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Determination of one unknown coefficient in a two-phase free boundary problem in an angular domain with variable thermal conductivity and specific heat

Julieta Bollati [a,b,*], María F. Natale [a], José A. Semitiel [a], Domingo A. Tarzia [a,b]

[a] Depto de Matemática, FCE-Universidad Austral, Paraguay 1950, 2000 Rosario, Argentina
[b] CONICET, Buenos Aires, Argentina

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A B S T R A C T

Two different two-phase free boundary Stefan problems in an angular domain with temperature-dependent thermal coefficients are considered. Analytical similarity solutions are obtained imposing a Dirichlet or Neumann type boundary condition, respectively, by solving functional problems. Moreover, formulas are obtained for the determination of one unknown thermal coefficient in the overspecified problem that consists in adding a Neumann condition to the problem with a Dirichlet one if and only if some restrictions on data are verified.

1. Introduction

Free boundary problems refer to boundary value problems for partial differential equations where, in addition to the unknown functions of the problem, an unknown surface called "free boundary", which separates two or more regions, has to be determined [1,14,21]. Such problems occur in a wide variety of industrial and natural applications, as the melting of the polar ice caps, sediment mass transport, solidification of lava from a volcano and tumor growth, etc. [4,6,7,9,22,29,31].

The Stefan problem (or phase change problem), studies the temperature in the space occupied by two phases of a material, usually a liquid phase and a solid phase. The functions representing the temperatures of the two phases satisfy the corresponding heat equations. An additional condition is imposed on the separation surface, which can vary over time and is at a constant temperature, arising from the principle of conservation of energy across the boundary. The interest and difficulty of the problem is due to the presence of the free boundary, whose determination is of fundamental importance in practice [25].

* Corresponding author.
E-mail address: jbollati@austral.edu.ar (J. Bollati).

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In the classic formulation of the Stefan problem, certain assumptions are supposed about the thermal coefficients involved in the phase change process in order to simplify the description of the model.

One of these hypotheses consists of considering constant thermal coefficients [12]. However, in the last decade, free boundary problems involving variable thermal coefficients, such as the thermal conductivity, specific heat or latent heat [8,11,19,20,27], have gained special importance due to its many applications. For example, a temperature-dependent thermal conductivity has been considered in [2] for the study of the flow of an incompressible fluid along a vertical porous plate. Some other models that involve a variable thermal conductivity can be found in [3,15]. Recently in [16,17], different approximations through the Tau method were studied, for a phase change problem of a semi-infinite material with a Dirichlet type condition at the fixed face and with variable thermal conductivity and specific heat given as potential functions of the temperature.

It should be noted that the classical Stefan problem is nonlinear even in its simplest form due to free boundary conditions. For this reason, in case the thermal coefficients depend on the temperature we will have a double non linearity.

Encouraged by [18,28,30], we consider a two-phase free boundary problem in an angular domain with non constants thermal coefficients in which shrinkage occurs. Pure substance which is initially in liquid state and when it is above the freezing temperature and cooling is applied at $x = 0$. The temperature of the liquid at the time $t = 0$ in $x = 0$ comes down to the freezing point and solidification begins, where $x = s(t)$ is the position of the interface. As the liquid solidifies it shrinks and appears an angular domain, i.e., a region between $x = 0$ and $x = rs(t)$ with

$$r = 1 - \frac{\rho_2}{\rho_1} \in (0, 1), \tag{1}$$

where ρ_i is the density of the region i where $i = 1$ is the solid region and $i = 2$ is the liquid region.

This leads us to the following free boundary problem with a Dirichlet type condition at $x = rs(t)$:

$$\frac{\partial}{\partial x}\left(k_1(u_1)\frac{\partial u_1}{\partial x}\right) = \rho_1 c_1(u_1)\left(\frac{\partial u_1}{\partial t} + r\dot{s}(t)\frac{\partial u_1}{\partial x}\right), \quad rs(t) < x < s(t), \quad t > 0, \tag{2}$$

$$\frac{\partial}{\partial x}\left(k_2(u_2)\frac{\partial u_2}{\partial x}\right) = \rho_2 c_2(u_2)\frac{\partial u_2}{\partial t}, \qquad x > s(t), \quad t > 0, \tag{3}$$

$$u_2(+\infty, t) = u_2(x, 0) = B > u^*, \qquad x > s(t), \tag{4}$$

$$u_1(s(t), t) = u_2(s(t), t) = u^*, \quad t > 0, \tag{5}$$

$$k_1(u_1(s(t), t))\frac{\partial u_1}{\partial x}(s(t), t) - k_2(u_2(s(t), t))\frac{\partial u_2}{\partial x}(s(t), t) = \rho_1 \ell \dot{s}(t), \quad t > 0, \tag{6}$$

$$s(0) = 0, \tag{7}$$

$$u_1(rs(t), t) = A < u^*, \tag{8}$$

where the temperature of the solid and the liquid, respectively are $u_i = u_i(x, t)$ for $i = 1, 2$, u^* is the freezing temperature, $\ell > 0$ is the latent heat of fusion by unit of mass, the thermal coefficients are variable and given by:

$$k_i(u_i) = k_i^* \left[1 + \beta_i \left(\frac{u_i - B}{u^* - B}\right)^{p_i}\right], \tag{9}$$

$$c_i(u_i) = c_i^* \left[1 + \beta_i \left(\frac{u_i - B}{u^* - B}\right)^{p_i}\right], \tag{10}$$

for $i = 1, 2$, with $\beta_i > 0$ and $p_i \geq 0$, where $k_i^* = k_i(u^*)$ and $c_i^* = c_i(u^*)$ are the reference thermal conductivity and the specific heat, respectively and the thermal diffusivity of the solid and liquid, respectively, are given by $\alpha_i = \dfrac{k_i^*}{\rho_i c_i^*}$ for $i = 1, 2$.

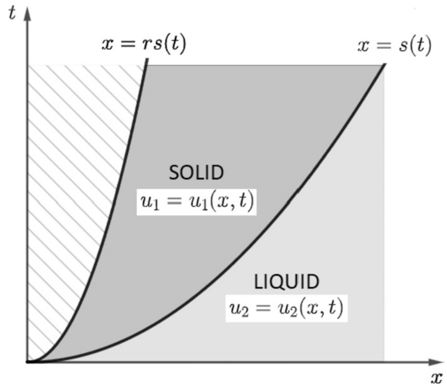


Fig. 1. Angular domain representation in the x, t plane.

In Fig. 1 we represent the angular domain in the x, t plane that corresponds to the problem (2)-(8) where shrinkage occurs.

The two-phase free boundary problem in the angular domain defined by (2)-(8) is also considered with a Neumann type condition. More precisely, we consider a two-phase free boundary problem in an angular domain which is defined by equations (2) and (3), conditions (4)-(6) of the problem (2)-(8) and the condition

$$k_1(u_1(rs(t),t))\frac{\partial u_1}{\partial x}(rs(t),t) = \frac{q_0}{\sqrt{t}}, \qquad q_0 > 0, \qquad t > 0, \tag{11}$$

instead condition (8) of the problem (2)-(8).

Moreover, in this paper, and inspired by the recent works [5,10,26] we impose to the problem (2)-(8) with a Dirichlet type condition, the over specified boundary condition which consists of the specification of a flux at the $x = rs(t)$ given by (11). The aim of the simultaneous determination of the temperature $u_i = u_i(x,t)$, $i = 1, 2$, the free boundary $x = s(t)$ and one unknown thermal coefficient among $\{\rho_1, \rho_2, c_1^*, c_2^*, k_1^*, k_2^*, \ell\}$.

The organization of this paper is as follows. In section 2, we examine the existence and uniqueness of solution to the problem with the Dirichlet type condition, (2)-(8). In section 3, we prove the existence and uniqueness of solution to the problem with the Neumann type condition, (2)-(7) and (11). Based on those results, in section 4, we present seven different cases for the phase-change process and we obtain formulae for the unknown coefficients as well as necessary and sufficient conditions for the existence of solution. Finally, in section 5, we give some conclusions.

2. Existence and uniqueness of solution to the two-phase free boundary problem with a Dirichlet type condition

In this section we will prove existence and uniqueness of solution of similarity type to the two-phase free boundary problem in an angular domain with a Dirichlet type condition defined by (2)-(8), where the temperatures $u_1 = u_1(x,t)$ and $u_2 = u_2(x,t)$ depends on the similarity variable given by

$$\eta = \frac{x}{2\lambda\sqrt{\alpha_2 t}}, \tag{12}$$

where $\lambda > 0$ is a dimensionless unknown coefficient to be determined. Through the following change of variables:

$$y_1(\eta) = \frac{B - u_1(x,t)}{B - u^*} \geq 0 \qquad \text{and} \qquad y_2(\eta) = \frac{B - u_2(x,t)}{B - u^*} \geq 0, \tag{13}$$

the phase front moves as

$$s(t) = 2\lambda\sqrt{\alpha_2 t}, \tag{14}$$

and thus we have the following result:

Theorem 2.1. *The Stefan problem (2)-(8) has a similarity solution (u_1, u_2, s) given by:*

$$u_1(x,t) = (u^* - B)\, y_1(\eta) + B, \quad rs(t) < x < s(t), \quad t > 0, \tag{15}$$

$$u_2(x,t) = (u^* - B)\, y_2(\eta) + B, \quad x > s(t), \quad t > 0, \tag{16}$$

$$s(t) = 2\lambda\sqrt{\alpha_2 t}, \qquad t > 0, \tag{17}$$

if and only if the functions y_1, y_2 and the parameter $\lambda > 0$ satisfy the following ordinary differential problems:

$$\tfrac{\alpha_1}{2\lambda^2\alpha_2}\left[(1+\beta_1 y_1^{p_1}(\eta))y_1'(\eta)\right]' + (\eta - r)\left(1+\beta_1 y_1^{p_1}(\eta)\right) y_1'(\eta) = 0, \; r < \eta < 1, \tag{18}$$

$$y_1(r) = \frac{A-B}{u^*-B}, \tag{19}$$

$$y_1(1) = 1, \tag{20}$$

and

$$\frac{1}{2\lambda^2}\left[(1+\beta_2 y_2^{p_2}(\eta))y_2'(\eta)\right]' + \eta\left(1+\beta_2 y_2^{p_2}(\eta)\right) y_2'(\eta) = 0, \qquad \eta > 1, \tag{21}$$

$$y_2(1) = 1, \tag{22}$$

$$y_2(\infty) = 0, \tag{23}$$

coupled through the following condition

$$\left(\frac{1+\beta_1}{1+\beta_2}\right)\frac{k_1^*}{k_2^*} y_1'(1) - y_2'(1) = \frac{-2\lambda^2}{(1+\beta_2)}\frac{\rho_1}{\rho_2 \mathrm{Ste}}, \tag{24}$$

where $\mathrm{Ste} = \frac{c_2^*(B-u^*)}{\ell} > 0$ *is the Stefan number.*

Proof. It follows by simple computations, recalling that the similarity variable η is given by (12). □

Lemma 2.2. *The triplet (y_1, y_2, λ) is a solution to the problem (18)-(24) if and only if it satisfies the following functional problem:*

$$\mathcal{F}_1(y_1(\eta)) = \mathcal{G}_1(\eta), \qquad r \le \eta \le 1, \tag{25}$$

$$\mathcal{F}_2(y_2(\eta)) = \mathcal{G}_2(\eta), \qquad \eta \ge 1, \tag{26}$$

$$\mathcal{M}(\lambda) = \mathcal{N}(\lambda), \tag{27}$$

where

$$\mathcal{F}_i(x) = x + \tfrac{\beta_i}{1+p_i} x^{1+p_i}, \qquad i = 1,2, \qquad x \ge 0, \tag{28}$$

$$\mathcal{G}_1(\eta) = \left(\mathcal{F}_1(1) - \mathcal{F}_1\left(\tfrac{A-B}{u^*-B}\right)\right)\frac{\mathrm{erf}\left(\lambda\sqrt{\frac{\alpha_2}{\alpha_1}}(\eta-r)\right)}{\mathrm{erf}\left(\lambda\sqrt{\frac{\alpha_2}{\alpha_1}}(1-r)\right)} + \mathcal{F}_1\left(\tfrac{A-B}{u^*-B}\right), r \le \eta \le 1, \tag{29}$$

$$\mathcal{G}_2(\eta) = \mathcal{F}_2(1)\tfrac{\mathrm{erfc}(\lambda\eta)}{\mathrm{erfc}(\lambda)}, \qquad \eta \ge 1, \tag{30}$$

$$\mathcal{M}(x) = \sqrt{\tfrac{\alpha_2}{\alpha_1}}\tfrac{k_1^*}{k_2^*}\left(\mathcal{F}_1(1) - \mathcal{F}_1\left(\tfrac{A-B}{u^*-B}\right)\right)\frac{\exp\left(-x^2\frac{\alpha_2}{\alpha_1}(1-r)^2\right)}{\operatorname{erf}\left(x\sqrt{\frac{\alpha_2}{\alpha_1}}(1-r)\right)}$$

$$+\mathcal{F}_2(1)\tfrac{\exp(-x^2)}{\operatorname{erfc}(x)}, \ x \geq 0, \tag{31}$$

$$\mathcal{N}(x) = -x\tfrac{\sqrt{\pi}}{\text{Ste}}\tfrac{\rho_1}{\rho_2}, \qquad x \geq 0. \tag{32}$$

Proof. Suppose that (y_1, y_2, λ) is a solution to the problem (18)-(24).

Let us define $v_1(\eta) = (1 + \beta_1 y_1^{p_1}(\eta))y_1'(\eta)$. Taking into account equation (18) we obtain that

$$v_1(\eta) = C_1 \exp\left(-\lambda^2\tfrac{\alpha_2}{\alpha_1}(\eta - r)^2\right), \qquad r < \eta < 1, \tag{33}$$

where $C_1 \in \mathbb{R}$. Then, integrating with respect to η we get that the general solution to the ordinary differential equation (18) must satisfy the following equation

$$y_1(\eta) + \tfrac{\beta_1}{1+p_1}y_1^{1+p_1}(\eta) = \tfrac{C_1}{\lambda}\sqrt{\tfrac{\alpha_1}{\alpha_2}}\tfrac{\sqrt{\pi}}{2}\operatorname{erf}\left(\lambda\sqrt{\tfrac{\alpha_2}{\alpha_1}}(\eta - r)\right) + D_1, \quad r < \eta < 1, \tag{34}$$

where D_1 is a real number. If we impose the boundary conditions (19) and (20) we get that

$$C_1 = \left(\mathcal{F}_1(1) - \mathcal{F}_1\left(\tfrac{A-B}{u^*-B}\right)\right)\tfrac{2}{\sqrt{\pi}}\sqrt{\tfrac{\alpha_2}{\alpha_1}}\frac{\lambda}{\operatorname{erf}\left(\lambda\sqrt{\frac{\alpha_2}{\alpha_1}}(1-r)\right)}, \quad D_1 = \mathcal{F}_1\left(\tfrac{A-B}{u^*-B}\right), \tag{35}$$

where $\mathcal{F}_1$ is given by (28). Therefore, y_1 satisfies equation (25).

In a similar way, we can define $v_2(\eta) = (1 + \beta_2 y_2^{p_2}(\eta))y_2'(\eta)$. From equation (21) we get

$$v_2(\eta) = C_2 \exp\left(-\lambda^2\eta^2\right), \qquad \eta > 1, \tag{36}$$

with $C_2 \in \mathbb{R}$. If we integrate the prior equation with respect to η, we obtain that the general solution to the ordinary differential equation (21) must satisfy the following equation

$$y_2(\eta) + \frac{\beta_2}{1+p_2}y_2^{1+p_2}(\eta) = \frac{C_2}{\lambda}\frac{\sqrt{\pi}}{2}\operatorname{erf}(\lambda\eta) + D_2, \qquad \eta > 1, \tag{37}$$

where D_2 is a real number. Then, imposing conditions (22) and (23) yields to

$$C_2 = \mathcal{F}_2(1)\frac{2}{\sqrt{\pi}}\frac{\lambda}{(\operatorname{erf}(\lambda) - 1)}, \qquad D_2 = -\frac{C_2}{\lambda}\frac{\sqrt{\pi}}{2}. \tag{38}$$

Therefore, y_2 satisfies equation (26).

Finally, from (33) and (36), taking into account (24) we easily get (27).

Reciprocally, assume that the triplet (y_1, y_2, λ) is a solution the system (25)-(27) then by elemental computations we can prove that it satisfies (18)-(24). □

Lemma 2.3. *The functional problem given by* (25)-(27) *has a unique solution* (y_1, y_2, λ).

Proof. Notice that for each $i = 1, 2$, $\mathcal{F}_i : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is a strictly increasing function. If we recall that $A < u^* < B$, then we get $\frac{A-B}{u^*-B} > 1$ and so $\mathcal{F}_1\left(\frac{A-B}{u^*-B}\right) > \mathcal{F}_1(1)$. Therefore, for each $\lambda > 0$, we obtain that the function $\mathcal{G}_1 = \mathcal{G}_1(\eta)$ given by (29) satisfies:

$$\mathcal{G}_1(\eta) = \mathcal{F}_1(1)\frac{\operatorname{erf}\left(\lambda\sqrt{\frac{\alpha_2}{\alpha_1}}(\eta-r)\right)}{\operatorname{erf}\left(\lambda\sqrt{\frac{\alpha_2}{\alpha_1}}(1-r)\right)} + \mathcal{F}_1\left(\tfrac{A-B}{u^*-B}\right)\left(1 - \frac{\operatorname{erf}\left(\lambda\sqrt{\frac{\alpha_2}{\alpha_1}}(\eta-r)\right)}{\operatorname{erf}\left(\lambda\sqrt{\frac{\alpha_2}{\alpha_1}}(1-r)\right)}\right) > \mathcal{F}_1(1) > 0. \tag{39}$$

Then, for each $\lambda > 0$ there exists a unique function $y_1 \in C^2(r, 1)$ solution to the equation (25) given by

$$y_1(\eta) = \mathcal{F}_1^{-1}(\mathcal{G}_1(\eta)), \qquad r \leq \eta \leq 1. \tag{40}$$

Similarly, taking into account that $\mathcal{G}_2(\eta) > 0$, for every $\eta > 1$, we get that there exists a unique function $y_2 \in C^2(1, +\infty)$ solution to the equation (26) given by

$$y_2(\eta) = \mathcal{F}_2^{-1}(\mathcal{G}_2(\eta)), \qquad \eta \geq 1. \tag{41}$$

Now we will analyse equation (27). Let us rewrite the function $\mathcal{M}$ given by (31) as:

$$\mathcal{M}(x) = \sqrt{\tfrac{\alpha_2}{\alpha_1}}\tfrac{k_1^*}{k_2^*}\left(\mathcal{F}_1(1) - \mathcal{F}_1\left(\tfrac{A-B}{u^*-B}\right)\right)\mathcal{M}_1\left(x\sqrt{\tfrac{\alpha_2}{\alpha_1}}(1-r)\right) + \mathcal{F}_2(1)\mathcal{M}_2(x), \tag{42}$$

with

$$\mathcal{M}_1(x) = \frac{\exp(-x^2)}{\operatorname{erf}(x)}, \qquad \mathcal{M}_2(x) = \frac{\exp(-x^2)}{\operatorname{erfc}(x)}. \tag{43}$$

It is easy to see that:

$$\begin{array}{llll} \mathcal{M}_1(0) = \infty, & \mathcal{M}_1(\infty) = 0, & \mathcal{M}_1'(x) < 0, & \forall\, x > 0, \\ \mathcal{M}_2(0) = 1, & \mathcal{M}_2(\infty) = \infty, & \mathcal{M}_2'(x) > 0, & \forall\, x > 0. \end{array} \tag{44}$$

Then, it follows straightforwardly that

$$\mathcal{M}(0) = -\infty, \qquad \mathcal{M}(\infty) = \infty, \qquad \mathcal{M}'(x) > 0, \quad \forall\, x > 0. \tag{45}$$

As

$$\mathcal{N}(0) = 0, \quad \mathcal{N}(\infty) = -\infty, \qquad \mathcal{N}'(x) < 0, \quad \forall\, x > 0, \tag{46}$$

we obtain that there exists a unique $\lambda > 0$ solution to equation $\mathcal{M}(x) = \mathcal{N}(x)$, $x > 0$.

In conclusion, there exists a unique triplet (y_1, y_2, λ) solution to the functional problem (25)-(27). □

Theorem 2.4. *The two-phase Stefan problem (2)-(8) has a unique similarity solution (u_1, u_2, s) given by (15)-(17) where (y_1, y_2, λ) is the unique solution to functional problem (25)-(27).*

Proof. It follows straightforwardly from Theorem 2.1 and Lemmas 2.2 and 2.3. □

Remark 2.5. *Taking into account that*

$$\mathcal{G}_1'(\eta) = \left(\mathcal{F}_1(1) - \mathcal{F}_1\left(\tfrac{A-B}{u^*-B}\right)\right)\frac{2\lambda\sqrt{\alpha_2}\exp\left(-\lambda^2\frac{\alpha_2}{\alpha_1}(\eta-r)^2\right)}{\sqrt{\pi\alpha_1}\operatorname{erf}\left(\lambda\sqrt{\frac{\alpha_2}{\alpha_1}}(1-r)\right)} < 0, \tag{47}$$

for each $r < \eta < 1$, we obtain that

$$\mathcal{F}_1(1) = \mathcal{G}_1(1) \leq \mathcal{G}_1(\eta) \leq \mathcal{G}_1(r) = \mathcal{F}_1\left(\tfrac{A-B}{u^*-B}\right). \tag{48}$$

In addition, as $\mathcal{F}_1^{-1} = \mathcal{F}_1^{-1}(\eta)$ is a increasing function we deduce that $y_1(\eta) = \mathcal{F}_1^{-1}(\mathcal{G}_1(\eta))$ is a decreasing function that satisfies the following inequality:

$$1 = y_1(1) < y_1(\eta) < y_1(r) = \tfrac{A-B}{u^*-B}, \qquad r < \eta < 1. \tag{49}$$

In virtue of this and Theorem 2.1, we have that

$$A < u_1(x,t) < u^*, \qquad rs(t) < x < s(t),\ t > 0. \tag{50}$$

In a similar way, we have that $\mathcal{G}_2'(\eta) = \dfrac{-2\lambda\mathcal{F}_2(1)\exp(\lambda^2\eta^2)}{\sqrt{\pi}\,\mathrm{erfc}(\lambda)} < 0$, *for* $\eta > 1$. *Then,*

$$\mathcal{G}_2(\infty) = 0 \le \mathcal{G}_2(\eta) \le \mathcal{G}_2(1) = \mathcal{F}_2(1). \tag{51}$$

Taking into account that $\mathcal{F}_2^{-1} = \mathcal{F}_2^{-1}(\eta)$ *is an increasing function we obtain that* $y_2(\eta) = \mathcal{F}_2^{-1}(\mathcal{G}_2(\eta))$ *is a decreasing function that satisfies the following inequality:*

$$0 = y_2(\infty) < y_2(\eta) < y_2(1) = 1, \qquad \eta > 1, \tag{52}$$

and from this and Theorem 2.1, we have that

$$u^* < u_2(x,t) < B, \qquad x > s(t),\ t > 0. \tag{53}$$

Remark 2.6. *(Particular case* $p_1 = p_2 = 0$*) If we consider* $p_1 = p_2 = 0$, *the thermal coefficients given in (9) and (10) are constants, i.e., the thermal conductivity and the specific heat are given by:*

$$k_i = k_i^*\left(1+\beta_i\right), \tag{54}$$

$$c_i = c_i^*\left(1+\beta_i\right), \tag{55}$$

respectively with $\beta_i > 0$ *for* $i = 1, 2$.

In this case, the unique solution $y_1 = y_1(\eta)$ *to the equation (25) is given by:*

$$y_1(\eta) = \left(\tfrac{u^*-A}{u^*-B}\right)\frac{\mathrm{erf}\left(\lambda\sqrt{\frac{\alpha_2}{\alpha_1}}(\eta-r)\right)}{\mathrm{erf}\left(\lambda\sqrt{\frac{\alpha_2}{\alpha_1}}(1-r)\right)} + \tfrac{A-B}{u^*-B}, \qquad r \le \eta \le 1, \tag{56}$$

the unique solution $y_2 = y_2(\eta)$ *to the equation (26) is given by:*

$$y_2(\eta) = \tfrac{\mathrm{erfc}(\lambda\eta)}{\mathrm{erfc}(\lambda)}, \qquad \eta \ge 1, \tag{57}$$

and from (27), $\lambda > 0$ *is the unique solution to the equation:*

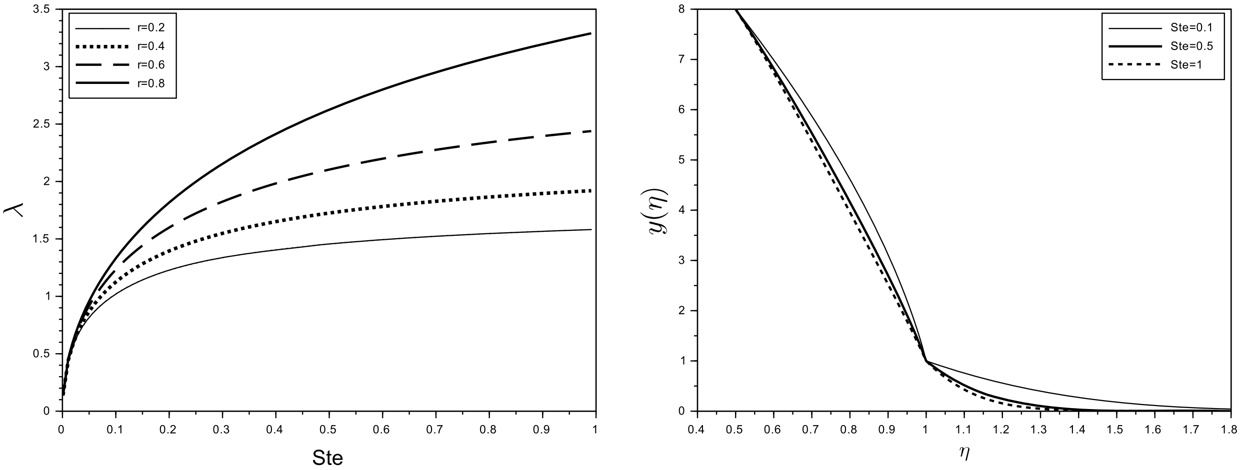
$$\sqrt{\tfrac{\alpha_2}{\alpha_1}}\tfrac{k_1}{k_2}\left(\tfrac{u^*-A}{u^*-B}\right)\frac{\exp\left(-x^2\frac{\alpha_2}{\alpha_1}(1-r)^2\right)}{\mathrm{erf}\left(x\sqrt{\frac{\alpha_2}{\alpha_1}}(1-r)\right)} + \tfrac{\exp(-x^2)}{\mathrm{erfc}(x)} = -x\tfrac{\sqrt{\pi}}{Ste}\tfrac{\rho_1}{\rho_2(1+\beta_2)}. \tag{58}$$

Therefore we have recovered the results obtained in [18].

Remark 2.7. *(Particular case* $p_1 = p_2 = 1$*) If we consider* $p_1 = p_2 = 1$, *the thermal coefficients given in (9) and (10) are linear, i.e., the thermal conductivity and the specific heat are given by:*

$$k_i(u_i) = k_i^*\left(1+\beta_i\left(\tfrac{u_i-B}{u^*-B}\right)\right), \tag{59}$$

$$c_i(u_i) = c_i^*\left(1+\beta_i\left(\tfrac{u_i-B}{u^*-B}\right)\right), \tag{60}$$



(a) Coefficient λ against Ste for different values of r

(b) $y(\eta)$ for different values of Ste.

Fig. 2. Plot of the coefficient λ and the function $y = y(\eta)$ for $\frac{k_1^*}{k_2^*} = 1$, $\frac{\alpha_1}{\alpha_2} = 1$, $\beta_1 = \beta_2 = 1$ and $\frac{A-B}{u^*-B} = 8$.

respectively with $\beta_i > 0$ *for* $i = 1, 2$.

In this case, the unique solution $y_1 = y_1(\eta)$ *to the equation* (25) *is given by:*

$$y_1(\eta) = \tfrac{1}{\beta_1}\left(-1 + \sqrt{1 + 2\beta_1 \hat{\mathcal{G}}_1(\eta)}\right), \qquad r \le \eta \le 1, \tag{61}$$

where

$$\hat{\mathcal{G}}_1(\eta) = \left(1 + \tfrac{\beta_1}{2} - \tfrac{A-B}{u^*-B} - \tfrac{\beta_1}{2}\left(\tfrac{A-B}{u^*-B}\right)^2\right) \frac{\operatorname{erf}\left(\lambda\sqrt{\frac{\alpha_2}{\alpha_1}}(\eta - r)\right)}{\operatorname{erf}\left(\lambda\sqrt{\frac{\alpha_2}{\alpha_1}}(1-r)\right)} + \tfrac{A-B}{u^*-B} + \tfrac{\beta_1}{2}\left(\tfrac{A-B}{u^*-B}\right)^2, \tag{62}$$

and the unique solution $y_2 = y_2(\eta)$ *to the equation* (26) *is given by:*

$$y_2(\eta) = \tfrac{1}{\beta_2}\left(-1 + \sqrt{1 + \beta_2\,(2 + \beta_2)\,\tfrac{\operatorname{erfc}(\lambda\eta)}{\operatorname{erfc}(\lambda)}}\right), \qquad \eta \ge 1, \tag{63}$$

and from (27), $\lambda > 0$ *is the unique solution to the equation:*

$$\sqrt{\tfrac{\alpha_2}{\alpha_1}}\tfrac{k_1^*}{k_2^*}\left(1 + \tfrac{\beta_1}{2} - \tfrac{A-B}{u^*-B} - \tfrac{\beta_1}{2}\left(\tfrac{A-B}{u^*-B}\right)^2\right) \frac{\exp\left(-x^2\frac{\alpha_2}{\alpha_1}(1-r)^2\right)}{\operatorname{erf}\left(x\sqrt{\frac{\alpha_2}{\alpha_1}}(1-r)\right)} + \\ + \left(1 + \tfrac{\beta_2}{2}\right)\frac{\exp(-x^2)}{\operatorname{erfc}(x)} = -x\tfrac{\sqrt{\pi}}{Ste}\tfrac{\rho_1}{\rho_2}. \tag{64}$$

In Fig. 2*a, we plot the solution* λ *to the equation* (64) *against Ste, for* $r = 0.2$, $r = 0.4$, $r = 0.6$ *and* $r = 0.8$. *Moreover in Fig.* 2*b, we plot the function*

$$y(\eta) = \begin{cases} y_1(\eta) & \text{if } r \le \eta \le 1 \\ y_2(\eta) & \text{if } \eta > 1 \end{cases}$$

where y_1 *is given by* (61) *and* y_2 *is defined by* (63), *for* $r = 0.5$.

3. Existence and uniqueness of solution to the two-phase free boundary problem with a Neumann type condition

In this section we will prove existence and uniqueness of solution of similarity type to the two-phase free boundary problem in an angular domain with a Neumann type condition defined by (2)-(7) and (11), where the temperatures u_1 and u_2 depends on the similarity variable given by

$$\eta = \frac{x}{2\mu\sqrt{\alpha_2 t}}, \tag{65}$$

where $\mu > 0$ is a dimensionless unknown coefficient to be determined. Through the following change of variables,

$$z_1(\eta) = \frac{B - u_1(x,t)}{B - u^*} \geq 0 \qquad \text{and} \qquad z_2(\eta) = \frac{B - u_2(x,t)}{B - u^*} \geq 0,$$

the phase front moves as

$$s(t) = 2\mu\sqrt{\alpha_2 t}, \tag{66}$$

and thus we have the following result:

Theorem 3.1. *The Stefan problem (2)-(7) and (11) has a similarity solution (u_1, u_2, s) given by:*

$$u_1(x,t) = (u^* - B)\, z_1(\eta) + B, \quad rs(t) < x < s(t), \quad t > 0, \tag{67}$$

$$u_2(x,t) = (u^* - B)\, z_2(\eta) + B, \quad x > s(t), \quad t > 0, \tag{68}$$

$$s(t) = 2\mu\sqrt{\alpha_2 t}, \qquad t > 0, \tag{69}$$

if and only if the functions z_1, z_2 and the parameter $\mu > 0$ satisfy the following ordinary differential problems:

$$\frac{\alpha_1}{2\mu^2\alpha_2}\left[(1+\beta_1 z_1^{p_1}(\eta))z_1'(\eta)\right]' + (\eta - r)\left(1+\beta_1 z_1^{p_1}(\eta)\right) z_1'(\eta) = 0, \qquad r < \eta < 1, \tag{70}$$

$$(1+\beta_1 z_1^{p_1}(r))z_1'(r) = \frac{2\mu\sqrt{\alpha_2}q_0}{(u^* - B)k_1^*}, \tag{71}$$

$$z_1(1) = 1, \tag{72}$$

and

$$\frac{1}{2\mu^2}\left[(1+\beta_2 z_2^{p_2}(\eta))z_2'(\eta)\right]' + \eta\left(1+\beta_2 z_2^{p_2}(\eta)\right) z_2'(\eta) = 0, \qquad \eta > 1, \tag{73}$$

$$z_2(1) = 1, \tag{74}$$

$$z_2(\infty) = 0, \tag{75}$$

coupled through the following condition

$$\left(\frac{1+\beta_1}{1+\beta_2}\right)\frac{k_1^*}{k_2^*} z_1'(1) - z_2'(1) = \frac{-2\mu^2}{(1+\beta_2)}\frac{\rho_1}{\rho_2 \text{Ste}}, \tag{76}$$

where $\text{Ste} = \frac{c_2^*(B-u^*)}{\ell} > 0$ *is the Stefan number.*

Proof. It follows by simple computations, recalling that the similarity variable η is given by (65). □

Lemma 3.2. *The triplet (z_1, z_2, μ) is a solution to the problem (70)-(76) if and only if it satisfies the functional problem:*

$$\mathcal{F}_1(z_1(\eta)) = \mathcal{G}_1^*(\eta), \qquad r \leq \eta \leq 1, \tag{77}$$

$$\mathcal{F}_2(z_2(\eta)) = \mathcal{G}_2^*(\eta), \qquad \eta \geq 1, \tag{78}$$

$$\mathcal{M}^*(\mu) = \mathcal{N}^*(\mu), \tag{79}$$

where

$$\mathcal{G}_1^*(\eta) = \mathcal{F}_1(1) + \tfrac{\sqrt{\alpha_1 \pi} q_0}{k_1^*(B-u^*)}\left[\operatorname{erf}\left(\mu\sqrt{\tfrac{\alpha_2}{\alpha_1}}(1-r)\right) - \operatorname{erf}\left(\mu\sqrt{\tfrac{\alpha_2}{\alpha_1}}(\eta-r)\right)\right], \qquad r \leq \eta \leq 1, \tag{80}$$

$$\mathcal{G}_2^*(\eta) = \mathcal{F}_2(1)\frac{\operatorname{erfc}(\mu\eta)}{\operatorname{erfc}(\mu)}, \qquad \eta \geq 1, \tag{81}$$

$$\mathcal{M}^*(x) = x\tfrac{\rho_1}{\rho_2 \mathrm{Ste}} + \frac{1}{\sqrt{\pi}}\mathcal{F}_2(1)\frac{\exp(-x^2)}{\operatorname{erfc}(x)}, \qquad x \geq 0, \tag{82}$$

$$\mathcal{N}^*(x) = \tfrac{\sqrt{\alpha_2} q_0}{(B-u^*)k_2^*}\exp\left(-x^2\tfrac{\alpha_2}{\alpha_1}(1-r)^2\right), \qquad x \geq 0. \tag{83}$$

Proof. It is similar to the proof of Lemma 2.2. □

Lemma 3.3. *The functional problem given by (77)-(79) has a unique solution (z_1, z_2, μ) if and only if*

$$q_0 > \frac{(B-u^*)k_2^*}{\sqrt{\alpha_2 \pi}}\left(1 + \frac{\beta_2}{p_2+1}\right). \tag{84}$$

Proof. Notice first that $\mathcal{G}_1^*(\eta) > 0$, $r < \eta < 1$. Then, for each $\mu > 0$ there exists a unique function $z_1 \in C^2[r, 1]$ solution to the equation (77) given by

$$z_1(\eta) = \mathcal{F}_1^{-1}(\mathcal{G}_1^*(\eta)), \qquad r \leq \eta \leq 1. \tag{85}$$

Similarly to the proof of Lemma 2.3, for every $\eta > 1$, we get that there exists a unique function $z_2 \in C^2[1, \infty)$ solution to the equation (78) given by

$$z_2(\eta) = \mathcal{F}_2^{-1}(\mathcal{G}_2^*(\eta)), \qquad \eta \geq 1. \tag{86}$$

It remains to prove that the equation (79) has a unique solution $\mu > 0$. If we rewrite the function $\mathcal{M}^*$ given by (82) as:

$$\mathcal{M}^*(x) = x\frac{\rho_1}{\rho_2 \mathrm{Ste}} + \frac{1}{\sqrt{\pi}}\mathcal{F}_2(1)\mathcal{M}_2(x), \tag{87}$$

and taking into account the properties of the function $\mathcal{M}_2$, it follows immediately that

$$\mathcal{M}^*(0) = \tfrac{1}{\sqrt{\pi}}\left(1 + \tfrac{\beta_2}{p_2+1}\right), \quad \mathcal{M}^*(\infty) = \infty, \quad \mathcal{M}^{*'}(x) > 0, \quad \forall\, x > 0. \tag{88}$$

As

$$\mathcal{N}^*(0) = \tfrac{\sqrt{\alpha_2} q_0}{(B-u^*)k_2^*}, \quad \mathcal{N}^*(\infty) = 0, \quad \mathcal{N}^{*'}(x) < 0, \quad \forall\, x > 0, \tag{89}$$

we obtain that there exists a unique $\mu > 0$ solution to equation $\mathcal{M}^*(x) = \mathcal{N}^*(x)$, $x > 0$ if and only if

$$\frac{\sqrt{\alpha_2} q_0}{(B - u^*) k_2^*} > \frac{1}{\sqrt{\pi}} \left(1 + \frac{\beta_2}{p_2 + 1}\right). \tag{90}$$

In conclusion, there exists a unique triplet (z_1, z_2, μ) solution to the functional problem (77)-(79) if and only if q_0 satisfies (84). □

Theorem 3.4. *The two-phase Stefan problem* (2)-(7) *and* (11) *has a unique similarity solution* (u_1, u_2, s) *given by* (67)-(69) *where* (z_1, z_2, μ) *is the unique solution to the functional problem* (77)-(79) *if and only if* (84) *holds.*

Proof. It follows straightforwardly from Theorem 3.1 and Lemmas 3.2 and 3.3. □

Remark 3.5. *(Particular case* $p_1 = p_2 = 0$*) If we consider* $p_1 = p_2 = 0$*, the thermal conductivity and the specific heat are given by* (54) *and* (55)*, respectively.*

In this case, the unique solution $z_1 = z_1(\eta)$ *to the equation* (77) *is given by:*

$$z_1(\eta) = 1 + \frac{\sqrt{\alpha_1 \pi} q_0}{k_1 (1+\beta_1)(B-u^*)} \left(\operatorname{erf}\left(\mu \sqrt{\frac{\alpha_2}{\alpha_1}} (1-r)\right) - \operatorname{erf}\left(\mu \sqrt{\frac{\alpha_2}{\alpha_1}} (\eta - r)\right)\right), r \leq \eta \leq 1, \tag{91}$$

and the unique solution $z_2 = z_2(\eta)$ *to the equation* (78) *is given by:*

$$z_2(\eta) = \frac{\operatorname{erfc}(\mu\eta)}{\operatorname{erfc}(\mu)}, \qquad \eta \geq 1, \tag{92}$$

and from (79)*,* $\mu > 0$ *is the unique solution to the equation:*

$$x \frac{\rho_1}{\rho_2 \text{Ste}} + \frac{1+\beta_2}{\sqrt{\pi}} \frac{\exp(-x^2)}{\operatorname{erfc}(x)} = \frac{\sqrt{\alpha_2} q_0}{(B-u^*) k_2^*} \exp\left(-x^2 \frac{\alpha_2}{\alpha_1} (1-r)^2\right), \qquad x \geq 0, \tag{93}$$

if and only if $q_0 > \frac{(B-u^*) k_2^* (1+\beta_2)}{\sqrt{\alpha_2 \pi}}$*. Therefore we can recover the results obtained in [18]. Moreover, if* $r = 0$ *(i.e.* $\rho_1 = \rho_2$*) we get the result obtain in [23].*

Remark 3.6. *(Particular case* $p_1 = p_2 = 1$*) If we consider de particular case* $p_1 = p_2 = 1$*, the thermal conductivity and the specific heat are given by* (59) *and* (60)*, respectively.*

In this case, the unique solution $z_1 = z_1(\eta)$ *to the equation* (77) *is given by:*

$$z_1(\eta) = \frac{1}{\beta_1} \left(-1 + \sqrt{1 + 2\beta_1 \hat{\mathcal{G}}_1^*(\eta)}\right), \qquad r \leq \eta \leq 1, \tag{94}$$

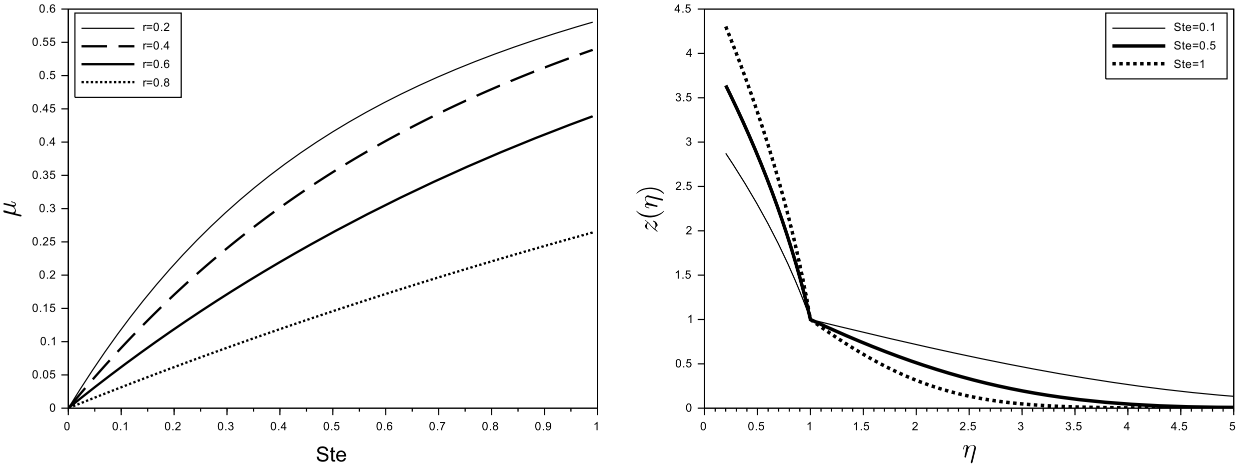
where

$$\hat{\mathcal{G}}_1^*(\eta) = 1 + \frac{\beta_1}{2} + \frac{\sqrt{\alpha_1 \pi} q_0}{k_1^* (B-u^*)} \left(\operatorname{erf}\left(\mu \sqrt{\frac{\alpha_2}{\alpha_1}} (1-r)\right) - \operatorname{erf}\left(\mu \sqrt{\frac{\alpha_2}{\alpha_1}} (\eta - r)\right)\right), \tag{95}$$

the unique solution $z_2 = z_2(\eta)$ *to the equation* (78) *is given by:*

$$z_2(\eta) = \frac{1}{\beta_2} \left(-1 + \sqrt{1 + \beta_2 (2 + \beta_2) \frac{\operatorname{erfc}(\mu\eta)}{\operatorname{erfc}(\mu)}}\right), \qquad \eta \geq 1, \tag{96}$$

and from (79)*,* $\mu > 0$ *is the unique solution to the equation:*



(a) Coefficient μ against Ste for different values of r

(b) $z(\eta)$ for different values of Ste with $r = 0.2$.

Fig. 3. Plot of the coefficient μ and the function $z = z(\eta)$ for $k_1^* = k_2^* = 1$, $\alpha_1 = 10$, $\alpha_2 = 5$, $\beta_1 = \beta_2 = 1$, $B - u^* = 5$ and $q_0 = 30$.

$$x \frac{\rho_1}{\rho_2 \text{Ste}} + \frac{2+\beta_2}{2\sqrt{\pi}} \frac{\exp(-x^2)}{\text{erfc}(x)} = \frac{\sqrt{\alpha_2} q_0}{(B-u^*)k_2^*} \exp\left(-x^2 \frac{\alpha_2}{\alpha_1}(1-r)^2\right), \quad x \geq 0, \tag{97}$$

if and only if $q_0 > \frac{(B-u^*)(2+\beta_2)k_2^*}{2\sqrt{\alpha_2 \pi}}$.

In Fig. 3a, we plot the solution λ *to the equation* (97) *against* Ste, *for* $r = 0.2$, $r = 0.4$, $r = 0.6$ *and* $r = 0.8$. *Moreover in Fig. 3b we plot the function*

$$z(\eta) = \begin{cases} z_1(\eta) & \text{if } r \leq \eta \leq 1 \\ z_2(\eta) & \text{if } \eta > 1 \end{cases}$$

where z_1 *is given by* (94) *and* z_2 *is defined by* (96).

Remark 3.7. *If the coefficient* q_0 *satisfies*

$$0 < q_0 \leq \frac{(B-u^*)k_2^*}{\sqrt{\alpha_2 \pi}} \left(1 + \frac{\beta_2}{p_2+1}\right),$$

then the free boundary problem (2)-(7) *and* (11) *is a classical heat transfer problem, without a phase change process, for the initial liquid phase given by:*

$$\begin{aligned} &\frac{\partial}{\partial x}\left(k_2(u_2)\frac{\partial u_2}{\partial x}\right) = \rho_2 c_2(u_2)\frac{\partial u_2}{\partial t}, && x > 0, \quad t > 0, \\ &u_2(+\infty, t) = u_2(x, 0) = B > u^*, && x > 0, \quad t > 0, \\ &k_2(u_2(0,t))\frac{\partial u_2}{\partial x}(0), t) = \frac{q_0}{\sqrt{t}}, && t > 0, \end{aligned}$$

whose unique similarity solution is

$$u_2(x,t) = (u^* - B)\, z_2\left(\frac{x}{2\sqrt{\alpha_2 t}}\right) + B, \quad x > 0, \quad t > 0,$$

where

$$z_2(\eta) = \mathcal{F}_2^{-1}\left(\widetilde{\mathcal{G}}_2(\eta)\right), \qquad \eta > 0,$$

with $\mathcal{F}_2$ *given by* (28) *and* $\widetilde{\mathcal{G}}_2(\eta) = \frac{\sqrt{\pi}\sqrt{\alpha_2} q_0 \,\text{erfc}(\eta)}{k_2^*(B-u^*)}$.

4. Determination of unknown thermal coefficients

In this section we are going to study a free boundary Stefan problem given by (2)-(8) together with the over specified Neumann condition (11) following the first idea given in [24]. This problem consists on finding the temperatures u_1, u_2, the free boundary $x = s(t)$ and one thermal coefficient among $\{\rho_1, \rho_2, c_1^*, c_2^*, k_1^*, k_2^*, \ell\}$.

According to Theorem 2.4, the similarity solution (u_1, u_2, s) to the problem (2)-(8) is given by (15)-(17) where the functions y_1, y_2 are obtained by (25)-(26) and the coefficient λ is defined by (27), i.e.

$$\frac{\sqrt{k_1^*\rho_1 c_1^*}}{\sqrt{k_2^*\rho_2 c_2^*}}\, m^*\, \mathcal{M}_1\left(\lambda\frac{\sqrt{k_2^*\rho_2 c_1^*}}{\sqrt{k_1^*\rho_1 c_2^*}}\right) - \mathcal{F}_2(1)\mathcal{M}_2(\lambda) = \frac{\lambda\sqrt{\pi}\ell\rho_1}{c_2^*(B-u^*)\rho_2}, \tag{98}$$

where $m^* = \mathcal{F}_1\left(\frac{A-B}{u^*-B}\right) - \mathcal{F}_1(1)$.

Moreover, if we request that the solution (u_1, u_2, s) to the problem (2)-(8) also satisfies the over specified condition (11) we obtain that:

$$y_1'(r) = \frac{2\lambda q_0\sqrt{\alpha_2}}{k_1^*\left(1+\beta_1\left(\frac{A-B}{u^*-B}\right)^{p_1}\right)(u^*-B)}.$$

Taking into account that y_1 was given by (25), the condition on y_1' leads to the following equality:

$$\frac{q_0\sqrt{\pi}}{\sqrt{k_1^*\rho_1 c_1^*}}\operatorname{erf}\left(\lambda\frac{\sqrt{k_2^*\rho_2 c_1^*}}{\sqrt{k_1^*\rho_1 c_2^*}}\right) = (B-u^*)m^*. \tag{99}$$

In view of this analysis, the Stefan problem with an over specified condition involves the determination of the coefficient λ that characterizes the free boundary and one thermal coefficient among $\{\rho_1, \rho_2, c_1^*, c_2^*, k_1^*, k_2^*, \ell\}$ that satisfy conditions (98)-(99).

We will divide the study into seven different cases:

† Case 1: Determination of λ, ρ_1.
† Case 2: Determination of λ, ρ_2.
† Case 3: Determination of λ, c_1^*.
† Case 4: Determination of λ, c_2^*.
† Case 5: Determination of λ, k_1^*.
† Case 6: Determination of λ, k_2^*.
† Case 7: Determination of λ, ℓ.

At the end of this section, in Table 1, we show a summary of all the cases studied. In each case, we specify the restriction on q_0 so that the corresponding unknown coefficients can be determined.

Theorem 4.1. ***(Case 1:** λ, ρ_1**)** *Let us define*

$$R_1 = \frac{\sqrt{k_1^* c_1^*}}{\sqrt{k_2^*\rho_2 c_2^*}} m^*, \quad Q_1 = \frac{\sqrt{k_2^*\rho_2 c_1^*}}{\sqrt{k_1^* c_2^*}}, \quad D_1 = \frac{\ell\sqrt{\pi}}{c_2^*\rho_2(B-u^*)}, \quad P_1 = \frac{\sqrt{k_1^* c_1^*}}{q_0\sqrt{\pi}}(B-u^*)m^*.$$

If we assume

$$q_0 > \left(1+\frac{\beta_2}{1+p_2}\right)\frac{\sqrt{k_2^*\rho_2 c_2^*}}{\sqrt{\pi}}(B-u^*), \tag{100}$$

then there is a unique solution (λ, ρ_1) to the system (98)-(99) given by

$$\lambda = \frac{\overline{z}_1 \operatorname{erf}^{-1}(\overline{z}_1)}{Q_1 P_1}, \qquad \rho_1 = \frac{\overline{z}_1^2}{P_1^2}, \tag{101}$$

where $\overline{z}_1$ is the unique solution to the following equation

$$\mathcal{L}_1(z) = \mathcal{R}_1(z), \qquad 0 < z < 1, \tag{102}$$

with

$$\mathcal{L}_1(z) = \tfrac{R_1}{P_1} \exp\left(- \left(\operatorname{erf}^{-1}(z) \right)^2 \right) - \mathcal{F}_2(1) \mathcal{M}_2 \left(\tfrac{z \operatorname{erf}^{-1}(z)}{Q_1 P_1} \right),$$

$$\mathcal{R}_1(z) = \tfrac{D_1}{Q_1 P_1^3} z^3 \operatorname{erf}^{-1}(z).$$

Proof. If we call $\overline{z}_1 = P_1 \sqrt{\rho_1}$, taking into account equation (99), we obtain that λ and ρ_1 are given by formula (101) as functions of $\overline{z}_1$ where we assume that $\overline{z}_1 \in (0, 1)$ (so as $\operatorname{erf}^{-1}(\overline{z}_1)$ makes sense). If we replace the value of λ in equation (98) we get that $\overline{z}_1$ must be a solution to the equation (102). On one hand, $\mathcal{L}_1$ is a decreasing function in z that satisfies: $\mathcal{L}_1(0) = \frac{R_1}{P_1} - \mathcal{F}_2(1)$ and $\mathcal{L}_1(1) = -\infty$. On the other hand, $\mathcal{R}_1$ is an increasing function that verifies: $\mathcal{R}_1(0) = 0$ and $\mathcal{R}_1(1) = +\infty$. Therefore, if we assume $\mathcal{L}_1(0) > 0$, which is equivalent to the hypothesis (100), we can guarantee that there exists a unique $\overline{z}_1 \in (0, 1)$ solution to equation (102). □

Theorem 4.2. ***(Case 2:** λ, ρ_2**)** Let us define*

$$R_2 = \tfrac{\sqrt{k_1^* \rho_1 c_1^*}}{\sqrt{k_2^* c_2^*}} m^*, \quad Q_2 = \tfrac{\sqrt{k_2^* c_1^*}}{\sqrt{k_1^* \rho_1 c_2^*}}, \quad D_2 = \tfrac{\ell \rho_1 \sqrt{\pi}}{c_2^* (B - u^*)}, \quad P_2 = \tfrac{\sqrt{k_1^* \rho_1 c_1^*}}{q_0 \sqrt{\pi}} (B - u^*) m^*.$$

We assume that

$$q_0 > \frac{\sqrt{k_1^* \rho_1 c_1^*}}{\sqrt{\pi} \operatorname{erf}(\overline{z}_2)} (B - u^*) m^*, \tag{103}$$

where $\overline{z}_2$ is the unique solution to the equation

$$G(z) = \tfrac{3}{2} \sqrt{\pi} \mathcal{F}_2(1), \qquad z > 0, \tag{104}$$

with

$$\begin{aligned} G(z) &= R_2 Q_2 \tfrac{\mathcal{M}_1(z)}{z} \mathcal{H}\left(\sqrt{g(z)} \right), \\ \mathcal{H}(z) &= \sqrt{\pi} z \operatorname{erfc}(z) \exp(z^2), \\ g(z) &= \tfrac{R_2}{3 Q_2 D_2} z \mathcal{M}_1(z). \end{aligned} \tag{105}$$

Then, there exists two solutions (λ, ρ_2) to the system (98)-(99). The coefficient ρ_2 is given by

$$\rho_2 = \left(\frac{\operatorname{erf}^{-1}(P_2)}{\lambda Q_2} \right)^2, \tag{106}$$

and λ is one positive solution to the following equation

$$\mathcal{L}_2(z) = \mathcal{R}_2(z), \quad z > 0, \tag{107}$$

with

$$\mathcal{L}_2(z) = z\tfrac{Q_2 R_2}{\operatorname{erf}^{-1}(P_2)}\mathcal{M}_1(\operatorname{erf}^{-1}(P_2)) - z^3\tfrac{Q_2^2 D_2}{(\operatorname{erf}^{-1}(P_2))^2}, \quad \mathcal{R}_2(z) = \mathcal{F}_2(1)\mathcal{M}_2(z).$$

Proof. Assuming $P_2 < 1$, from equation (99) we get that the coefficient ρ_2 is given in function of λ by expression (106). By equation (98), it follows that λ must satisfy (107). On one hand, $\mathcal{R}_2$ is an increasing function in λ that verifies $\mathcal{R}_2(0) = \mathcal{F}_2(1)$ and $\mathcal{R}_2(+\infty) = +\infty$. On the other hand, $\mathcal{L}_2(0) = 0$ and $\mathcal{L}_2(+\infty) = -\infty$. Moreover, defining $\overline{\lambda}$ as

$$\overline{\lambda} = \left(\frac{R_2 \operatorname{erf}^{-1}(P_2)\mathcal{M}_1(\operatorname{erf}^{-1}(P_2))}{3Q_2 D_2}\right)^{1/2},$$

we obtain that $\mathcal{L}_2$ is increasing function in $(0, \overline{\lambda})$ and a decreasing function in $(\overline{\lambda}, +\infty)$. Therefore, if we assume

$$\mathcal{L}_2(\overline{\lambda}) > \mathcal{R}_2(\overline{\lambda}), \tag{108}$$

then there exists two solutions to the equation (107).

Working algebraically, it is easy to see that $\overline{\lambda} = \sqrt{g(\operatorname{erf}^{-1}(P_2))}$ and therefore condition (108) is equivalent to

$$G(\operatorname{erf}^{-1}(P_2)) = R_2 Q_2 \tfrac{\mathcal{M}_1(\operatorname{erf}^{-1}(P_2))}{\operatorname{erf}^{-1}(P_2)}\,\mathcal{H}\left(\sqrt{g(\operatorname{erf}^{-1}(P_2))}\right) > \tfrac{3}{2}\sqrt{\pi}\mathcal{F}_2(1), \tag{109}$$

where G, $\mathcal{H}$ and g are given by (105).

From [13] we know that $\mathcal{H}$ is an increasing function that satisfies $\mathcal{H}(0) = 0$ and $\mathcal{H}(+\infty) = 1$ and g is a decreasing function such that $g(0) = \frac{\sqrt{\pi}}{2}$ and $g(+\infty) = 0$. Then G is a decreasing function that satisfies $G(0) = +\infty$ and $G(+\infty) = 0$. As a consequence inequality (109) holds if and only if $\operatorname{erf}^{-1}(P_2) < \overline{z}_2$ where $\overline{z}_2$ is the unique solution to the equation (104). Therefore, provided that $P_2 < \min\{1, \operatorname{erf}(\overline{z}_2)\} = \operatorname{erf}(\overline{z}_2)$ (equivalent to inequality (103)), we obtain that there exists at least one solution λ to the equation (107). □

Theorem 4.3. *(Case 3: λ, c_1^*) Let us define*

$$R_3 = \tfrac{\sqrt{k_1^*\rho_1}}{\sqrt{k_2^*\rho_2 c_2^*}}m^*, \quad Q_3 = \tfrac{\sqrt{k_2^*\rho_2}}{\sqrt{k_1^*\rho_1 c_2^*}}, \quad D_3 = \tfrac{\ell\rho_1\sqrt{\pi}}{c_2^*\rho_2(B-u^*)}, \quad P_3 = \tfrac{\sqrt{k_1^*\rho_1}}{q_0\sqrt{\pi}}(B-u^*)m^*.$$

If we assume

$$q_0 > q_0^*, \tag{110}$$

where $q_0^* > 0$ is the unique solution to the equation

$$z = \tfrac{\ell\rho_1^2 k_1^*}{2\rho_2}\tfrac{1}{z} + \sqrt{\frac{k_2^* c_2^* \rho_2}{\pi}}\mathcal{F}_2(1)\mathcal{M}_2\left(\tfrac{\rho_1 k_1^*(B-u^*)m^*}{2z}\sqrt{\tfrac{c_2^*}{k_2^*\rho_2}}\right), \quad z > 0, \tag{111}$$

then there is a unique solution (λ, c_1^*) to the system (98)-(99) given by

$$\lambda = \frac{P_3}{Q_3}\frac{\operatorname{erf}^{-1}(\overline{z}_3)}{\overline{z}_3}, \qquad c_1^* = \frac{\overline{z}_3^2}{P_3^2}, \tag{112}$$

where $\overline{z}_3$ is the unique solution to the following equation

$$\mathcal{L}_3(z) = \mathcal{R}_3(z), \quad 0 < z < 1, \tag{113}$$

with

$$\mathcal{L}_3(z) = \tfrac{R_3}{P_3} \exp\left(-\left(\operatorname{erf}^{-1}(z)\right)^2\right),$$

$$\mathcal{R}_3(z) = \tfrac{P_3 D_3}{Q_3} \tfrac{\operatorname{erf}^{-1}(z)}{z} + \mathcal{F}_2(1)\mathcal{M}_2\left(\tfrac{P_3}{Q_3} \tfrac{\operatorname{erf}^{-1}(z)}{z}\right).$$

Proof. If we define $\overline{z}_3 = P_3\sqrt{c_1^*}$ and taking into account equation (99), we can easily obtain that λ and c_1^* are given by (112) where $\overline{z}_3 \in (0,1)$ in order that $\operatorname{erf}^{-1}(\overline{z}_3)$ makes sense. Replacing the value of λ in (98) it follows that $\overline{z}_3$ must be a solution to equation (113). Notice that $\mathcal{L}_3$ is a decreasing function that verifies $\mathcal{L}_3(0) = \frac{R_3}{P_3}$ and $\mathcal{L}_3(1) = 0$, while $\mathcal{R}_3$ is an increasing function such that $\mathcal{R}_3(0) = \frac{P_3 D_3 \sqrt{\pi}}{2Q_3} + \mathcal{F}_2(1)\mathcal{M}_2\left(\frac{\sqrt{\pi}P_3}{2Q_3}\right)$ and $\mathcal{R}_3(1) = +\infty$. Therefore, under the assumption $\mathcal{L}_3(0) > \mathcal{R}_3(0)$, which is equivalent to $q_0 > q_0^*$ where $q_0^* > 0$ is the unique solution to the equation (111), we can guarantee that there exists a unique $\overline{z}_3 \in (0,1)$ solution to equation (113). □

Theorem 4.4. ***(Case 4:*** λ, c_2^****)*** *Let us define*

$$R_4 = \tfrac{\sqrt{k_1^* \rho_1 c_1^*}}{\sqrt{k_2^* \rho_2}} m^*, \quad Q_4 = \tfrac{\sqrt{k_2^* \rho_2 c_1^*}}{\sqrt{k_1^* \rho_1}}, \quad D_4 = \tfrac{\ell \rho_1 \sqrt{\pi}}{\rho_2 (B-u^*)}, \quad P_4 = \tfrac{\sqrt{k_1^* \rho_1 c_1^*}}{q_0 \sqrt{\pi}} (B-u^*) m^*.$$

Assume that

$$q_0 > \frac{\sqrt{k_1^* \rho_1 c_1^*}(B-u^*)m^*}{\operatorname{erf}(\overline{z}_4)\sqrt{\pi}}, \tag{114}$$

where $\overline{z}_4$ *is the unique solution to the equation*

$$h(z) = \frac{R_4 Q_4}{D_4}, \qquad z > 0, \tag{115}$$

with

$$h(z) = \frac{z}{\mathcal{M}_1(z)}. \tag{116}$$

Then there is a unique solution (λ, c_2^*) *to the system* (98)-(99)*. The coefficient* c_2^* *is given by*

$$c_2^* = \left(\frac{\lambda Q_4}{\operatorname{erf}^{-1}(P_4)}\right)^2, \tag{117}$$

and λ *is the unique positive solution to the following equation*

$$\mathcal{R}_4(z) = \mathcal{L}_4, \quad z > 0, \tag{118}$$

where

$$\mathcal{L}_4 = \left[R_4 \mathcal{M}_1(\operatorname{erf}^{-1}(P_4)) - \tfrac{D_4}{Q_4} \operatorname{erf}^{-1}(P_4)\right] \frac{\operatorname{erf}^{-1}(P_4)}{Q_4}, \quad \mathcal{R}_4(z) = \mathcal{F}_2(1) z \mathcal{M}_2(z).$$

Proof. Taking into account equation (99), if $P_4 < 1$ is assumed, then expression (117) for c_2^* is derived. Replacing this value in equation (98), the non-linear equation (118) for λ is obtained. As $\mathcal{R}_4$ is an increasing function that satisfies $\mathcal{R}_4(0) = 0$ and $\mathcal{R}_4(+\infty) = +\infty$, if we assume $\mathcal{L}_4 > 0$, then existence and uniqueness of solution to equation (118) is guaranteed. Notice that $\mathcal{L}_4 > 0$ is equivalent to

$$h(\operatorname{erf}^{-1}(P_4)) < \frac{R_4 Q_4}{D_4}, \tag{119}$$

where h is given by (116). According to [13], $h'(z) > 0$ for all $z > 0$, $h(0) = 0$ and $h(+\infty) = +\infty$. Therefore, inequality (119) holds if and only if $\operatorname{erf}^{-1}(P_4) < \overline{z}_4$ where $\overline{z}_4$ is the unique solution to equation (115). As a consequence, if we assume $P_4 < \min\{1, \operatorname{erf}(\overline{z}_4)\} = \operatorname{erf}(\overline{z}_4)$ (equivalent to inequality (114)) there exists a unique solution to equation (118). □

Theorem 4.5. ***(Case 5:*** λ, k_1^****)*** *Let us define*

$$R_5 = \frac{\sqrt{\rho_1 c_1^*}}{\sqrt{k_2^* \rho_2 c_2^*}} m^*, \quad Q_5 = \frac{\sqrt{k_2^* \rho_2 c_1^*}}{\sqrt{\rho_1 c_2^*}}, \quad D_5 = \frac{\ell \rho_1 \sqrt{\pi}}{c_2^* \rho_2 (B - u^*)}, \quad P_5 = \frac{\sqrt{\rho_1 c_1^*}}{q_0 \sqrt{\pi}} (B - u^*) m^*.$$

If we assume

$$q_0 > \left(1 + \frac{\beta_2}{1 + p_2}\right)(B - u^*) \frac{\sqrt{k_2^* c_2^* \rho_2}}{\sqrt{\pi}}, \tag{120}$$

then there is a unique solution (λ, k_1^*) *to the system* (98)-(99) *given by*

$$\lambda = \frac{\overline{z}_5 \operatorname{erf}^{-1}(\overline{z}_5)}{Q_5 P_5}, \qquad k_1^* = \frac{\overline{z}_5^2}{P_5^2}, \tag{121}$$

where $\overline{z}_5$ *is the unique solution to the following equation*

$$\mathcal{L}_5(z) = \mathcal{R}_5(z), \quad 0 < z < 1, \tag{122}$$

with

$$\mathcal{L}_5(z) = \tfrac{R_5}{P_5} \exp\left(-\left(\operatorname{erf}^{-1}(z)\right)^2\right) - \mathcal{F}_2(1) \mathcal{M}_2\left(\frac{z \operatorname{erf}^{-1}(z)}{Q_5 P_5}\right),$$
$$\mathcal{R}_5(z) = \tfrac{D_5}{Q_5 P_5} z \operatorname{erf}^{-1}(z).$$

Proof. If we define $\overline{z}_5 = P_5 \sqrt{k_1^*}$ and take into account equation (99), it follows that λ and k_1^* are given by (121) where $\overline{z}_5 \in (0, 1)$. Then, from equation (98) it follows that $\overline{z}_5$ must satisfy equation (122). Notice that, $\mathcal{R}_5$ is an increasing function such that $\mathcal{R}_5(0) = 0$ and $\mathcal{R}_5(1) = +\infty$. Moreover, $\mathcal{L}_5$ is a decreasing function such that $\mathcal{L}_5(0) = \frac{R_5}{P_5} - \mathcal{F}_2(1)$ and $\mathcal{L}_5(1) = -\infty$. As a consequence, under assumption $\mathcal{L}_5(0) > 0$ (equivalent to inequality (120)), there exists a unique solution to the equation (122) in the interval $(0, 1)$. □

Theorem 4.6. ***(Case 6:*** λ, k_2^****)*** *Let us define*

$$R_6 = \frac{\sqrt{k_1^* \rho_1 c_1^*}}{\sqrt{\rho_2 c_2^*}} m^*, \quad Q_6 = \frac{\sqrt{\rho_2 c_1^*}}{\sqrt{k_1^* \rho_1 c_2^*}}, \quad D_6 = \frac{\ell \rho_1 \sqrt{\pi}}{c_2^* \rho_2 (B - u^*)}, \quad P_6 = \frac{\sqrt{k_1^* \rho_1 c_1^*}}{q_0 \sqrt{\pi}} (B - u^*) m^*.$$

Assume that

$$q_0 > \frac{\sqrt{k_1^* \rho_1 c_1^*}(B - u^*) m^*}{\operatorname{erf}(\overline{w}_6) \sqrt{\pi}}, \tag{123}$$

where $\overline{w}_6$ *is the unique solution to the equation*

$$h(z) = \frac{R_6 Q_6}{D_6 + \sqrt{\pi} \mathcal{F}_2(1)}, \qquad z > 0, \tag{124}$$

with h given by (116).

Then there is a unique solution (λ, k_2^*) to the system (98)-(99). The coefficient k_2^* is given by

$$k_2^* = \left(\frac{\operatorname{erf}^{-1}(P_6)}{\lambda Q_6}\right)^2, \tag{125}$$

and λ is the unique positive solution to the following equation

$$\mathcal{L}_6(z) = \mathcal{R}_6, \quad z > 0, \tag{126}$$

with

$$\mathcal{L}_6(z) = \frac{z}{\mathcal{M}_2(z)}, \qquad \mathcal{R}_6 = \frac{\mathcal{F}_2(1)}{Q_6 R_6 \frac{\mathcal{M}_1(\operatorname{erf}^{-1}(P_6))}{\operatorname{erf}^{-1}(P_6)} - D_6}.$$

Proof. First, we assume $P_6 < 1$ in order that equation (99) is well defined. Then, from this equation, the coefficient k_2^* given by (125) is immediately obtained. If we replace the value of k_2^* in (98), we get the equation (126) for the coefficient λ.

From [13], we know that $\mathcal{L}_6$ is an increasing function such that $\mathcal{L}_6(0) = 0$ and $\mathcal{L}_6(+\infty) = \frac{1}{\sqrt{\pi}}$. Therefore, we can guarantee that there exists a unique solution λ to the equation (126) if the constant $\mathcal{R}_6$ verifies the following inequality $0 < \mathcal{R}_6 < \frac{1}{\sqrt{\pi}}$. This inequality is equivalent to

$$h(\operatorname{erf}^{-1}(P_6)) < \frac{R_6 Q_6}{D_6 + \sqrt{\pi}\mathcal{F}_2(1)} < \frac{R_6 Q_6}{D_6}, \tag{127}$$

where h is the function defined by (116). As h is an increasing function that satisfies $h(0) = 0$ and $h(+\infty) = +\infty$, it follows that there exists a unique value $\overline{w}_6 > 0$ such that:

$$h(\overline{w}_6) = \frac{R_6 Q_6}{D_6 + \mathcal{F}_2(1)\sqrt{\pi}}, \qquad h(z) < \frac{R_6 Q_6}{D_6 + \mathcal{F}_2(1)\sqrt{\pi}}, \quad \forall z < \overline{w}_6$$

We conclude that inequality (127) is equivalent to $\operatorname{erf}^{-1}(P_6) < \overline{w}_6$ (equivalent to inequality (123) for q_0). Then, under this assumption, we get that there exists a unique pair (λ, k_2^*) to the system (98)-(99). □

Theorem 4.7. ***(Case 7:** λ, ℓ**)** *Let us define*

$$R_7 = \frac{\sqrt{k_1^* \rho_1 c_1^*}}{\sqrt{k_2^* \rho_2 c_2^*}} m^*, \quad Q_7 = \frac{\sqrt{k_2^* \rho_2 c_1^*}}{\sqrt{k_1^* \rho_1 c_2^*}}, \quad D_7 = \frac{\rho_1 \sqrt{\pi}}{c_2^* \rho_2 (B - u^*)}, \quad P_7 = \frac{\sqrt{k_1^* \rho_1 c_1^*}}{q_0 \sqrt{\pi}} (B - u^*) m^*.$$

Assume that

$$q_0 > \frac{(B - u^*) m^* \sqrt{k_1^* \rho_1 c_1^*}}{\operatorname{erf}(\overline{z}_7)\sqrt{\pi}}, \tag{128}$$

where $\overline{z}_7$ is the unique solution to the equation

$$j(z) = 0, \qquad z > 0, \tag{129}$$

with

$$j(z) = R_7 \mathcal{M}_1(z) - \mathcal{F}_2(1) \mathcal{M}_2\left(\frac{z}{Q_7}\right). \tag{130}$$

Then there is a unique solution (λ, ℓ) to the system (98)-(99) given by

Table 1
Summary of cases.

Cases	Restriction	Solution
Case 1: $\lambda,\ \rho_1$	$q_0 > \left(1+\frac{\beta_2}{1+p_2}\right)\frac{\sqrt{k_2^*\rho_2 c_2^*}}{\sqrt{\pi}}(B-u^*)$	$\lambda = \frac{\overline{z}_1 \operatorname{erf}^{-1}(\overline{z}_1)}{Q_1 P_1}, \qquad \rho_1 = \frac{\overline{z}_1^2}{P_1^2}$ where $\overline{z}_1$ is the unique solution to eq. (102)
Case 2: $\lambda,\ \rho_2$	$q_0 > \frac{\sqrt{k_1^*\rho_1 c_1^*}}{\sqrt{\pi}\operatorname{erf}(\overline{z}_2)}(B-u^*)m^*$ where $\overline{z}_2$ is the unique solution to eq. (104)	$\rho_2 = \left(\frac{\operatorname{erf}^{-1}(P_2)}{\lambda Q_2}\right)^2$ λ is one positive solution to eq. (107)
Case 3: $\lambda,\ c_1^*$	$q_0 > q_0^*$ where $q_0^* > 0$ is the unique solution to eq. (111)	$\lambda = \frac{P_3}{Q_3}\frac{\operatorname{erf}^{-1}(\overline{z}_3)}{\overline{z}_3}, \qquad c_1^* = \frac{\overline{z}_3^2}{P_3^2}$ where $\overline{z}_3$ is the unique solution to eq. (113)
Case 4: $\lambda,\ c_2^*$	$q_0 > \frac{\sqrt{k_1^*\rho_1 c_1^*}(B-u^*)m^*}{\operatorname{erf}(\overline{z}_4)\sqrt{\pi}}$ where $\overline{z}_4$ is the unique solution to eq. (115)	$c_2^* = \left(\frac{\lambda Q_4}{\operatorname{erf}^{-1}(P_4)}\right)^2$ λ is the unique positive solution to eq. (118)
Case 5: $\lambda,\ k_1^*$	$q_0 > \left(1+\frac{\beta_2}{1+p_2}\right)(B-u^*)\frac{\sqrt{k_2^* c_2^*\rho_2}}{\sqrt{\pi}}$	$\lambda = \frac{\overline{z}_5 \operatorname{erf}^{-1}(\overline{z}_5)}{Q_5 P_5}, \qquad k_1^* = \frac{\overline{z}_5^2}{P_5^2}$ where $\overline{z}_5$ is the unique solution to eq. (122)
Case 6: $\lambda,\ k_2^*$	$q_0 > \frac{\sqrt{k_1^*\rho_1 c_1^*}(B-u^*)m^*}{\operatorname{erf}(\overline{w}_6)\sqrt{\pi}}$ where $\overline{w}_6$ is the unique solution to eq. (124)	$k_2^* = \left(\frac{\operatorname{erf}^{-1}(P_6)}{\lambda Q_6}\right)^2$ λ is the unique positive solution to eq. (126)
Case 7: $\lambda,\ \ell$	$q_0 > \frac{(B-u^*)m^*\sqrt{k_1^*\rho_1 c_1^*}}{\operatorname{erf}(\overline{z}_7)\sqrt{\pi}}$ where $\overline{z}_7$ is the unique solution to eq. (129)	$\lambda = \frac{\operatorname{erf}^{-1}(P_7)}{Q_7}$, $\ell = \frac{\left[R_7\mathcal{M}_1(\operatorname{erf}^{-1}(P_7)) - \mathcal{F}_2(1)\mathcal{M}_2\left(\frac{\operatorname{erf}^{-1}(P_7)}{Q_7}\right)\right]Q_7}{D_7 \operatorname{erf}^{-1}(P_7)}$

$$\lambda = \frac{\operatorname{erf}^{-1}(P_7)}{Q_7}, \qquad \ell = \left[R_7\mathcal{M}_1(\operatorname{erf}^{-1}(P_7)) - \mathcal{F}_2(1)\mathcal{M}_2\left(\frac{\operatorname{erf}^{-1}(P_7)}{Q_7}\right)\right]\frac{Q_7}{D_7 \operatorname{erf}^{-1}(P_7)}. \tag{131}$$

Proof. If we assume $P_7 < 1$, the equation (99) leads λ to be given by (131). From (98) it follows that ℓ is defined by (131) where in order that $\ell > 0$ it is necessary that $j(\operatorname{erf}^{-1}(P_7)) > 0$ where j is given by formula (130). Taking into account that j is a decreasing function, $j(0) = +\infty$ and $j(+\infty) = -\infty$, we get there exists a unique $\overline{z}_7$ such that $j(\overline{z}_7) = 0$, $j(z) > 0$ for $0 < z < \overline{z}_7$ and $j(z) < 0$ for $z > \overline{z}_7$. Therefore the inequality $j(\operatorname{erf}^{-1}(P_7)) > 0$ is equivalent to $\operatorname{erf}^{-1}(P_7) < \overline{z}_7$. As a consequence, if we assume $P_7 < \min\{1, \operatorname{erf}(\overline{z}_7)\} = \operatorname{erf}(\overline{z}_7)$ we can guarantee that there exists a unique pair (λ, ℓ) solution to the system (98)-(99). □

5. Conclusions

Two different two-phase free boundary Stefan problems in an angular domain with temperature-dependent thermal coefficients were considered. Analytical similarity solutions were obtained imposing a Dirichlet or Neumann type condition at the boundary $x = rs(t)$ where $0 < r = 1 - \frac{\rho_2}{\rho_1} < 1$ and $x = s(t)$ is the position of the free boundary.

In addition, some formulas were obtained in the determination of one unknown thermal coefficient among $\{\rho_1, \rho_2, c_1^*, c_2^*, k_1^*, k_2^*, \ell\}$ in the over specified problem that consists in adding a Neumann condition to the problem with a Dirichlet one.

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