{2_N}}(\eta),\ P_j^{[0,\widetilde{\lambda}_N]}(\eta)\right\rangle = \int_0^{\widetilde{\lambda}_N} R_{N,\,\widetilde{y}_{2_N}}(\eta)\ P_j^{[0,\widetilde{\lambda}_N]}(\eta)\,d\eta = 0, \tag{126}$$

$$\left\langle R_{N,\,\widetilde{y}_{1_N}}(\eta),\ P_j^{[\widetilde{\lambda}_N,\widetilde{\mu}_N]}(\eta)\right\rangle = \int_{\widetilde{\lambda}_N}^{\widetilde{\mu}_N} R_{N,\,\widetilde{y}_{1_N}}(\eta)\ P_j^{[\widetilde{\lambda}_N,\widetilde{\mu}_N]}(\eta)\,d\eta = 0, \tag{127}$$

for $j = 0, 1, ..., N-2$.

Furthermore, by substituting equations (122) and (123) into the boundary conditions (114)-(119), we obtain the following constraints, respectively:

$$\mathbf{A}_1\,\mathbf{T}\,\mathbf{W}_{\widetilde{\lambda}_N,\widetilde{\mu}_N}\mathbf{Z}(\widetilde{\mu}_N)+C=0, \tag{128}$$

$$\mathbf{A}_1\,\mathbf{T}\,\mathbf{W}_{\widetilde{\lambda}_N,\widetilde{\mu}_N}\mathbf{B}\,\mathbf{Z}(\widetilde{\mu}_N)=0, \tag{129}$$

$$\mathbf{A}_2\,\mathbf{T}\,\mathbf{W}_{0,\widetilde{\lambda}_N}\,(2\mathrm{Bi}\,\mathbf{I}-\mathbf{B})\,\mathbf{Z}(0)-B_\infty=0, \tag{130}$$

$$\left(\mathbf{A}_1\,\mathbf{T}\,\mathbf{W}_{\widetilde{\lambda}_N,\widetilde{\mu}_N}-\tfrac{k_2}{k_1}\mathbf{A}_2\,\mathbf{T}\,\mathbf{W}_{0,\widetilde{\lambda}_N}\right)\mathbf{B}\,\mathbf{Z}(\widetilde{\lambda}_N)-\tfrac{2}{\mathrm{Ste}_2}\tfrac{k_2}{k_1}\widetilde{\lambda}_N=0, \tag{131}$$

$$\mathbf{A}_2\,\mathbf{T}\,\mathbf{W}_{0,\widetilde{\lambda}_N}\,\mathbf{Z}(\widetilde{\lambda}_N)=0, \tag{132}$$

$$\mathbf{A}_1\,\mathbf{T}\,\mathbf{W}_{\widetilde{\lambda}_N,\widetilde{\mu}_N}\,\mathbf{Z}(\widetilde{\lambda}_N)=0. \tag{133}$$

Consequently, computing the vectors $\mathbf{A}_1$ and $\mathbf{A}_2$ of dimension $(N+1)$, together with the positive parameters $\widetilde{\lambda}_N$ and $\widetilde{\mu}_N$, requires solving the nonlinear system of $2N+4$ equations defined by (126)-(133).

5.2.2. τ-Method based on shifted Legendre polynomials

To approximate the solution to the differential problem (112)-(119), the τ-method is implemented with shifted Legendre polynomials (τ-MLP). The unknown functions $\widetilde{y}_1(\eta)$ and $\widetilde{y}_2(\eta)$ are expanded in a series of shifted Legendre polynomials, in accordance with (92).

$$\widetilde{y}_2(\eta)\approx\widetilde{y}_{2_N}(\eta)=\sum_{j=0}^{N}a_{4j}L_j^{[0,\widetilde{\lambda}_N]}(\eta)=\mathbf{A}_2\mathbf{L}^{[0,\widetilde{\lambda}_N]}(\eta),\qquad \eta\in[0,\widetilde{\lambda}_N], \tag{134}$$

$$\widetilde{y}_1(\eta)\approx\widetilde{y}_{1_N}(\eta)=\sum_{j=0}^{N}a_{3j}L_j^{[\widetilde{\lambda}_N,\widetilde{\mu}_N]}(\eta)=\mathbf{A}_1\mathbf{L}^{[\widetilde{\lambda}_N,\widetilde{\mu}_N]}(\eta),\qquad \eta\in[\widetilde{\lambda}_N,\widetilde{\mu}_N], \tag{135}$$

where $\mathbf{A}_2=[a_{40}\;a_{41}\;...\;a_{4N}]$ and $\mathbf{A}_1=[a_{30}\;a_{31}\;...\;a_{3N}]$ are row $(N+1)$-vectors to be determinated.

Based on expressions (134) and (135), and the derivative approximation given in (95), the residual $R_{N,\,\widetilde{y}_{i_N}}(\eta)$, for $i=1,2$, is defined as in (82):

$$R_{N,\,\widetilde{y}_{2_N}}(\eta)=\mathbf{A}_2\mathbf{D}_{0,\widetilde{\lambda}_N}\left(\mathbf{D}_{0,\widetilde{\lambda}_N}+2\eta\mathbf{I}\right)\mathbf{L}^{[0,\widetilde{\lambda}_N]}(\eta)-4\tfrac{1}{\mathrm{Ste}_2}\exp\left(-(\eta+d_2)^2\right), \tag{136}$$

$$R_{N,\,\widetilde{y}_{1_N}}(\eta)=\mathbf{A}_1\mathbf{D}_{\widetilde{\lambda}_N,\widetilde{\mu}_N}\left(\mathbf{D}_{\widetilde{\lambda}_N,\widetilde{\mu}_N}+2\tfrac{a_2^2}{a_1^2}\eta\mathbf{I}\right)\mathbf{L}^{[\widetilde{\lambda}_N,\widetilde{\mu}_N]}(\eta)+4\tfrac{a_2^2}{a_1^2}\tfrac{1}{\mathrm{Ste}_1}\exp(-(\tfrac{a_2}{a_1}\eta+d_1)^2). \tag{137}$$

In accordance with the τ-method, minimizing the residuals (136) and (137), according to the definition given in (83), leads to the following system of $2N-2$ nonlinear equations:

$$\left\langle R_{N,\,\widetilde{y}_{2_N}}(\eta),\;L_j^{[0,\widetilde{\lambda}_N]}(\eta)\right\rangle=\int_0^{\widetilde{\lambda}_N}R_{N,\,\widetilde{y}_{2_N}}(\eta)\;L_j^{[0,\widetilde{\lambda}_N]}(\eta)\,d\eta=0, \tag{138}$$

$$\left\langle R_{N,\,\widetilde{y}_{1_N}}(\eta),\;L_j^{[\widetilde{\lambda}_N,\widetilde{\mu}_N]}(\eta)\right\rangle=\int_{\widetilde{\lambda}_N}^{\widetilde{\mu}_N}R_{N,\,\widetilde{y}_{1_N}}(\eta)\;L_j^{[\widetilde{\lambda}_N,\widetilde{\mu}_N]}(\eta)\,d\eta=0, \tag{139}$$

for $j=0,1,...,N-2$.

Furthermore, substituting equations (134) and (135) into the boundary conditions (114)–(119) yields the following, respectively:

$$\mathbf{A}_1 \mathbf{L}^{[\widetilde{\lambda}_N,\widetilde{\mu}_N]}(\widetilde{\mu}_N) + C = 0, \tag{140}$$

$$\mathbf{A}_1\, \mathbf{D}_{\widetilde{\lambda}_N,\widetilde{\mu}_N} \mathbf{L}^{[\widetilde{\lambda}_N,\widetilde{\mu}_N]}(\widetilde{\mu}_N) = 0, \tag{141}$$

$$\mathbf{A}_2 \left(2\mathrm{Bi}\, \mathbf{I} - \mathbf{D}_{0,\widetilde{\lambda}_N}\right) \mathbf{L}^{[0,\widetilde{\lambda}_N]}(0) - 2\mathrm{Bi} = 0, \tag{142}$$

$$\tfrac{k_1}{k_2}\mathbf{A}_1\, \mathbf{D}_{\widetilde{\lambda}_N,\widetilde{\mu}_N} \mathbf{L}^{[\widetilde{\lambda}_N,\widetilde{\mu}_N]}(\widetilde{\lambda}_N) - \mathbf{A}_2\, \mathbf{D}_{0,\widetilde{\lambda}_N} \mathbf{L}^{[0,\widetilde{\lambda}_N]}(\widetilde{\lambda}_N) - \tfrac{2}{\mathrm{Ste}_2}\widetilde{\lambda}_N = 0, \tag{143}$$

$$\mathbf{A}_2\, \mathbf{L}^{[0,\widetilde{\lambda}_N]}(\widetilde{\lambda}_N) = 0, \tag{144}$$

$$\mathbf{A}_1\, \mathbf{L}^{[\widetilde{\lambda}_N,\widetilde{\mu}_N]}(\widetilde{\lambda}_N) = 0. \tag{145}$$

As a result, determining the vectors $\mathbf{A}_1$ and $\mathbf{A}_2$ of dimension $(N+1)$, along with the positive parameters $\widetilde{\lambda}_N$ and $\widetilde{\mu}_N$, involves solving the nonlinear system of $2N+4$ equations given by (138)–(145).

5.3. Numerical results

Based on the equivalences established in the previous theorems, this section examines the accuracy of the approximate solutions to problem (112)–(119) obtained using the τ-MCP and the τ-MLP, by comparison with the exact solution presented in Section 3. To this end, the nonlinear systems (126)–(133) or (138)–(145) are solved numerically following the computational procedure outlined in Algorithm 1.

Algorithm 1 Numerical solution of the nonlinear system (126)–(133) or (138)–(145).

Input: Integer N; initial guess $\mathbf{v}_0 = (\mathbf{A}_{1_0}, \mathbf{A}_{2_0}, \widetilde{\lambda}_0, \widetilde{\mu}_0) \in \mathbb{R}^{2N+4}$
Define the function $\mathbf{F}\colon \mathbb{R}^{2N+4} \to \mathbb{R}^{2N+4}$ as follows:
    Let $\mathbf{v} = (\mathbf{A}_1, \mathbf{A}_2, \widetilde{\lambda}, \widetilde{\mu}) \in \mathbb{R}^{2N+4}$, where $\mathbf{A}_1, \mathbf{A}_2 \in \mathbb{R}^{N+1}$ and $\widetilde{\lambda}, \widetilde{\mu} \in \mathbb{R}$
    Return the vector $\mathbf{F}(\mathbf{v}) \in \mathbb{R}^{2N+4}$ corresponding to the nonlinear system given by (126)–(133) or (138)–(145)
Use the `fsolve` routine from Scilab with:
    Initial guess $\mathbf{v}_0$
    Function $\mathbf{F}$
Obtain the numerical solution $\mathbf{F}(\mathbf{v}) = \mathbf{0}$
Output: $\mathbf{v}_{\mathrm{sol}} = (\mathbf{A}_1, \mathbf{A}_2, \widetilde{\lambda}_N, \widetilde{\mu}_N)$

The presented code is implemented in Scilab. The `fsolve` routine typically employs variants of the Newton-Raphson or quasi-Newton methods. Future improvements to the numerical implementation may include enhanced control over the nonlinear solver or the development of customized algorithms specifically tailored to the structure of this system.

Approximate solutions will be obtained by applying the τ-MCP and τ-MLP to the paraffin melting process, whose exact solution was presented in Example 3.5. The parameter values are taken from Tables 1 and 2, with an initial temperature of $-C = -250$ K and an ambient temperature of $B_\infty = 300$ K. The exact value of the coefficient characterizing the free boundary in this case is $\lambda = 0.0921568$.

The approximate solution $(\widetilde{y}_{1_N}, \widetilde{y}_{2_N}, \widetilde{\lambda}_N, \widetilde{\mu}_N)$ is first computed by applying the τ-MCP, that is, by solving the nonlinear system given in (126)–(133) for $N = 5, 6, \ldots, 10$. The corresponding results are reported in Table 3. In addition, the same procedure is carried out by applying the τ-MLP, which requires solving the nonlinear system (138)–(145) for $N = 5, 6, \ldots, 10$, and the results are shown in Table 4.

To highlight the differences between the approximate solution to the problem given by (112)–(119) and the exact solution of the problem defined by (2)–(10), for $N = 5$ to 10, we report the errors in three separate tables.

Table 5 presents the relative error of $\widetilde{\lambda}_N$, defined as

$$e_N = \tfrac{|\lambda - \widetilde{\lambda}_N|}{\lambda}. \tag{146}$$

Table 3
Approximate values of $\mathbf{A}_1$, $\mathbf{A}_2$, $\widetilde{\lambda}_N$ and $\widetilde{\mu}_N$, for $N = 5, 6, .., 10$ applying τ-MCP.

N	$\mathbf{v}_{\text{sol}} = \left(\mathbf{A}_1, \mathbf{A}_2, \widetilde{\lambda}_N, \widetilde{\mu}_N\right)$
5	$\mathbf{A}_1 = [-1.6974751 \times 10^2 \; -1.2290697 \times 10^2 \; 5.0937390 \times 10^1 \; -4.4075767 \; -6.1898788 \; 2.3145465]$
	$\mathbf{A}_2 = [6.5778942 \; -6.7771402 \; 0.1985210 \; 7.772 \times 10^{-4} \; -5.22 \times 10^{-5} \; -5.693 \times 10^{-8}]$
	$\widetilde{\lambda}_5 = 0.0922236$
	$\widetilde{\mu}_5 = 1.9479942$
6	$\mathbf{A}_1 = [-1.6677442 \times 10^2 \; -1.2460675 \times 10^2 \; 4.7830039 \times 10^1 \; -2.2414447 \; -6.2628617 \; 1.8481919 \; 2.072384 \times 10^{-1}]$
	$\mathbf{A}_2 = [6.5748054 \; -6.7738158 \; 0.1982865 \; 7.761 \times 10^{-4} \; -5.21 \times 10^{-5} \; -5.682 \times 10^{-8} \; 9.135 \times 10^{-9}]$
	$\widetilde{\lambda}_6 = 0.0921696$
	$\widetilde{\mu}_6 = 1.8386562$
7	$\mathbf{A}_1 = [-1.9409461 \times 10^2 \; -9.8906562 \times 10^1 \; 6.7181997 \times 10^1 \; -3.2006681 \times 10^1 \; 6.8298994 \; 4.3235914 \; -4.9172886 \; 1.5896514]$
	$\mathbf{A}_2 = [6.5736544 \; -6.7725770 \; 0.1981991 \; 7.757 \times 10^{-4} \; -5.21 \times 10^{-5} \; -5.678 \times 10^{-8} \; 9.118 \times 10^{-9} \; 9.175 \times 10^{-12}]$
	$\widetilde{\lambda}_7 = 0.0921495$
	$\widetilde{\mu}_7 = 3.6526516$
8	$\mathbf{A}_1 = [-1.7633103 \times 10^2 \; -1.1825038 \times 10^2 \; 5.7575033 \times 10^1 \; -1.0850862 \times 10^1 \; -5.5403196 \; 4.4456268 \; -0.8936753 \; -0.3443846 \; 0.1899938]$
	$\mathbf{A}_2 = [6.5746296 \; -6.7736266 \; 0.1982731 \; 7.760 \times 10^{-4} \; -5.21 \times 10^{-5} \; -5.682 \times 10^{-8} \; 9.125 \times 10^{-9} \; -1.194 \times 10^{-12} \; -1.198 \times 10^{-12}]$
	$\widetilde{\lambda}_8 = 0.0921666$
	$\widetilde{\mu}_8 = 2.2473084$
9	$\mathbf{A}_1 = [-1.7756586 \times 10^2 \; -1.1720438 \times 10^2 \; 5.8667517 \times 10^1 \; -1.2199960 \times 10^1 \; -5.2107027 \; 4.7295073 \; -1.1089307 \; -0.3055436 \; 0.2179720 \; -0.0196242]$
	$\mathbf{A}_2 = [6.5747821 \; -6.7737908 \; 0.1982847 \; 7.761 \times 10^{-4} \; -5.21 \times 10^{-5} \; -5.682 \times 10^{-8} \; 9.127 \times 10^{-9} \; -1.197 \times 10^{-12} \; -1.199 \times 10^{-12} \; 9.811 \times 10^{-16}]$
	$\widetilde{\lambda}_9 = 0.0921692$
	$\widetilde{\mu}_9 = 2.3126354$
10	$\mathbf{A}_1 = [-1.8027084 \times 10^2 \; -1.1475117 \times 10^2 \; 6.0877583 \times 10^1 \; -1.5298940 \times 10^1 \; -4.2311383 \; 5.2979614 \; -1.6824696 \; -0.1857260 \; 0.3283799 \; -0.0621794 \; -0.0215604]$
	$\mathbf{A}_2 = [6.5744799 \; -6.7732350 \; 0.1982284 \; 7.645 \times 10^{-4} \; -5.26 \times 10^{-5} \; 1.1 \times 10^{-6} \; 6.0 \times 10^{-7} \; 5.0 \times 10^{-7} \; 4.690 \times 10^{-8} \; 4.0 \times 10^{-7} \; 2.0 \times 10^{-7}]$
	$\widetilde{\lambda}_{10} = 0.0921590$
	$\widetilde{\mu}_{10} = 2.4682868$

Table 4
Approximate values of $\mathbf{A}_1$, $\mathbf{A}_2$, $\widetilde{\lambda}_N$ and $\widetilde{\mu}_N$, for $N = 5, 6, \ldots, 10$ applying τ-MLP.

N	$\mathbf{v}_{\text{sol}} = \left(\mathbf{A}_1, \mathbf{A}_2, \widetilde{\lambda}_N, \widetilde{\mu}_N\right)$
5	$\mathbf{A}_1 = [-1.8623286 \times 10^2 \;\; -1.2066210 \times 10^2 \; 7.2539594 \times 10^1 \;\; -9.0198962 \;\; -1.1306729 \times 10^1 \; 4.6819992]$ $\mathbf{A}_2 = [6.5131217 \;\; -6.7791510 \; 2.648806 \times 10^{-1} \; 1.2444 \times 10^{-3} \;\; -9.56 \times 10^{-5} \;\; -1 \times 10^{-7}]$ $\widetilde{\lambda}_5 = 0.0922488$ $\widetilde{\mu}_5 = 1.9454481$
6	$\mathbf{A}_1 = [-1.8216169 \times 10^2 \;\; -1.2362835 \times 10^2 \; 6.8338680 \times 10^1 \;\; -5.0951832 \;\; -1.1640659 \times 10^1 \; 3.7235330 \; 4.636669 \times 10^{-1}]$ $\mathbf{A}_2 = [6.5094205 \;\; -6.7750632 \; 2.644959 \times 10^{-1} \; 1.2422 \times 10^{-3} \;\; -9.53 \times 10^{-5} \;\; -1 \times 10^{-7} \; 2.024 \times 10^{-8}]$ $\widetilde{\lambda}_6 = 0.0921824$ $\widetilde{\mu}_6 = 1.8347483$
7	$\mathbf{A}_1 = [-1.8911686 \times 10^2 \;\; -1.1845705 \times 10^2 \; 7.5681336 \times 10^1 \;\; -1.3042370 \times 10^1 \;\; -1.1329597 \times 10^1 \; 7.3077200 \;\; -2.345630 \times 10^{-1}]$ $\mathbf{A}_2 = [-8.082173 \times 10^{-1} \; 6.5125967 \;\; -6.7784984 \; 2.647688 \times 10^{-1} \; 1.3905 \times 10^{-3} \;\; -1.456 \times 10^{-4} \; 1.844 \times 10^{-4} \;\; -8.17 \times 10^{-5} \; 1.284 \times 10^{-4}]$ $\widetilde{\lambda}_7 = 0.0920181$ $\widetilde{\mu}_7 = 2.0322634$
8	$\mathbf{A}_1 = [-1.9452142 \times 10^2 \;\; -1.1301806 \times 10^2 \; 8.0643157 \times 10^1 \;\; -2.0345343 \times 10^1 \;\; -9.5558316 \; 9.1841364 \;\; -2.0206850 \;\; -8.207370 \times 10^{-1} \; 4.547758 \times 10^{-1}]$ $\mathbf{A}_2 = [6.5079826 \;\; -6.7734752 \; 2.643465 \times 10^{-1} \; 1.2414 \times 10^{-3} \;\; -9.52 \times 10^{-5} \;\; -1 \times 10^{-7} \; 2.021 \times 10^{-8} \;\; -2.844 \times 10^{-12} \;\; -3.050 \times 10^{-12}]$ $\widetilde{\lambda}_8 = 0.0921567$ $\widetilde{\mu}_8 = 2.2235562$
9	$\mathbf{A}_1 = [-1.9571112 \times 10^2 \;\; -1.1171538 \times 10^2 \; 8.1598511 \times 10^1 \;\; -2.2093561 \times 10^1 \;\; -9.0186777 \; 9.5946124 \;\; -2.3740421 \;\; -7.510211 \times 10^{-1} \; 5.053251 \times 10^{-1} \;\; -3.46539 \times 10^{-2}]$ $\mathbf{A}_2 = [6.5079930 \;\; -6.7734859 \; 2.643468 \times 10^{-1} \; 1.2414 \times 10^{-3} \;\; -9.52 \times 10^{-5} \;\; -1 \times 10^{-7} \; 2.021 \times 10^{-8} \;\; -2.849 \times 10^{-12} \;\; -3.049 \times 10^{-12} \; 2.756 \times 10^{-15}]$ $\widetilde{\lambda}_9 = 0.0921567$ $\widetilde{\mu}_9 = 2.2702686$
10	$\mathbf{A}_1 = [-1.9921550 \times 10^2 \;\; -1.0759346 \times 10^2 \; 8.4070108 \times 10^1 \;\; -2.7502892 \times 10^1 \;\; -6.9171098 \; 1.0713500 \times 10^1 \;\; -3.6841925 \;\; -4.827316 \times 10^{-1} \; 8.077871 \times 10^{-1} \;\; -1.343833 \times 10^{-1} \;\; -6.10445 \times 10^{-2}]$ $\mathbf{A}_2 = [6.5096383 \;\; -6.7740626 \; 2.642411 \times 10^{-1} \; 1.1383 \times 10^{-3} \;\; -1.547 \times 10^{-4} \;\; -6.14 \times 10^{-5} \;\; -1.35 \times 10^{-5} \;\; -1.47 \times 10^{-5} \; 9.0 \times 10^{-6} \; 2.28 \times 10^{-5} \; 7.5 \times 10^{-6}]$ $\widetilde{\lambda}_{10} = 0.0921764$ $\widetilde{\mu}_{10} = 2.4201374$

Table 5
Relative error of $\widetilde{\lambda}_N$ for τ-MCP and τ-MLP, for $N = 5$ to 10.

N	e_N for τ-MCP	e_N for τ-MLP
5	0.0007246	0.000998
6	0.0001386	0.0002775
7	0.0000795	0.0015053
8	0.000106	0.0000014
9	0.0001343	0.0000014
10	0.0000236	0.0002124

Table 6
Absolute error of $\widetilde{y}_{1_N}$ for τ-MCP and τ-MLP, for $N = 5$ to 10.

N	$\|\|y_1 - \widetilde{y}_{1_N}\|\|_\infty$ for τ-MCP	$\|\|y_1 - \widetilde{y}_{1_N}\|\|_\infty$ for τ-MLP
5	1.0351337	1.0242008
6	1.0370891	0.7509413
7	1.0378172	0.4560169
8	1.0371978	0.0600211
9	1.0371036	0.0582998
10	1.0374731	0.0507035

Table 7
Absolute error of $\widetilde{y}_{2_N}$ for τ-MCP and τ-MLP, for $N = 5$ to 10.

N	$\|\|y_2 - \widetilde{y}_{2_N}\|\|_\infty$ for τ-MCP	$\|\|y_2 - \widetilde{y}_{2_N}\|\|_\infty$ for τ-MLP
5	0.0000157	0.0000003
6	0.0069996	0.0000054
7	0.0096054	0.0265373
8	0.0073885	0.0000038
9	0.0070514	0.0000176
10	0.0083738	0.0010211

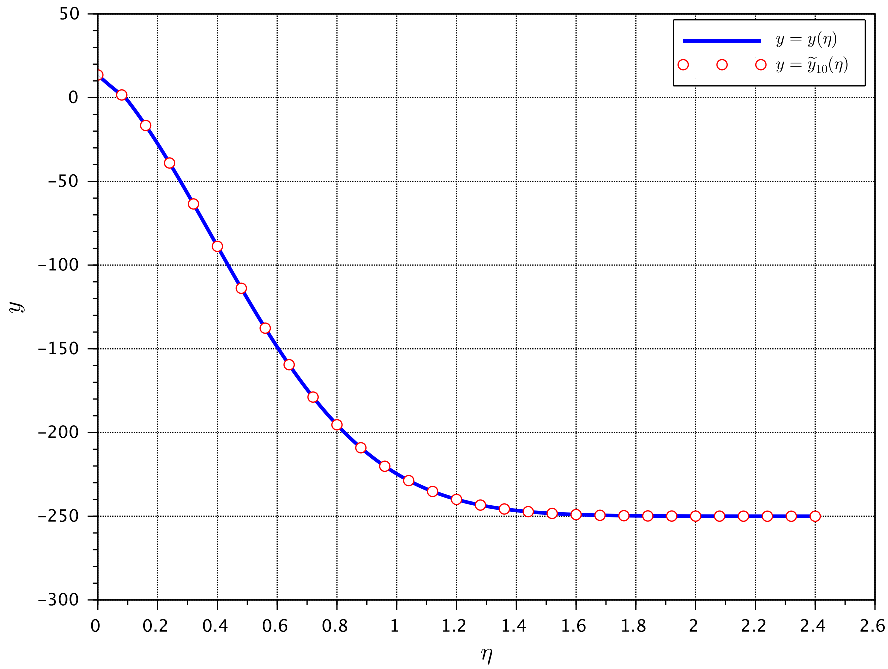

Fig. 5. Exact and approximate solution obtain using τ-MLP for $N = 10$.

Table 6 shows the absolute errors, measured in the supremum norm, between $\widetilde{y}_{1_N}$ and y_1, computed over the interval where both functions are defined, namely $(\lambda_N^{\max}, \widetilde{\mu}_N)$, with $\lambda_N^{\max} = \max(\lambda, \widetilde{\lambda}_N)$.

Finally, Table 7 provides the absolute errors for y_2 and $\widetilde{y}_{2_N}$, evaluated over the interval $(0, \lambda_N^{\min})$, where $\lambda_N^{\min} = \min(\lambda, \widetilde{\lambda}_N)$.

The findings presented in Tables 5-7 shed light on how the two numerical methods perform. On one hand, τ-MCP shows a stable and consistently low relative error for λ, but it doesn't make much progress in accurately estimating y_1 and actually worsens for y_2 as N grows. On the other hand, τ-MLP delivers significantly better approximations for both y_1 and y_2 as N increases, even though there are some occasional bumps in the error for λ. All in all, τ-MLP seems to be the better option when looking at the complete solution profile of the problem. That's why Fig. 5 illustrates the solution for the approximate problem (112)–(119) with $N = 10$, which was obtained using τ-MLP, alongside the exact solution for the problem (2)-(10).

6. Conclusion

A precise similarity-type solution was obtained for the two-phase Stefan problem that includes a convective boundary condition at the fixed face and an exponential internal source. It was shown that as the Biot number approaches infinity, this solution converges to the unique solution of the two-phase Stefan problem where the fixed boundary has a temperature condition. Numerical experiments indicated that both spectral methods, based on the τ-method with shifted Chebyshev and Legendre polynomials, provide reliable approximations. The Chebyshev method displayed strong stability in estimating the free boundary parameter, whereas the Legendre method proved more accurate in reproducing the complete profiles of the solution to the ordinary differential problem. Overall, the Legendre formulation was found to be more effective in capturing the general structure of the solution. These approximation methods, which exhibit close agreement with exact solutions when available, can also be successfully applied to phase-change problems for which exact solutions are not reported in the literature.

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References

[1] W.M. Abd-Elhameed, Y.H. Youssri, E.H. Doha, A novel operational matrix method based on shifted Legendre polynomials for solving second-order boundary value problems involving singular, singularly perturbed and Bratu-type equations, Math. Sci. 9 (2) (2015) 93–102.

[2] M. Abdelhakem, Mona Fawzy, M. El-Kady, Hanaa Moussa, An efficient technique for approximated BVPs via the second derivative Legendre polynomials pseudo-Galerkin method: certain types of applications, Results Phys. 43 (2022) 106067.

[3] D. Abdelhamied, M. Abdelhakem, M. El-Kady, M. Youssri, Adapted shifted ChebyshevU operational matrix of derivatives: two algorithms for solving even-order BVPs, Appl. Math. Inf. Sci. 17 (3) (2023) 575–581.

[4] V. Alexiades, A.D. Solomon, Mathematical Modeling of Melting and Freezing Processes, Hemisphere Publishing Corporation, Washington, 1993.

[5] Y. Belhamadia, G.O. Cassol, S. Dubljevic, Numerical modelling of hyperbolic phase change problems: application to continuous casting, Int. J. Heat Mass Transf. 209 (2023) 124042.

[6] J. Bollati, M.F. Natale, J.A. Semitiel, D.A. Tarzia, Exact solution for non-classical one-phase Stefan problem with variable thermal coefficients and two different heat source terms, Comput. Appl. Math. 41 (2022) 1–11.

[7] J. Bollati, M.F. Natale, J.A. Semitiel, D.A. Tarzia, A two-phase Stefan problem with power-type temperature-dependent thermal conductivity. Existence of a solution by two fixed points and numerical results, AIMS Math. 9 (8) (2024) 21189–21211.

[8] J. Bollati, M.T. Cao-Rial, M.F. Natale, J. Semitiel, D.A. Tarzia, Tau method implementation for approximating the solution to a two-phase change problem with temperature-dependent thermal coefficients, J. Comput. Appl. Math. 475 (2026) 117001.

[9] L. Bougoffa, A. Khanfer, Solution of non-classical Stefan problem with nonlinear thermal coefficients and a Robin boundary condition, AIMS Math. 6 (2021) 6569–6579.

[10] L. Bougoffa, R.C. Rach, A. Mennouni, On the existence, uniqueness, and new analytic approximate solution of the modified error function in two-phase Stefan problems, Math. Methods Appl. Sci. 44 (2021) 10948–10956.

[11] A.C. Briozzo, M.F. Natale, Non-classical Stefan problem with nonlinear thermal coefficients and a Robin boundary condition, Nonlinear Anal., Real World Appl. 49 (2019) 159–168.
[12] A.C. Briozzo, D.A. Tarzia, Existence and uniqueness of a one-phase Stefan problem for a non-classical heat equation with temperature boundary condition at the fixed face, Electron. J. Differ. Equ. 21 (2006) 1-16.
[13] A.C. Briozzo, M.F. Natale, D.A. Tarzia, Explicit solutions for a two-phase unidimensional Lamé-Clapeyron-Stefan problem with source terms in both phases, J. Math. Anal. Appl. 329 (2007) 145–162.
[14] F. Brosa Planella, A simple model for latent thermal energy storage systems with encapsulated phase-change material, Int. J. Heat Mass Transf. 239 (2025) 126533.
[15] F. Brosa Planella, C.P. Please, R.A. Van Gorder, Extended Stefan problem for the solidification of binary alloys in a sphere, Eur. J. Appl. Math. 32 (2) (2021) 242–279.
[16] V. Chaurasiya, Numerical simulation of a non-linear sublimation process with temperature-dependent permeability and volumetric heat source: a phase change problem, Comput. Math. Appl. 176 (2024) 55–66.
[17] J. Crank, Free and Moving Boundary Problems, Clarendon Press, Oxford, 1984.
[18] M.P. Dalwadi, S.L. Waters, H.M. Byrne, I.J. Hewitt, A mathematical framework for developing freezing protocols in the cryopreservation of cells, SIAM J. Appl. Math. 80 (2020) 667–689.
[19] S.C. Gupta, The Classical Stefan Problem. Basic Concepts, Modelling and Analysis, Elsevier, Amsterdam, 2018.
[20] E. Hasrati, A. Jain, Non-monotonic phase change front propagation during one-dimensional melting/solidification in the conduction-dominated regime, Int. J. Therm. Sci. 217 (2025) 109970.
[21] E. Hesameddini, M. Riahi, Shifted Chebyshev polynomial method for solving systems of linear and nonlinear Fredhom-Volterra integro-differential equations, J. Math. Ext. 12 (3) (2018) 55–79.
[22] S. Koga, M. Krstic, Materials Phase Change PDE Control & Estimation, Birkhäuser, 2020.
[23] V.J. Lunardini, Heat Transfer with Freezing and Thawing, Elsevier Science Publishers B. V., 1991.
[24] J.L. Menaldi, D.A. Tarzia, Generalized Lamé-Clapeyron solution for a one-phase source Stefan problem, Comput. Appl. Math. 12 (1993) 123–142.
[25] N. Perez, Phase Transformation in Metals. Mathematics, Theory and Practice, Springer, 2020.
[26] K. Raslan, K. Ali, E. Mohamed, J. Younis, M. Abd El salam, An operational matrix technique based on Chebyshev polynomials for solving mixed Volterra-Fredholm delay integro-differential equations of variable-order, J. Funct. Spaces 2022 (6203440) (2022) 1–15.
[27] L. Rubinstein, The Stefan Problem, American Mathematical Society, Providence, 1971.
[28] M.N. Sahlan, H. Feyzollahzadeh, Operational matrices of Chebyshev polynomials for solving singular Volterra integral equations, Math. Sci. 11 (2017) 165–171.
[29] E.P. Scott, An analytical solution and sensitivity study of sublimation-dehydration within a porous medium with volumetric heating, J. Heat Transf. 116 (1994) 686–693.
[30] N.A. Shah, N. Kumar, H.C. Yadav, V. Chaurasiya, Gegenbauer wavelet collocation method to analyze one-dimensional solid–liquid phase change in a functionally graded material with liquid fraction distribution, Int. Commun. Heat Mass Transf. 162 (2025) 108620.
[31] A.D. Solomon, D.G. Wilson, V. Alexiades, A mushy zone model with an exact solution, Lett. Heat Mass Transf. 9 (1982) 319–324.
[32] A.D. Solomon, D.G. Wilson, V. Alexiades, Explicit solutions to phase change problems, Q. Appl. Math. 41 (1983) 237–243.
[33] A.D. Solomon, D.G. Wilson, V. Alexiades, The quasi-stationary approximation for the Stefan problem with a convective boundary condition, Int. J. Math. Math. Sci. 7 (1984) 549–563.
[34] A.D. Solomon, V. Alexiades, D.G. Wilson, The initial velocity of the emerging free boundary in a two-phase Stefan problem with imposed flux, SIAM J. Math. Anal. 18 (1987) 1438–1452.
[35] D.A. Tarzia, Relationship between Neumann solutions for two-phase Lamé-Clapeyron-Stefan problems with convective and temperature boundary conditions, Therm. Sci. 21 (1) (2017) 187–197.
[36] L.D. Venturato, M.B. Cirelli, D.A. Tarzia, Explicit solutions related to the Rubinstein binary-alloy solidification problem with a heat flux or a convective condition at the fixed face, Math. Methods Appl. Sci. 47 (2024) 6770–6788.
[37] A. Visintin, Models of Phase Transitions, Birkhäuser, 1996.
[38] T. Zhou, H. Xiao, Z. Rao, Inverse method for simultaneously estimating temperature-dependent solid- and liquid-phase thermal conductivities during phase transition, Int. Commun. Heat Mass Transf. 137 (2022) 106220.