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Relationship among solutions for three-phase change problems with Robin, Dirichlet and Neumann boundary conditions

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ABSTRACT

This study presents a novel approach to the melting process in a three-phase Stefan problem, applied to a semi-infinite material with a convective boundary condition at the fixed face. By using a similarity-type transformation, the problem is simplified and solved explicitly, yielding a unique solution. Additionally, a computational example is provided to illustrate the temperature distribution and the evolution of the free boundaries in a melting semi-infinite material with an intermediate zone. The principal key contribution lies in revealing new equivalences among solutions to three distinct three-phase Stefan problems, each with different boundary conditions (Robin, Dirichlet and Neumann). These equivalences are established under specific data relationships, providing fresh insights into phase change behavior across varying boundary conditions. This research significantly advances the understanding of multi-phase heat transfer problems.

1. Introduction

Stefan problems are a significant area of study because they occur in various important engineering and industrial contexts. They are crucial for understanding phase transition phenomena, especially in scenarios involving heat transfer and processes of solidification or melting. The goal of Stefan problems is to describe the liquid and solid phases of a material undergoing a phase change and to identify the location of the sharp interface that separates these phases, known as the free boundaries. Transient heat conduction issues that include one or more phase changes are found in a number of practical areas. Applications of Stefan-type problems include the solidification of binary alloys [1–4], continuous casting of steel [5] and cryopreservation of cells [6]. So many applications of phase-change processes can be seen in the books [7–14].

In the classical formulation of Stefan's problem, several assumptions are made regarding the physical factors influencing the phase change process to simplify the model. One such assumption is that the thermal properties of the material are treated as constant positive values. However, thermodynamic considerations suggest the necessity of addressing Stefan's problems with variable thermal coefficients. For instance, in [15], the authors solve a Stefan problem involving a moving phase change material, size-dependent thermal conductivity and a periodic time-dependent boundary condition, utilizing the finite difference scheme. The results are compared with the exact solution, showing excellent agreement and the dependence of the moving boundary and temperature distribution on various parameters is thoroughly analyzed. In [16], a phase change problem in a one-dimensional domain with time-dependent speed is studied. A Dirichlet boundary condition is applied, with thermal properties assumed to be linear functions of temperature. An exact solution is derived for specific parameters, while an approximate solution is provided for other cases. The results demonstrate good accuracy and examine how different parameters influence both the temperature profile and the movement of the phase front. In [17], a numerical study employing the finite difference method is presented to analyze a moving boundary in phase change. The model incorporates temperature-dependent thermal coefficients and a mixed convective boundary condition. The numerical solution is compared with the exact solution to ensure its accuracy. Additionally, the study explores the stability and consistency of the solution, as well as the impact of various parameters on the temperature distribution and the evolution of the interface.

During the solidification or melting process, the material can be divided into three distinct regions: a solid region, an intermediate zone (called mushy zone in [18,19]) and a liquid region. In the case of polymorphous materials such as metallic iron and silica, multiple crystalline forms exist in the solid phase, resulting in several free boundaries between different phases. For example, metallic iron has three main crystalline forms, while silica exists in several distinct forms like quartz, tridymite and cristobalite under high pressure. When these

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polymorphous materials freeze or melt, various phases are separated by multiple moving interfaces [18,20].

Under certain boundary conditions, it is possible to find similarity type solutions to multiphase Stefan problems. In particular, in [21,22], it was considered a n-phase Stefan problem for a semi-infinite material imposing a constant temperature at the fixed face. A similar study was carried out in [23] with a Neumann type condition at the fixed face. A multiphase implicit Stefan problem was studied in [24] for a one-dimensional non-Darcy flow in a semi-infinite porous media. However, the existence of analytical solutions to phase-change problems is challenging to ascertain due to the inherent non-linearity of these issues. For the analysis of more complex scenarios, numerical methods seem to be highly efficient. In [18], a hybrid numerical method is employed that combines the Laplace transform technique, control-volume formulation and Taylor series approximations. In [19], an approximate analytical solution is derived for a non-linear multiphase Stefan problem and the accuracy of this approximate method is assessed by comparison with the available exact solution.

Heat transfer in three-phase systems presents significant challenges that are critical for a variety of applications. For instance, in [25], researchers developed an analytical solution for the time-dependent heat transfer equation that accounts for phase change. This solution enables a new numerical algorithm to analyze temperature and heat flux variations in a three-layer building wall under transient ambient conditions. Additionally, [26] examines a numerical simulation of a triplex-tube thermal energy system that combines multiple phase change materials with porous metal foam. Another relevant application is presented in [27], where an analytical solution for coupled heat and mass transfer during the freezing of high-water-content materials is developed.

Moreover, a new numerical method for modeling phase change problems involving three phases: solid, liquid and gas is presented in [28]. It focuses on simulating melting and solidification of phase change materials with variable density and thermophysical properties. The method accounts for free surface dynamics and density changes during the phase transition. It revisits the two-phase Stefan problem, which involves a density jump between phases and proposes a way to incorporate kinetic energy changes into the Stefan condition.

A specific case of a three-phase system is the three-phase Stefan problem that consists of the solidification of an alloy. An alloy undergoes at least two phase changes when it solidifies: one when the temperature falls below the liquid temperature and another when it falls below the solid temperature. In contrast to the case of a pure metal, there are now two free boundaries corresponding to the liquid and solid temperatures.

This paper introduces a groundbreaking approach to analyzing the melting process in a three-phase Stefan problem, applied to a semi-infinite material subject to a convective boundary condition at the fixed face. Through the application of a similarity-type transformation, the complex problem is reduced to a more manageable form, enabling the derivation of an explicit solution. The primary contribution of this work lies in uncovering novel equivalences between the solutions of three distinct three-phase Stefan problems, each governed by different boundary conditions: Robin, Dirichlet and Neumann. These equivalences are established under a specific set of relationships between the problem data, offering a deeper and more comprehensive understanding of the behavior of phase change processes under varying boundary conditions. By providing new insights into how different boundary conditions influence multi-phase systems, this research contributes significantly to advancing the field of heat transfer and the study of phase change phenomena, laying the groundwork for future studies and practical applications in materials science and engineering.

The equivalence between Stefan problems with convective, Dirichlet and Neumann boundary conditions facilitates the analysis of heat conduction by allowing the transformation of one condition into another depending on the context. In a problem with convective boundary conditions, which describes heat transfer through a moving fluid, it is possible to equate it to a Dirichlet problem if the fluid temperature and the convection coefficient are known. Similarly, a Dirichlet problem, where the temperature is specified at the boundary, can be transformed into a Neumann problem, which defines the heat flux, by using the relationship between temperature and the rate of heat flux. These equivalences allow for more efficient modeling of systems with complex boundary conditions. A practical example [29] is the monitoring of soil surface temperature, which, although it can be obtained through long-term observations, is affected by factors such as surface cover, animal activities and extreme weather, making it difficult to obtain accurate data for calculating the soil temperature field. The surface temperature is primarily influenced by radiation and heat convection. To overcome these challenges, analytical solutions are applied to solve heat conduction problems with combined boundary conditions, transforming, for instance, periodic heat flux and convective boundary conditions into a simpler form, such as Dirichlet boundary conditions. These equivalences simplify the analysis and enhance the understanding of the thermal behavior of the soil, improving the accuracy of thermal calculations in various practical applications.

The aim of this work is twofold. First, the existence and uniqueness of a similarity-type solution to a three-phase melting Stefan problem is established, specifically under a Robin type boundary condition at the fixed face $x = 0$. Second, the connections between this problem and those arising from the imposition of Dirichlet or Neumann boundary conditions at the fixed face, are explored.

The organization of this paper is as follows. Section 2 presents one-dimensional Stefan problems with different boundary conditions related to the melting of a semi-infinite material in the region $x \geq 0$, undergoing three-phase changes. Additionally, the existence and uniqueness of a similarity-type solution are proven by imposing a Robin-type condition at the fixed face $x = 0$. Furthermore, similarity-type solutions from the existing literature are retrieved for cases where Dirichlet and Neumann boundary conditions are applied at the fixed boundary $x = 0$. Computational examples of the posed problems are also provided in Section 2. Finally, these solutions are then used in Section 3 to establish a relationship among them.

2. Three-phase Stefan problems with different boundary conditions

In this section, the analysis of three Stefan problems involving three phases for the melting of a semi-infinite material $x \geq 0$ with an intermediate zone is focused on, with each problem characterized by different conditions at the fixed face $x = 0$. The aim is to determine the temperature

$$\Phi(x,t) = \begin{cases} \Phi_3(x,t) & \text{if} \quad 0 < x < y_2(t), \quad t > 0, \\ \Phi_2(x,t) & \text{if} \quad y_2(t) < x < y_1(t), \quad t > 0, \\ \Phi_1(x,t) & \text{if} \quad y_1(t) < x, \quad t > 0, \end{cases} \tag{2.1}$$

and the free boundaries $x = y_i(t)$, $i = 1, 2$, $t > 0$ that separates the three regions, that satisfy:

$$\frac{\partial \Phi_3}{\partial t} = \alpha_3 \frac{\partial^2 \Phi_3}{\partial x^2}, \qquad 0 < x < y_2(t), \quad t > 0, \tag{2.2}$$

$$\frac{\partial \Phi_2}{\partial t} = \alpha_2 \frac{\partial^2 \Phi_2}{\partial x^2}, \qquad y_2(t) < x < y_1(t), \quad t > 0, \tag{2.3}$$

$$\frac{\partial \Phi_1}{\partial t} = \alpha_1 \frac{\partial^2 \Phi_1}{\partial x^2}, \qquad x > y_1(t), \quad t > 0, \tag{2.4}$$

$$\Phi_3(y_2(t),t) = \Phi_2(y_2(t),t) = B, \qquad t > 0, \tag{2.5}$$

$$\Phi_2(y_1(t),t) = \Phi_1(y_1(t),t) = C, \qquad t > 0, \tag{2.6}$$

$$\Phi_1(x,0) = D, \qquad x > 0, \tag{2.7}$$

$$\Phi_1(+\infty,t) = D, \qquad t > 0, \tag{2.8}$$

$$k_2 \frac{\partial \Phi_2}{\partial x}(y_2(t),t) - k_3 \frac{\partial \Phi_3}{\partial x}(y_2(t),t) = \delta_2 \dot{y}_2(t), \qquad t > 0, \tag{2.9}$$

$$k_1 \frac{\partial \Phi_1}{\partial x}(y_1(t),t) - k_2 \frac{\partial \Phi_2}{\partial x}(y_1(t),t) = \delta_1 \dot{y}_1(t), \qquad t > 0, \tag{2.10}$$

$$y_1(0) = y_2(0) = 0, \tag{2.11}$$

where the positive constants $\alpha_i = \frac{k_i}{\rho c_i}$, k_i and c_i represent the thermal diffusivity, thermal conductivity and specific heat, respectively, for phase $i = 1,2,3$, with ρ being the common mass density. It is assumed throughout the paper that

$$\alpha_2 \geq \alpha_3. \tag{2.12}$$

The latent heat per unit volume used for passing from phase i to $i+1$ for $i = 1,2$ is $\delta_i = \rho \ell_i$ where $\ell_i > 0$ represents the latent heat per unit mass. The phase change temperatures B and C, and the initial temperature D verify the condition

$$B > C > D. \tag{2.13}$$

In the following subsections, different boundary conditions will be applied at the fixed face $x = 0$. The analysis begins by considering a convective boundary condition given by

$$k_3 \frac{\partial \Phi_3}{\partial x}(0,t) = \frac{h_0}{\sqrt{t}} \left(\Phi_3(0,t) - A_\infty \right), \qquad t > 0, \tag{2.14}$$

where $h_0 > 0$ is the coefficient that characterizes the heat transfer at the fixed face, and $A_\infty > B$ is the bulk temperature. In this case, the existence and uniqueness of a similarity-type solution are demonstrated.

Next, the similarity-type solution obtained by imposing a Dirichlet boundary condition at the fixed face $x = 0$, as described in [22], is presented, considering only three phases. This condition is given by

$$\Phi_3(0,t) = A, \qquad t > 0, \tag{2.15}$$

with $A > B$.

Finally, the similarity-type solution for a Neumann boundary condition from [23] is recovered for the three-phase Stefan problem with a heat flux imposed at the fixed face $x = 0$ of the form:

$$k_3 \frac{\partial \Phi_3}{\partial x}(0,t) = -\frac{q_0}{\sqrt{t}}, \qquad t > 0, \tag{2.16}$$

where $q_0 > 0$.

In Fig. 1, the schematic representation of the three-phase Stefan problems are provided. This figure illustrates the conditions to be solved, clearly depicting the initial setup and the distribution of the phases at time zero.

2.1. Existence and uniqueness of similarity-type solution by imposing a convective condition at the fixed face

A similarity-type solution is proposed to the problem (2.2)–(2.11) and (2.14), represented in the following manner:

$$v_3(x,t) = A_3 + B_3 \operatorname{erf}\left(\frac{x}{2\sqrt{\alpha_3 t}} \right), \qquad 0 < x < w_2(t), \quad t > 0, \tag{2.17}$$

$$v_2(x,t) = A_2 + B_2 \operatorname{erf}\left(\frac{x}{2\sqrt{\alpha_2 t}} \right), \qquad w_2(t) < x < w_1(t), \quad t > 0, \tag{2.18}$$

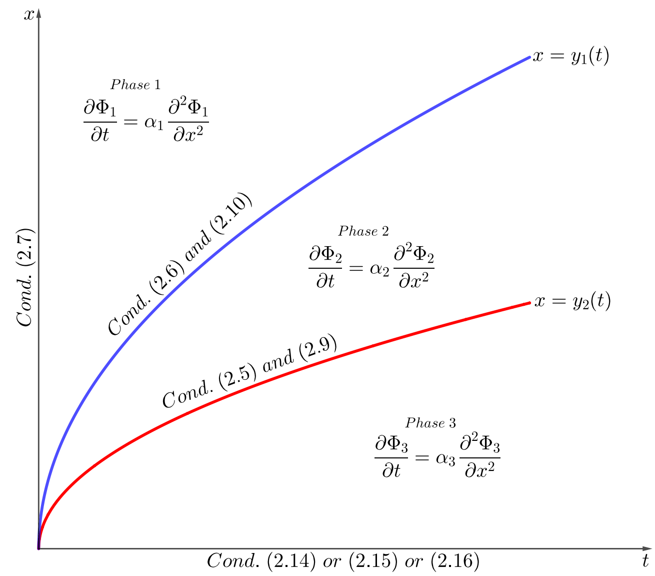

Fig. 1. Three-phase Stefan problems.

$$v_1(x,t) = A_1 + B_1 \operatorname{erf}\left(\frac{x}{2\sqrt{\alpha_1 t}} \right), \qquad x > w_1(t), \quad t > 0, \tag{2.19}$$

$$w_2(t) = 2\xi_2 \sqrt{\alpha_1 t}, \qquad t > 0, \tag{2.20}$$

$$w_1(t) = 2\xi_1 \sqrt{\alpha_1 t}, \qquad t > 0, \tag{2.21}$$

where A_i and B_i are unknown constants to be determined for $i = 1,2,3$, and ξ_1 and ξ_2 are positive dimensionless parameters that characterizes the free boundaries and must also be determined.

Condition (2.5) yields:

$$A_3 = B - B_3 \operatorname{erf}\left(\xi_2 \sqrt{\frac{\alpha_1}{\alpha_3}} \right), \tag{2.22}$$

and, taking into account (2.14) and (2.22), the expression for B_3 is obtained as:

$$B_3 = \frac{A_\infty - B}{\frac{k_3}{h_0\sqrt{\pi \alpha_3}} + \operatorname{erf}\left(\xi_2 \sqrt{\frac{\alpha_1}{\alpha_3}} \right)}. \tag{2.23}$$

Substituting (2.23) into (2.22) results in the expression for A_3:

$$A_3 = \frac{\frac{B k_3}{h_0\sqrt{\pi \alpha_3}} + A_\infty \operatorname{erf}\left(\xi_2 \sqrt{\frac{\alpha_1}{\alpha_3}} \right)}{\frac{k_3}{h_0\sqrt{\pi \alpha_3}} + \operatorname{erf}\left(\xi_2 \sqrt{\frac{\alpha_1}{\alpha_3}} \right)}. \tag{2.24}$$

Likewise, using (2.5) and (2.6), an expression for B_2 is derived:

$$B_2 = -\frac{B - C}{\operatorname{erf}\left(\xi_1 \sqrt{\frac{\alpha_1}{\alpha_2}} \right) - \operatorname{erf}\left(\xi_2 \sqrt{\frac{\alpha_1}{\alpha_2}} \right)}, \tag{2.25}$$

and substituting this into (2.6) gives the value of A_2:

$$A_2 = \frac{-C \operatorname{erf}\left(\xi_2 \sqrt{\frac{\alpha_1}{\alpha_2}} \right) + B \operatorname{erf}\left(\xi_1 \sqrt{\frac{\alpha_1}{\alpha_2}} \right)}{\operatorname{erf}\left(\xi_1 \sqrt{\frac{\alpha_1}{\alpha_2}} \right) - \operatorname{erf}\left(\xi_2 \sqrt{\frac{\alpha_1}{\alpha_2}} \right)}. \tag{2.26}$$

Condition (2.6) provides the relation for A_1:

$$A_1 = C - B_1 \operatorname{erf}\left(\xi_1 \right). \tag{2.27}$$

With (2.7) and using (2.27), the expression for B_1 is obtained as:

$$B_1 = -\frac{C - D}{\operatorname{erfc}(\xi_1)}, \tag{2.28}$$

and A_1 is consequently expressed as:

$$A_1 = \frac{C - D \operatorname{erf}(\xi_1)}{\operatorname{erfc}(\xi_1)}. \tag{2.29}$$

Therefore, based on the previous calculations, the expressions for the temperatures in the three phases become:

$$v_3(x,t) = \frac{\frac{Bk_3}{h_0\sqrt{\pi\alpha_3}} + A_\infty \operatorname{erf}\left(\xi_2\sqrt{\frac{\alpha_1}{\alpha_3}}\right) - (A_\infty - B)\operatorname{erf}\left(\frac{x}{2\sqrt{\alpha_3 t}}\right)}{\frac{k_3}{h_0\sqrt{\pi\alpha_3}} + \operatorname{erf}\left(\xi_2\sqrt{\frac{\alpha_1}{\alpha_3}}\right)}, \qquad 0 < x < w_2(t),\ t > 0, \tag{2.30}$$

$$v_2(x,t) = \frac{-C\operatorname{erf}\left(\xi_2\sqrt{\frac{\alpha_1}{\alpha_2}}\right) + B\operatorname{erf}\left(\xi_1\sqrt{\frac{\alpha_1}{\alpha_2}}\right) - (B-C)\operatorname{erf}\left(\frac{x}{2\sqrt{\alpha_2 t}}\right)}{\operatorname{erf}\left(\xi_1\sqrt{\frac{\alpha_1}{\alpha_2}}\right) - \operatorname{erf}\left(\xi_2\sqrt{\frac{\alpha_1}{\alpha_2}}\right)}, \qquad w_2(t) < x < w_1(t),\ t > 0, \tag{2.31}$$

$$v_1(x,t) = \frac{C\left(1 - \operatorname{erf}\left(\frac{x}{2\sqrt{\alpha_1 t}}\right)\right) + D\left(\operatorname{erf}\left(\frac{x}{2\sqrt{\alpha_1 t}}\right) - \operatorname{erf}(\xi_1)\right)}{\operatorname{erfc}(\xi_1)}, \qquad x > w_1(t),\ t > 0, \tag{2.32}$$

where erf and erfc denote the error function and the complementary error function, respectively, defined by

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}}\int_0^z \exp(-\eta^2)\,d\eta, \qquad \operatorname{erfc}(z) = 1 - \operatorname{erf}(z), \qquad z \geq 0.$$

The conditions (2.9) and (2.10) are satisfied if ξ_1 and ξ_2 fulfill the following equalities:

$$\xi_2 = \frac{k_3}{\delta_2\sqrt{\pi\alpha_1\alpha_3}}\frac{(A_\infty - B)\exp\left(-\xi_2^2\frac{\alpha_1}{\alpha_3}\right)}{\frac{k_3}{h_0\sqrt{\pi\alpha_3}} + \operatorname{erf}\left(\xi_2\sqrt{\frac{\alpha_1}{\alpha_3}}\right)} - \frac{k_2}{\delta_2\sqrt{\pi\alpha_1\alpha_2}}\frac{(B-C)\exp\left(-\xi_2^2\frac{\alpha_1}{\alpha_2}\right)}{\operatorname{erf}\left(\xi_1\sqrt{\frac{\alpha_1}{\alpha_2}}\right) - \operatorname{erf}\left(\xi_2\sqrt{\frac{\alpha_1}{\alpha_2}}\right)}, \tag{2.33}$$

$$\xi_1 = -\frac{k_1}{\delta_1\alpha_1\sqrt{\pi}}\frac{(C-D)\exp\left(-\xi_1^2\right)}{\operatorname{erfc}(\xi_1)} - \frac{k_2}{\delta_1\sqrt{\pi\alpha_1\alpha_2}}\frac{(B-C)\exp\left(-\xi_1^2\frac{\alpha_1}{\alpha_2}\right)}{\operatorname{erf}\left(\xi_1\sqrt{\frac{\alpha_1}{\alpha_2}}\right) - \operatorname{erf}\left(\xi_2\sqrt{\frac{\alpha_1}{\alpha_2}}\right)}. \tag{2.34}$$

Expression (2.34) can be rewritten as follows

$$\operatorname{erf}\left(\xi_2\sqrt{\frac{\alpha_1}{\alpha_2}}\right) = H(\xi_1), \tag{2.35}$$

where the real function H is defined by

$$H(z) = \operatorname{erf}\left(z\sqrt{\frac{\alpha_1}{\alpha_2}}\right) - \frac{\mathrm{Ste}_2}{\sqrt{\pi}}\frac{\ell_2}{\ell_1}\sqrt{\frac{k_2c_1}{k_1c_2}}\frac{\exp\left(-z^2\frac{\alpha_1}{\alpha_2}\right)}{\varphi(z)}, \qquad z \geq 0, \tag{2.36}$$

$$\varphi(z) = z + \frac{\mathrm{Ste}_1}{\sqrt{\pi}}\frac{\exp(-z^2)}{\operatorname{erfc}(z)}, \qquad z \geq 0, \tag{2.37}$$

and the Stefan numbers are defined by:

$$\mathrm{Ste}_1 = \frac{c_1(C-D)}{\ell_1} \qquad , \qquad \mathrm{Ste}_2 = \frac{c_2(B-C)}{\ell_2}. \tag{2.38}$$

Taking into account that H is an increasing function that satisfies

$$H(0) = -\frac{\mathrm{Ste}_2}{\mathrm{Ste}_1}\frac{\ell_2}{\ell_1}\sqrt{\frac{k_2c_1}{k_1c_2}} < 0, \qquad H(+\infty) = 1,$$

then, there exists a unique $z_0 > 0$ such that

$$z_0 = H^{-1}(0). \tag{2.39}$$

Therefore

$$\xi_2 = \sqrt{\frac{\alpha_1}{\alpha_2}}\operatorname{erf}^{-1}\left(H(\xi_1)\right), \quad \xi_1 > z_0. \tag{2.40}$$

Notice that

$$\operatorname{erf}\left(\xi_2\sqrt{\frac{\alpha_1}{\alpha_2}}\right) = H(\xi_1) = \operatorname{erf}\left(\xi_1\sqrt{\frac{\alpha_1}{\alpha_2}}\right) - \frac{\mathrm{Ste}_2}{\sqrt{\pi}}\frac{\ell_2}{\ell_1}\sqrt{\frac{k_2c_1}{k_1c_2}}\frac{\exp\left(-\xi_1^2\frac{\alpha_1}{\alpha_2}\right)}{\varphi(\xi_1)}$$
$$< \operatorname{erf}\left(\xi_1\sqrt{\frac{\alpha_1}{\alpha_2}}\right),$$

then $\xi_2 < \xi_1$.

Isolating $\left(\operatorname{erf}\left(\xi_1\sqrt{\frac{\alpha_1}{\alpha_2}}\right) - \operatorname{erf}\left(\xi_2\sqrt{\frac{\alpha_1}{\alpha_2}}\right)\right)^{-1}$ from (2.33) and (2.34) leads to the conclusion that ξ_1 must satisfy the following equation:

$$Q(z) = U(z), \qquad z > z_0, \tag{2.41}$$

where

$$Q(z) = \frac{\ell_1}{\ell_2}\varphi(z)\exp\left(z^2\frac{\alpha_1}{\alpha_2}\right), \qquad z \geq 0, \tag{2.42}$$

$$U(z) = T\left(\sqrt{\frac{\alpha_2}{\alpha_1}}\operatorname{erf}^{-1}(H(z))\right), \qquad z > z_0. \tag{2.43}$$

and

$$T(z) = \frac{\mathrm{Ste}_2}{\sqrt{\pi}c_2}\frac{A_\infty - B}{B-C}\sqrt{\frac{k_3c_1c_3}{k_1}}\frac{\exp\left(-z^2\alpha_1\left(\frac{1}{\alpha_3} - \frac{1}{\alpha_2}\right)\right)}{\frac{k_3}{h_0\sqrt{\pi\alpha_3}} + \operatorname{erf}\left(z\sqrt{\frac{\alpha_1}{\alpha_3}}\right)} - z\exp\left(z^2\frac{\alpha_1}{\alpha_2}\right), \qquad z > z_0. \tag{2.44}$$

Considering (2.12), it follows that U is a strictly decreasing function. Given that $U(z_0)$ is a positive constant, $U(+\infty) = -\infty$ and Q is a strictly increasing function such that $Q(0) = \frac{\ell_1}{\ell_2}\frac{\mathrm{Ste}_1}{\sqrt{\pi}}$ and $Q(+\infty) = +\infty$, it can be inferred that the solution ξ_1 to Eq. (2.41) is unique in $(z_0, +\infty)$ if and only if $U(z_0) > Q(z_0)$. Since $H(z_0) = 0$, this inequality is equivalent to the following condition on the parameters of the problem:

$$(A_\infty - B)h_0\sqrt{\frac{c_1c_3\alpha_3}{k_1k_3}} > \ell_1\varphi(z_0)\exp\left(z_0^2\frac{\alpha_1}{\alpha_2}\right). \tag{2.45}$$

Moreover, from (2.37), the previous inequality can be expressed in an equivalent way

$$(A_\infty - B)h_0\sqrt{\frac{c_3\alpha_3}{k_3}} > \sqrt{\frac{k_2c_2}{\pi}}\frac{B-C}{\operatorname{erf}\left(z_0\sqrt{\frac{\alpha_1}{\alpha_2}}\right)}, \tag{2.46}$$

or else

$$h_0 > \frac{B-C}{A_\infty - B}\sqrt{\frac{k_2k_3c_2}{\pi c_3\alpha_3}}\frac{1}{\operatorname{erf}\left(z_0\sqrt{\frac{\alpha_1}{\alpha_2}}\right)}. \tag{2.47}$$

The previous analysis leads to the following theorem

Theorem 2.1. *Assuming $h_0 > h_2$ with*

$$h_2 = \frac{B-C}{A_\infty - B}\sqrt{\frac{k_2k_3c_2}{\pi c_3\alpha_3}}\frac{1}{\operatorname{erf}\left(z_0\sqrt{\frac{\alpha_1}{\alpha_2}}\right)}, \tag{2.48}$$

there exists a unique similarity-type solution to the problem (2.2)–(2.11) and (2.14). The temperature v_i in each phase, for $i = 1, 2, 3$, is described by (2.30), (2.31) and (2.32), respectively. The free boundaries w_2 and w_1, given by (2.20) and (2.21), are characterized by the dimensionless parameters ξ_1 and ξ_2. The parameter ξ_2 is defined by (2.40), while ξ_1 is the unique solution to Eq. (2.41).

Remark 2.2. In [30], a two-phase Stefan problem with a convective condition at the fixed face was studied. It was shown that for a unique similarity-type solution to exist, the coefficient h_0 must satisfy the following inequality:

$$h_0 > h_1 := \frac{k_1}{\sqrt{\pi\alpha_1}} \cdot \frac{C-D}{A_\infty - C}.$$

In the case of three phases, it has been proven that $h_0 > h_2$, and it is easy to see that the following relation holds:

$$h_0 > h_2 > h_1.$$

Moreover, it can be stated that:

(i) If $0 < h_0 \leq h_1$ then the problem defined by (2.2)–(2.11) and (2.14) becomes a classical heat transfer problem for the initial solid phase.
(ii) If $h_1 < h_0 \leq h_2$ then the problem defined by (2.2)–(2.11) and (2.14) becomes a two-phase Stefan problem.
(iii) If $h_0 > h_2$ then the problem defined by (2.2)–(2.11) and (2.14) is a three-phase Stefan problem, whose unique similarity-type solution is given by Theorem 2.1.

Remark 2.3. This Stefan problem could represent the evaporation of a solid, where the phase change occurs at the interface between the solid and its surrounding environment. The model would capture the dynamics of the solid–liquid or solid-vapor interfaces, depending on the specific conditions and describe the evolution of the phase boundary over time. The Stefan problem framework is particularly well-suited to

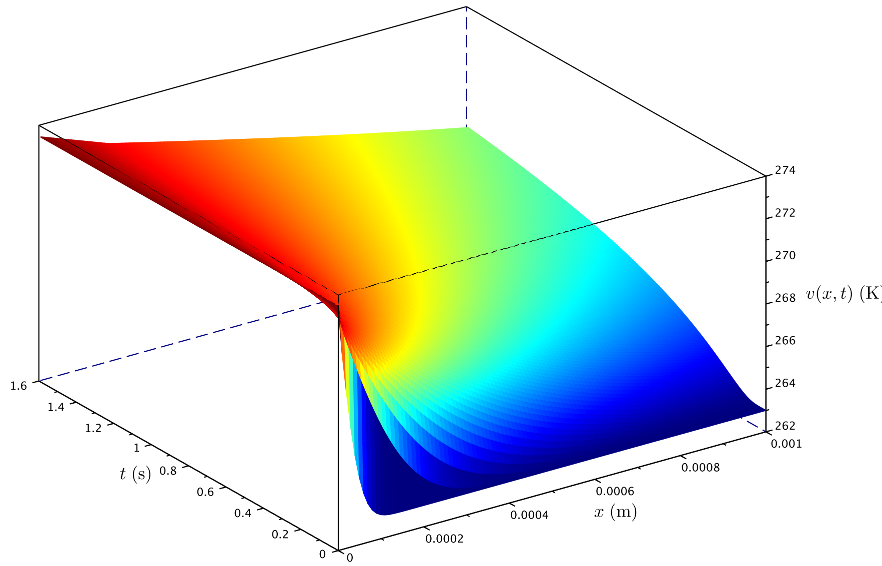

Fig. 2. Temperature distribution of the three-phase Stefan problem with a Robin type condition.

Table 1
Thermo-physical coefficients.

Thermal conductivity	k_i $i = 1,2,3$	2 W m^{-1}K^{-1}
Specific heat	c_i $i = 1,2,3$	2.5×10^6J kg^{-1}K^{-1}
Latent heat	ℓ_i $i = 1,2$	100×10^6 J kg^{-1}
Density	ρ	1 kg m^{-3}
Phase change temperature (phase 3 - phase 2)	B	273.5 K
Phase change temperature (phase 2 - phase 1)	C	272.3 K
Initial temperature	D	263 K

Table 2
Deduced parameters from Table 1.

Thermal diffusivity	α_i $i = 1,2,3$	8×10^{-7} m^2 s^{-1}
Latent heat per unit volume	δ_i $i = 1,2$	1×10^8 kg m^{-1} s^{-2}
Stefan number 1	Ste_1	0.2325
Stefan number 2	Ste_2	0.03
–	z_0	0.0704469

this type of process, as it accounts for the heat transfer, mass transfer and the latent heat involved in the phase transition from solid to vapor, providing a mathematical basis for understanding the rate and progression of the evaporation process. In this case, the Stefan number Ste_2 given by (2.38), would represent the Jacob number Ja, i.e

$$\mathrm{Ja} = \mathrm{Ste}_2 = \frac{c_2(B-C)}{\ell_2}. \tag{2.49}$$

Example 1. A computational example to show the applicability of the three-phase Stefan problem imposing a convective condition at the fixed face analyzed in Theorem 2.1 is presented. This example involves a half-space melting with an intermediate region and assuming constant thermal properties for the material, as taken from [18,19], and shown in Table 1.

Considering the data provided in Table 1 and the definitions of thermal diffusivity, latent heat per unit volume, Stefan numbers and z_0 as given by (2.39), the parameters listed in Table 2 are derived.

A solid is initially at a uniform temperature 263 K which is lower than the phase-change temperature (272.3 K $< v(x,t) <$ 273.5 K). At time $t = 0$, the boundary surface temperature at $x = 0$ is upper to a temperature 273.5 K and maintained at that temperature for all times $t > 0$. As a result there are three phases, i.e., solid, intermediate (mushy) and liquid phases, existing during the melting process. In the present example, the liquid, mushy and solid phases are respectively designated by phase 1, 2 and 3.

Given the bulk temperature $A_\infty = 274$ K and the data provided in Tables 1 and 2, it follows that $h_2 = 38152.451$ kg K^{-1}s$^{-5/2}$. Considering the assumptions of Theorem 2.1, $h_0 = 38153.451$ kg K^{-1}s$^{-5/2}$ is selected. Fig. 2 illustrates the temperature distribution $v = v(x,t)$ described by (2.30)–(2.32) while Fig. 3 presents a color map of the temperature.

The free boundaries, $x = w_2(t)$ and $x = w_1(t)$, as defined by Eqs. (2.20) and (2.21), are characterized by the parameters $\xi_2 = 6\times10^{-7}$ and $\xi_1 = 0.0704473$, respectively. Due to the very small coefficient ξ_2 associated with the free boundary $x = w_2(t)$, the initial phase change occurs almost instantaneously. To better illustrate this behavior, Table 3 shows the position and the velocity of the free boundaries, $x = w_1(t)$ and $x = w_2(t)$, at several time instants t between 0 and 1.5 s.

2.2. Similarity-type solutions with a temperature and a flux condition at the fixed face

The aim of this section is to connect the previously studied three-phase Stefan problem with the problems that involve Dirichlet and Neumann boundary conditions. These problems have been previously studied in the literature for the more general case of n phases. Below, existence and uniqueness results for $n = 3$ are presented.

From [22], the existence and uniqueness of a similarity-type solution to the three-phase Stefan problem with a Dirichlet-type condition at the fixed face $x = 0$ can be ensure, as shown in the following theorem.

Theorem 2.4. *Under the assumption*

$$A > B, \tag{2.50}$$

the problem defined by (2.2)–(2.11) and (2.15) has a unique similarity-type solution given by

$$u_3(x,t) = A\,\frac{\operatorname{erf}\left(\mu_2\sqrt{\frac{\alpha_1}{\alpha_3}}\right)-\operatorname{erf}\left(\frac{x}{2\sqrt{\alpha_3 t}}\right)}{\operatorname{erf}\left(\mu_2\sqrt{\frac{\alpha_1}{\alpha_3}}\right)} + B\,\frac{\operatorname{erf}\left(\frac{x}{2\sqrt{\alpha_3 t}}\right)}{\operatorname{erf}\left(\mu_2\sqrt{\frac{\alpha_1}{\alpha_3}}\right)}, \qquad 0 < x < r_2(t),\ t > 0, \tag{2.51}$$

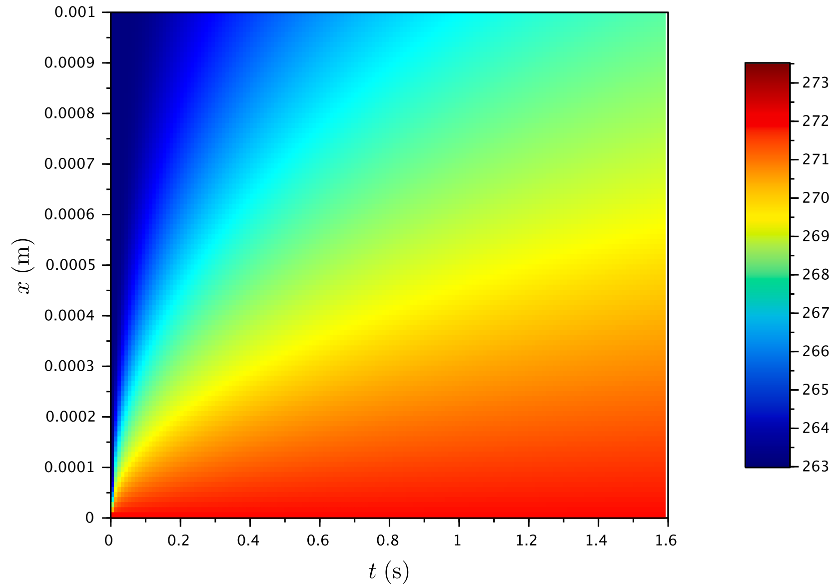

Fig. 3. Color map of the temperature of the three-phase Stefan problem with a Robin type condition.

Table 3
Position and velocity of the interfaces in the three-phase Stefan problem with a Robin type condition.

Time t	Position $x = w_1(t)$	Velocity $\dot{w}_1(t)$	Position $x = w_2(t)$	Velocity $\dot{w}_2(t)$
0.0 s	0 m	–	0 m	–
0.1 s	3.99×10^{-5} m	1.993×10^{-4} ms^{-1}	3.462×10^{-10} m	1.731×10^{-9} ms^{-1}
0.3 s	6.9×10^{-5} m	1.15×10^{-4} ms^{-1}	5.996×10^{-10} m	9.993×10^{-10} ms^{-1}
0.5 s	8.91×10^{-5} m	8.91×10^{-5} ms^{-1}	7.74×10^{-10} m	7.74×10^{-10} ms^{-1}
0.7 s	1.054×10^{-4} m	7.53×10^{-5} ms^{-1}	9.159×10^{-10} m	6.542×10^{-10} ms^{-1}
0.9 s	1.196×10^{-4} m	6.64×10^{-5} ms^{-1}	1.038×10^{-9} m	5.769×10^{-10} ms^{-1}
1.1 s	1.322×10^{-4} m	6.01×10^{-5} ms^{-1}	1.148×10^{-9} m	5.219×10^{-10} ms^{-1}
1.3 s	1.437×10^{-4} m	5.53×10^{-5} ms^{-1}	1.248×10^{-9} m	4.8×10^{-10} ms^{-1}
1.5 s	1.543×10^{-4} m	5.14×10^{-5} ms^{-1}	1.341×10^{-9} m	4.469×10^{-10} ms^{-1}

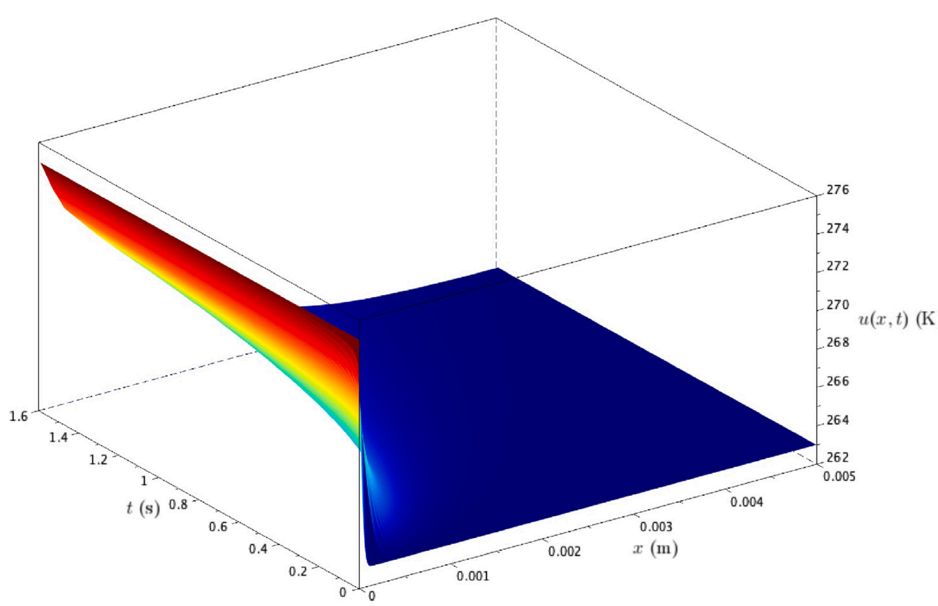

Fig. 4. Temperature distribution of the three-phase Stefan problem with a Dirichlet type condition.

$$u_2(x,t) = \frac{(C-B)\,\mathrm{erf}\left(\frac{x}{2\sqrt{\alpha_2 t}}\right) - C\,\mathrm{erf}\left(\mu_2\sqrt{\frac{\alpha_1}{\alpha_2}}\right) + B\,\mathrm{erf}\left(\mu_1\sqrt{\frac{\alpha_1}{\alpha_2}}\right)}{\mathrm{erf}\left(\mu_1\sqrt{\frac{\alpha_1}{\alpha_2}}\right) - \mathrm{erf}\left(\mu_2\sqrt{\frac{\alpha_1}{\alpha_2}}\right)}, \quad r_2(t) < x < r_1(t),\ t > 0, \tag{2.52}$$

$$u_1(x,t) = D + (C-D)\frac{\mathrm{erfc}\left(\frac{x}{2\sqrt{\alpha_1 t}}\right)}{\mathrm{erfc}(\mu_1)}, \qquad x > r_1(t),\ t > 0, \tag{2.53}$$

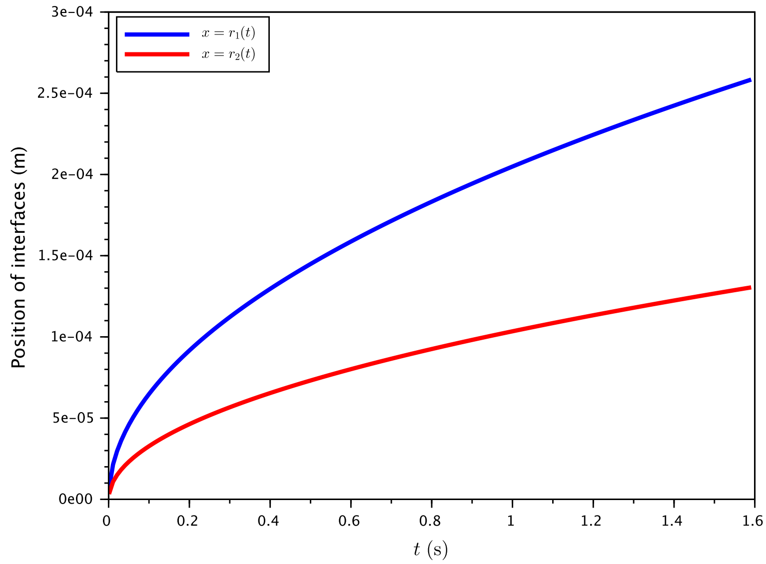

Fig. 5. Free boundaries of the three-phase Stefan problem with a Dirichlet type condition.

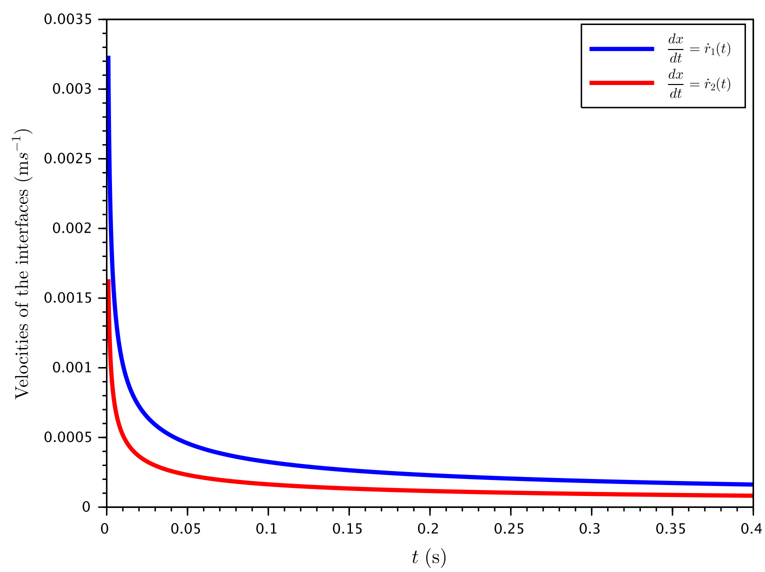

Fig. 6. Velocities of the free boundaries of the three-phase Stefan problem with a Dirichlet type condition.

$$r_2(t) = 2\mu_2\sqrt{\alpha_1 t}, \qquad t > 0, \tag{2.54}$$

$$r_1(t) = 2\mu_1\sqrt{\alpha_1 t}, \qquad t > 0, \tag{2.55}$$

where

$$\mu_2 = \sqrt{\frac{\alpha_2}{\alpha_1}}\,\mathrm{erf}^{-1}\left(H(\mu_1)\right), \tag{2.56}$$

and μ_1 *is the unique solution to*

$$Q(z) = V\left(\sqrt{\frac{\alpha_2}{\alpha_1}}\,\mathrm{erf}^{-1}\left(H(z)\right)\right), \qquad z > z_0, \tag{2.57}$$

where $z_0 = H^{-1}(0)$ *with* H *defined by* (2.36), Q *is given by* (2.42) *and*

$$V(z) = \frac{A-B}{\ell_2}\sqrt{\frac{c_1 c_3 k_3}{\pi k_1}}\,\frac{\exp\left(-z^2\left(\frac{\alpha_1}{\alpha_3}-\frac{\alpha_1}{\alpha_2}\right)\right)}{\mathrm{erf}\left(z\sqrt{\frac{\alpha_1}{\alpha_3}}\right)} - z\exp\left(z^2\frac{\alpha_1}{\alpha_2}\right), \qquad z > 0. \tag{2.58}$$

Example 2. This computational example involves a half-space melting three-phase Stefan problem, where a temperature condition is applied at the fixed face. Based on the assumptions outlined in Theorem 2.4 and the data provided in Tables 1 and 2, a Dirichlet boundary condition of $A = 275$ K is imposed at the fixed face located at $x = 0$.

Fig. 4 illustrates the temperature distribution $u = u(x,t)$ described by (2.51)–(2.53) while Fig. 5 shows the free boundaries $x = r_2(t)$ and $x = r_1(t)$, as given by (2.54) and (2.55), that are characterized by the parameters $\mu_2 = 0.0578268$ and $\mu_1 = 0.1145284$, respectively. Additionally, Fig. 6 shows the velocity of the interfaces.

Based on [23], a similarity-type solution to the three-phase Stefan problem with a Neumann boundary condition at the fixed face $x = 0$ can be represented as:

Theorem 2.5. *If the following inequality holds*

$$q_0 > q_2 := \frac{k_2(B-C)}{\sqrt{\alpha_2\pi}\,\mathrm{erf}\left(z_0\sqrt{\frac{\alpha_1}{\alpha_2}}\right)}, \tag{2.59}$$

where z_0 is defined by (2.39), then the problem (2.2)–(2.11) and (2.16) has a unique similarity-type solution given by

$$\theta_3(x,t) = B + \frac{q_0\sqrt{\pi\alpha_3}}{k_3}\left(\operatorname{erf}\left(\lambda_2\sqrt{\frac{\alpha_1}{\alpha_3}}\right) - \operatorname{erf}\left(\frac{x}{2\sqrt{\alpha_3 t}}\right)\right), \qquad 0 < x < s_2(t),\ t > 0, \tag{2.60}$$

$$\theta_2(x,t) = C + (B-C)\frac{\operatorname{erf}\left(\lambda_1\sqrt{\frac{\alpha_1}{\alpha_2}}\right) - \operatorname{erf}\left(\frac{x}{2\sqrt{\alpha_2 t}}\right)}{\operatorname{erf}\left(\lambda_1\sqrt{\frac{\alpha_1}{\alpha_2}}\right) - \operatorname{erf}\left(\lambda_2\sqrt{\frac{\alpha_1}{\alpha_2}}\right)}, \qquad s_2(t) < x < s_1(t),\ t > 0, \tag{2.61}$$

$$\theta_1(x,t) = D + (C-D)\frac{\operatorname{erfc}\left(\frac{x}{2\sqrt{\alpha_1 t}}\right)}{\operatorname{erfc}(\lambda_1)}, \qquad x > s_1(t),\ t > 0. \tag{2.62}$$

$$s_2(t) = 2\lambda_2\sqrt{\alpha_1 t}, \qquad t > 0, \tag{2.63}$$

$$s_1(t) = 2\lambda_1\sqrt{\alpha_1 t}, \qquad t > 0, \tag{2.64}$$

where

$$\lambda_2 = \sqrt{\frac{\alpha_2}{\alpha_1}}\operatorname{erf}^{-1}\left(H(\lambda_1)\right), \tag{2.65}$$

and λ_1 is the unique solution to

$$Q(z) = P\left(\sqrt{\frac{\alpha_2}{\alpha_1}}\operatorname{erf}^{-1}(H(z))\right), \qquad z > z_0, \tag{2.66}$$

with Q given by (2.42) and

$$P(z) = \exp\left(z^2\frac{\alpha_1}{\alpha_2}\right)\left[-z + \frac{q_0}{\ell_2}\sqrt{\frac{c_1}{\rho k_1}}\exp\left(-z^2\frac{\alpha_1}{\alpha_3}\right)\right], \qquad z \geq 0. \tag{2.67}$$

Remark 2.6. In [31], a two-phase Stefan problem with a Neumann condition at the fixed face was studied. It was proved that for a unique similarity-type solution to exist, q_0 must satisfy the following inequality $q_0 > q_1 := \frac{k_1(C-D)}{\sqrt{\pi\alpha_1}}$. In the case of three phases, it has been proven that $q_0 > q_2$ and it is easy to see that the following relation is satisfied:

$$q_0 > q_2 > q_1.$$

Furthermore, it can be asserted that:

1. If $0 < q_0 \leq q_1$ then the problem defined by (2.2)–(2.11) and (2.16) reduces to a classical heat transfer problem for the initial solid phase.
2. If $q_1 < q_0 \leq q_2$ then the problem defined by (2.2)–(2.11) and (2.16) becomes a two-phase Stefan problem.
3. If $q_0 > q_2$ then the problem defined by (2.2)–(2.11) and (2.16) is a three-phase Stefan problem whose unique similarity-type solution is given in Theorem 2.5.

Example 3. The problem involves a half-space melting three-phase Stefan problem, where a flux condition is applied at the fixed face $x = 0$. This computational example uses the data from Tables 1 and 2 and, based on Theorem 2.5, the value of q_2 is computed as $19076.225\ \mathrm{kg\,m^2\,K\,s^{-7/2}}$. Following the hypothesis in Theorem 2.5, it is assumed that $q_0 = 19078\ \mathrm{kg\,m^2\,K\,s^{-7/2}}$. Fig. 7 shows the temperature distribution $\theta = \theta(x,t)$ as described by (2.60)–(2.62).

The free boundaries, $x = s_2(t)$ and $x = s_1(t)$, defined by Eqs. (2.63) and (2.64), are characterized by the parameters $\lambda_2 = 1.07 \times 10^{-5}$ and $\lambda_1 = 0.0704546$, respectively. As in Example 1, due to the very small coefficient λ_2 associated with the free boundary $x = s_2(t)$, the initial phase change occurs nearly instantaneously. To better illustrate this behavior, Table 4 presents the positions and velocities of the free boundaries, $x = s_1(t)$ and $x = s_2(t)$, at various time instants t between 0 and 1 s.

3. Relationship among problems

From this point forward, the problem governed by (2.2)–(2.11) and (2.14) is denoted as **(P1)**. If the Robin boundary condition (2.14) is replaced by a temperature boundary condition, the problem defined by (2.2)–(2.11) and (2.15) arises, denoted as **(P2)**. Similarly, problem **(P3)** is defined by (2.2)–(2.11) and (2.16), which results from replacing condition (2.14) with a Neumann boundary condition.

Having established the three problems, the equivalence among them will now be demonstrated. Equivalence refers to the condition in which, if the data of both problems satisfy a specific relationship, they will yield the same solution. This will be shown through a detailed analysis of the boundary conditions and their implications for the solutions of the respective problems. The results presented in this section are theoretical and applicable to all phase-change materials, with the possibility of experimental verification.

3.1. Equivalence between problems (P1) and (P2)

Linking a Stefan problem characterized by a temperature boundary condition to the one with a convective boundary condition is essential for a deeper comprehension of heat transfer processes. Exploring how these conditions interact offers a broader perspective on the thermal dynamics of the system.

In the subsequent theorem, the necessary relationships between the parameters of both problems are defined to guarantee their equivalence.

Theorem 3.1.

(a) *Let h_0 and A_∞ with $h_0 > h_2$ be the given constants of the convective condition in the problem **(P1)** where h_2 is defined by (2.48). If the following inequality holds:*

$$\frac{\frac{Bk_3}{h_0\sqrt{\pi\alpha_3}} + A_\infty\operatorname{erf}\left(\xi_2\sqrt{\frac{\alpha_1}{\alpha_3}}\right)}{\frac{k_3}{h_0\sqrt{\pi\alpha_3}} + \operatorname{erf}\left(\xi_2\sqrt{\frac{\alpha_1}{\alpha_3}}\right)} > B, \tag{3.1}$$

*where ξ_2 is given by (2.40), then the solution to problem **(P2)** with*

$$A = \frac{\frac{Bk_3}{h_0\sqrt{\pi\alpha_3}} + A_\infty\operatorname{erf}\left(\xi_2\sqrt{\frac{\alpha_1}{\alpha_3}}\right)}{\frac{k_3}{h_0\sqrt{\pi\alpha_3}} + \operatorname{erf}\left(\xi_2\sqrt{\frac{\alpha_1}{\alpha_3}}\right)}, \tag{3.2}$$

*coincides with the solution to problem **(P1)**.*

(b) *Let $A > B$ be the data of the temperature boundary condition of the problem **(P2)**. If the following inequality holds:*

$$\frac{A-B}{(A_\infty - A)\operatorname{erf}\left(\mu_2\sqrt{\frac{\alpha_1}{\alpha_3}}\right)} > \frac{B-C}{A_\infty - B}\sqrt{\frac{k_2c_2}{k_3c_3}}\frac{1}{\operatorname{erf}\left(z_0\sqrt{\frac{\alpha_1}{\alpha_2}}\right)}, \tag{3.3}$$

*where μ_2 is given by (2.56) and $A_\infty > A$, then the solution to problem **(P1)** with*

$$h_0 = \frac{k_3}{\sqrt{\alpha_3\pi}}\frac{A-B}{A_\infty - A}\frac{1}{\operatorname{erf}\left(\mu_2\sqrt{\frac{\alpha_1}{\alpha_3}}\right)}, \tag{3.4}$$

*coincides with the solution to problem **(P2)**.*

Proof.

(a) Based on the solution to the problem **(P1)**, which involves the temperatures v_3, v_2 and v_1 and the free boundaries w_2 and w_1 established by Theorem 2.1, it is possible to determine the temperature at $x = 0$:

$$v_3(0,t) = \frac{\frac{Bk_3}{h_0\sqrt{\pi\alpha_3}} + A_\infty\operatorname{erf}\left(\xi_2\sqrt{\frac{\alpha_1}{\alpha_3}}\right)}{\frac{k_3}{h_0\sqrt{\pi\alpha_3}} + \operatorname{erf}\left(\xi_2\sqrt{\frac{\alpha_1}{\alpha_3}}\right)}, \tag{3.5}$$

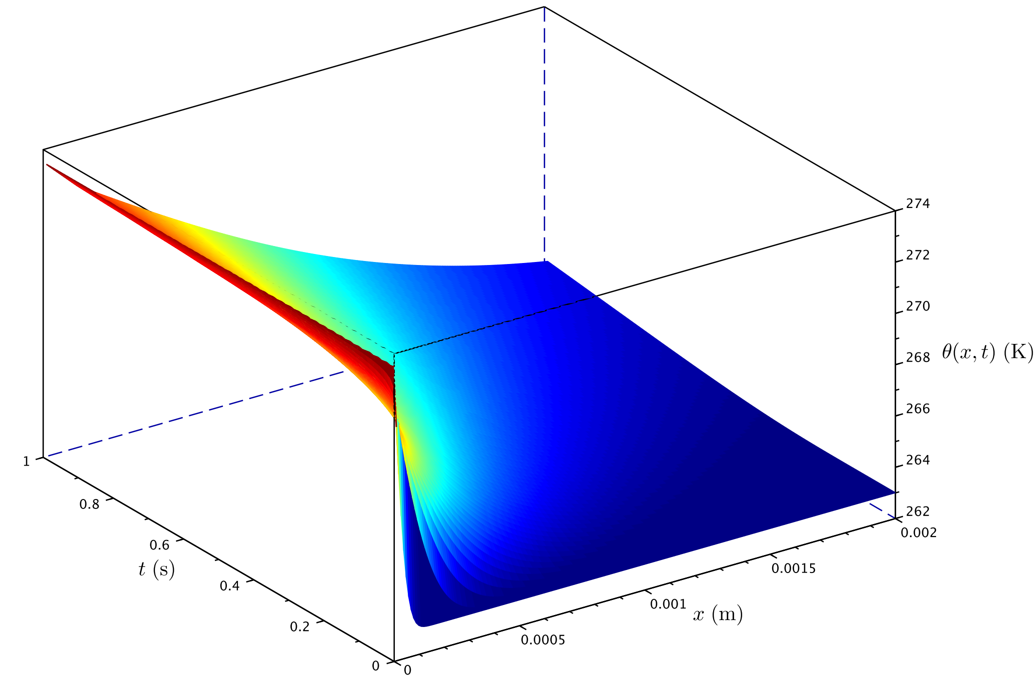

Fig. 7. Temperature distribution of the three-phase Stefan problem with a Neumann type condition.

Table 4
Position and velocity of the interfaces in the three-phase Stefan problem with a Neumann type condition.

Time t	Position $x = s_1(t)$	Velocity $\dot{s}_1(t)$	Position $x = s_2(t)$	Velocity $\dot{s}_2(t)$
0.0 s	0 m	–	0 m	–
0.2 s	5.64×10^{-5} m	1.409×10^{-4} ms^{-1}	8.547×10^{-9} m	2.137×10^{-8} ms^{-1}
0.4 s	7.97×10^{-5} m	9.96×10^{-5} ms^{-1}	1.209×10^{-8} m	1.511×10^{-8} ms^{-1}
0.6 s	9.76×10^{-5} m	8.14×10^{-5} ms^{-1}	1.48×10^{-8} m	1.234×10^{-8} ms^{-1}
0.8 s	1.127×10^{-4} m	7.05×10^{-5} ms^{-1}	1.709×10^{-8} m	1.068×10^{-8} ms^{-1}
1.0 s	1.260×10^{-4} m	6.3×10^{-5} ms^{-1}	1.911×10^{-8} m	9.556×10^{-9} ms^{-1}

then

$$v_3(0,t) - B = \frac{(A_\infty - B)\,\mathrm{erf}\left(\xi_2\sqrt{\frac{\alpha_1}{\alpha_3}}\right)}{\frac{k_3}{h_0\sqrt{\pi\alpha_3}} + \mathrm{erf}\left(\xi_2\sqrt{\frac{\alpha_1}{\alpha_3}}\right)} > 0, \tag{3.6}$$

Taking into account that $v_3(0,t) > B$, the three-phase Stefan problem **(P2)** can be formulated with a temperature condition at $x = 0$ given by $A = v_3(0,t)$, as defined by (3.2). Then, for this data, the solution to problem **(P2)** can be rewritten as

$$u_3(x,t) = \frac{\frac{Bk_3}{h_0\sqrt{\pi\alpha_3}} + A_\infty\,\mathrm{erf}\left(\xi_2\sqrt{\frac{\alpha_1}{\alpha_3}}\right)}{\frac{k_3}{h_0\sqrt{\pi\alpha_3}} + \mathrm{erf}\left(\xi_2\sqrt{\frac{\alpha_1}{\alpha_3}}\right)} \frac{\mathrm{erf}\left(\mu_2\sqrt{\frac{\alpha_1}{\alpha_3}}\right) - \mathrm{erf}\left(\frac{x}{2\sqrt{\alpha_3 t}}\right)}{\mathrm{erf}\left(\mu_2\sqrt{\frac{\alpha_1}{\alpha_3}}\right)} + B\frac{\mathrm{erf}\left(\frac{x}{2\sqrt{\alpha_3 t}}\right)}{\mathrm{erf}\left(\mu_2\sqrt{\frac{\alpha_1}{\alpha_3}}\right)},$$
$$0 < x < r_2(t),\ t > 0,$$

and (2.52)–(2.57).

Taking into account (2.38), (3.2) and the Eqs. (2.56), (2.57), it follows that the coefficients μ_1 and μ_2 constitute the unique solution to the following system of equations

$$\begin{cases} \frac{\ell_1}{\ell_2}\varphi(z_1)\exp\left(z_1^2\frac{\alpha_1}{\alpha_2}\right) \\ \quad = \frac{\mathrm{Ste}_2}{c_2}\,\frac{A_\infty - B}{B-C}\sqrt{\frac{c_1c_3k_3}{k_1\pi}}\,\frac{\mathrm{erf}\left(\xi_2\sqrt{\frac{\alpha_1}{\alpha_3}}\right)}{\mathrm{erf}\left(z_2\sqrt{\frac{\alpha_1}{\alpha_3}}\right)}\,\frac{\exp\left(-z_2^2\alpha_1\left(\frac{1}{\alpha_3}-\frac{1}{\alpha_2}\right)\right)}{\frac{k_3}{h_0\sqrt{\pi\alpha_3}} + \mathrm{erf}\left(\xi_2\sqrt{\frac{\alpha_1}{\alpha_3}}\right)} - z_2\exp\left(z_2^2\frac{\alpha_1}{\alpha_2}\right), \\ z_2 = \sqrt{\frac{\alpha_2}{\alpha_1}}\,\mathrm{erf}^{-1}(H(z_1)). \end{cases} \tag{3.7}$$

Considering that ξ_2 is given by (2.40) and that ξ_1 is the unique solution to (2.34), it follows that ξ_1 and ξ_2 constitute a solution of the system (3.7). By uniqueness, it follows that $\xi_1 = \mu_1$ and $\xi_2 = \mu_2$. From this fact, it follows immediately that $v_i(x,t) = u_i(x,t)$ for $i = 1,2,3$.

(b) From the temperatures u_3, u_2 and u_1, along with the free boundaries r_2 and r_1 that represent the unique solution to problem **(P2)** as specified by (2.51)–(2.58), the following is obtained:

$$k_3\frac{\partial u_3}{\partial x}(0,t) = \frac{-(A-B)k_3}{\sqrt{\pi\alpha_3}\,\mathrm{erf}\left(\mu_2\sqrt{\frac{\alpha_1}{\alpha_3}}\right)}\frac{1}{\sqrt{t}},$$

and therefore, the convective condition at the fixed face $x = 0$, described by (2.14), with h_0 defined in (3.4), can be computed for some value $A_\infty > A$ such that (3.3) holds. Then the unique solution to the problem **(P1)** with h_0 defined by (3.4) is given by

$$v_3(x,t) = \frac{\frac{B(A_\infty - A)}{A-B}\,\mathrm{erf}\left(\mu_2\sqrt{\frac{\alpha_1}{\alpha_3}}\right) + A_\infty\,\mathrm{erf}\left(\xi_2\sqrt{\frac{\alpha_1}{\alpha_3}}\right) - (A_\infty - B)\,\mathrm{erf}\left(\frac{x}{2\sqrt{\alpha_3 t}}\right)}{\frac{A_\infty - A}{A-B}\,\mathrm{erf}\left(\mu_2\sqrt{\frac{\alpha_1}{\alpha_3}}\right) + \mathrm{erf}\left(\xi_2\sqrt{\frac{\alpha_1}{\alpha_3}}\right)},$$
$$0 < x < w_2(t),\ t > 0, \tag{3.8}$$

and (2.20)–(2.21) and (2.31)–(2.32). In addition, taking into account (2.38), (2.40) and (2.41), the coefficients ξ_1 and ξ_2 constitute the unique solution to the following system of equations

$$\begin{cases} \frac{\ell_1}{\ell_2}\varphi(z_1)\exp\left(z_1^2\frac{\alpha_1}{\alpha_2}\right) \\ \quad = \frac{\mathrm{Ste}_2}{c_2}\frac{A_\infty-B}{B-C}\sqrt{\frac{c_1c_3k_3}{k_1\pi}}\frac{\exp\left(-z_2^2\alpha_1\left(\frac{1}{\alpha_3}-\frac{1}{\alpha_2}\right)\right)}{\mathrm{erf}\left(z_2\sqrt{\frac{\alpha_1}{\alpha_3}}\right)+\frac{A_\infty-A}{A-B}\mathrm{erf}\left(\mu_2\sqrt{\frac{\alpha_1}{\alpha_3}}\right)} - z_2\exp\left(z_2^2\frac{\alpha_1}{\alpha_2}\right), \\ z_2=\sqrt{\frac{\alpha_2}{\alpha_1}}\mathrm{erf}^{-1}(H(z_1)). \end{cases} \tag{3.9}$$

Considering that μ_2 is given by (2.56) and that μ_1 is the unique solution to (2.57), it follows that μ_1 and μ_2 constitute a solution to (3.9). By uniqueness, it follows that $\mu_1=\xi_1$ and $\mu_2=\xi_2$. From this fact, it can be deduced immediately that $u_i(x,t)=v_i(x,t)$ for $i=1,2,3$. □

From the previous theorem, the equivalence of problems **(P1)** and **(P2)** arises under certain conditions regarding the data; therefore, the following relationships can be established. This insight highlights that, under specific circumstances, two distinct problems share a common solution framework, which can be crucial for simplifying analyses or applications in various fields.

Corollary 3.2. *Let $A>B$ be the data of the temperature boundary condition of the problem **(P2)**. The coefficient μ_2 that characterizes the free boundary r_2 given by (2.56) satisfies the following inequality:*

$$\mathrm{erf}\left(\mu_2\sqrt{\frac{\alpha_1}{\alpha_3}}\right) < \sqrt{\frac{k_3c_3}{k_2c_2}\frac{A-B}{B-C}\frac{A_\infty-B}{A_\infty-A}}\,\mathrm{erf}\left(z_0\sqrt{\frac{\alpha_1}{\alpha_2}}\right), \qquad \forall A_\infty > A, \tag{3.10}$$

where z_0 is given by (2.39).

Proof. From part (b) of the previous Theorem, it follows that h_0, as given by (3.4), must satisfy $h_0>h_2$, where h_2 is defined by (2.48). Therefore, the coefficients μ_2 that characterize the interface $x=r_2(t)$ of the solution to the problem **(P2)**, must also satisfy the inequality (3.10). □

Remark 3.3. The function defined by the right hand side in (3.10) is a strictly decreasing function of the variable A_∞. Then, by taking the limit $A_\infty\to+\infty$ in inequality (3.10), it is obtained that:

$$\mathrm{erf}\left(\mu_2\sqrt{\frac{\alpha_1}{\alpha_3}}\right) < \sqrt{\frac{k_3c_3}{k_2c_2}\frac{A-B}{B-C}}\,\mathrm{erf}\left(z_0\sqrt{\frac{\alpha_1}{\alpha_2}}\right). \tag{3.11}$$

Remark 3.4. The inequality (3.10) holds a physical significance for the solution (2.51)–(2.57) when the parameters of the problem **(P2)** satisfy the inequality:

$$\sqrt{\frac{k_3c_3}{k_2c_2}\frac{A-B}{B-C}\frac{A_\infty-B}{A_\infty-A}}\,\mathrm{erf}\left(z_0\sqrt{\frac{\alpha_1}{\alpha_2}}\right) < 1, \qquad A_\infty > A > B > C > D. \tag{3.12}$$

Remark 3.5. The coefficient h_0 defined by (3.4) can be considered a function of A. By setting the bulk temperature $A_\infty = 274$ K and using the parameters provided in Tables 1 and 2, h_0 can be plotted as a function of A. Fig. 8 clearly illustrates that h_0 is a strictly increasing function, with a vertical asymptote at $A=A_\infty$.

Corollary 3.6. *Let h_0 and A_∞ with $h_0>h_2$ be the data of the convective boundary condition of the problem **(P1)**. The value A that characterizes the temperature boundary condition at the fixed face $x=0$ to the problem **(P2)**, satisfies the following inequality:*

$$B < A < A_\infty. \tag{3.13}$$

Proof. From part (a) of Theorem 3.1, it follows that A, as given by (3.2), satisfy $A-A_\infty<0$ and $A-B>0$. Therefore, A must also satisfy the inequality (3.13). □

Corollary 3.7. *The value $A=A(h_0,A_\infty)$ given by (3.1) is an increasing function of h_0.*

Proof. Notice that

$$A(h_0,A_\infty) = B\frac{1+A_\infty\frac{h_0\sqrt{\pi\alpha_3}}{Bk_3}\mathrm{erf}\left(\xi_2\sqrt{\frac{\alpha_1}{\alpha_3}}\right)}{1+\frac{h_0\sqrt{\pi\alpha_3}}{k_3}\mathrm{erf}\left(\xi_2\sqrt{\frac{\alpha_1}{\alpha_3}}\right)} = B\Psi\left(h_0\,\mathrm{erf}\left(\xi_2\sqrt{\frac{\alpha_1}{\alpha_3}}\right)\right)$$

where $\Psi(z)=\frac{1+\nu_1 z}{1+\nu_2 z}$ with $\nu_1=\frac{\sqrt{\pi\alpha_3}}{k_3}\frac{A_\infty}{B}$ and $\nu_2=\frac{\sqrt{\pi\alpha_3}}{k_3}$. Taking into account that $A_\infty>B$, it follows that $\nu_1-\nu_2=\frac{\sqrt{\pi\alpha_3}}{k_3}\left(\frac{A_\infty}{B}-1\right)>0$ and therefore $\Psi'(z)=\frac{\nu_1-\nu_2}{(1+\nu_2 z)^2}>0$ for all z. This means that Ψ is an increasing function in z.

In addition, notice that ξ_2 given by (2.40) depends on ξ_1. In turn, ξ_1 is the unique solution to Eq. (2.41). On one hand, the function T defined by (2.44) is an increasing function in h_0. As a consequence, by (2.43), the function U increases when h_0 becomes greater. On the other hand, the function Q given by (2.42) does not depend on h_0. Then it follows that the unique solution $\xi_1>z_0$ to Eq. (2.41) increases in h_0. Consequently, ξ_2 also becomes an increasing function in h_0.

Putting all of the above together, it is easy to see that A is an increasing function in h_0. □

3.2. Equivalence between problems (P2) and (P3)

Connecting a Stefan problem with a temperature boundary condition to the one with a flux boundary condition is crucial for understanding thermal phenomena. Analyzing their interaction provides a more comprehensive view of the thermal behavior of the system. Additionally, this relationship can reveal important mathematical properties of the underlying equations.

In the following theorem, the relationships to impose between the data of both problems are established to ensure their equivalence.

Theorem 3.8.

(a) *Let $q_0>q_2$ be the given constant of the flux condition of the problem **(P3)** where q_2 is defined by (2.59). If the problem **(P2)** is considered with a temperature boundary condition defined by*

$$A = B + \frac{q_0\sqrt{\pi\alpha_3}}{k_3}\mathrm{erf}\left(\lambda_2\sqrt{\frac{\alpha_1}{\alpha_3}}\right), \tag{3.14}$$

*where λ_2 is given by (2.65), then the solution to problem **(P2)** coincides with the solution to problem **(P3)**.*

(b) *Let $A>B$ be the data of the temperature boundary condition of the problem **(P2)**. If the following inequality holds:*

$$\frac{k_3(A-B)}{\sqrt{\alpha_3}\,\mathrm{erf}\left(\mu_2\sqrt{\frac{\alpha_1}{\alpha_3}}\right)} > \frac{k_2(B-C)}{\sqrt{\alpha_2}\,\mathrm{erf}\left(z_0\sqrt{\frac{\alpha_1}{\alpha_2}}\right)}, \tag{3.15}$$

*where μ_2 is given by (2.56), then the solution to problem **(P3)** with*

$$q_0 = \frac{k_3}{\sqrt{\pi\alpha_3}}\frac{(A-B)}{\mathrm{erf}\left(\mu_2\sqrt{\frac{\alpha_1}{\alpha_3}}\right)}, \tag{3.16}$$

*coincides with the solution to problem **(P2)**.*

Proof.

(a) Using the solution to the problem **(P3)**, which involves the temperatures θ_3, θ_2 and θ_1, along with the free boundaries s_2 and s_1 established in Section 2.2, it is possible to determine the temperature at $x=0$:

$$\theta_3(0,t) = B + \frac{q_0\sqrt{\pi\alpha_3}}{k_3}\mathrm{erf}\left(\lambda_2\sqrt{\frac{\alpha_1}{\alpha_3}}\right). \tag{3.17}$$

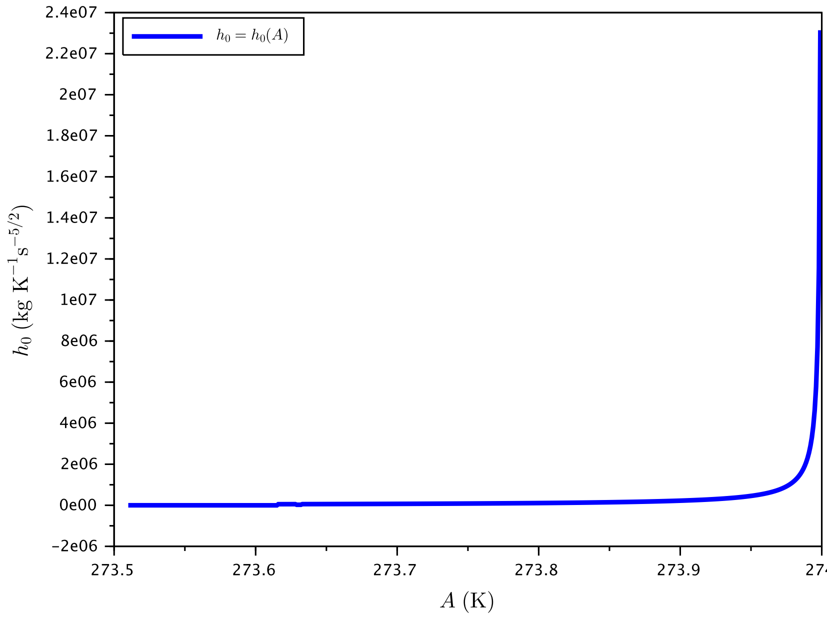

Fig. 8. Coefficient h_0 characterizing heat transfer under convective conditions as a function of A.

Taking into account that $\theta_3(0,t) > B$, the three-phase problem **(P2)** can be defined with a temperature condition at $x = 0$ specified as $A = \theta_3(0,t)$, i.e. A is given by (3.14).
Then, for this data, the solution to problem **(P2)** can be rewritten for $0 < x < r_2(t)$, $t > 0$ as

$$u_3(x,t) = \left(B + \frac{q_0\sqrt{\pi\alpha_3}\,\operatorname{erf}\left(\lambda_2\sqrt{\frac{\alpha_1}{\alpha_3}}\right)}{k_3}\right)\frac{\operatorname{erf}\left(\mu_2\sqrt{\frac{\alpha_1}{\alpha_3}}\right)-\operatorname{erf}\left(\frac{x}{2\sqrt{\alpha_3 t}}\right)}{\operatorname{erf}\left(\mu_2\sqrt{\frac{\alpha_1}{\alpha_3}}\right)} + B\frac{\operatorname{erf}\left(\frac{x}{2\sqrt{\alpha_3 t}}\right)}{\operatorname{erf}\left(\mu_2\sqrt{\frac{\alpha_1}{\alpha_3}}\right)},$$

and (2.52)–(2.57).
From the values given by (2.38) and (3.14) and the Eqs. (2.56) and (2.57), it follows that the coefficients μ_1 and μ_2 constitute the unique solution to the following system of equations

$$\begin{cases} \frac{\ell_1}{\ell_2}\varphi(z_1)\exp\left(z_1^2\frac{\alpha_1}{\alpha_2}\right) = \frac{q_0}{\ell_2}\sqrt{\frac{c_1}{\rho k_1}}\,\frac{\operatorname{erf}\left(\lambda_2\sqrt{\frac{\alpha_1}{\alpha_3}}\right)\exp\left(-z_2^2\alpha_1\left(\frac{1}{\alpha_3}-\frac{1}{\alpha_2}\right)\right)}{\operatorname{erf}\left(z_2\sqrt{\frac{\alpha_1}{\alpha_3}}\right)} \\ \quad -z_2\exp\left(z_2^2\frac{\alpha_1}{\alpha_2}\right), \\ z_2 = \sqrt{\frac{\alpha_2}{\alpha_1}}\operatorname{erf}^{-1}(H(z_1)). \end{cases} \tag{3.18}$$

Given that λ_2 is defined by (2.65) and λ_1 is the unique solution to (2.66), it follows that λ_1 and λ_2 form a solution to the system (3.18). Due to uniqueness, $\lambda_1 = \mu_1$ and $\lambda_2 = \mu_2$. Consequently, it can be immediately deduced that $\theta_i(x,t) = u_i(x,t)$ for $i = 1,2,3$.

(b) Using the temperatures u_3, u_2 and u_1, along with the free boundaries r_2 and r_1, which represent the unique solution to problem **(P2)** as defined by (2.51)–(2.58), the following expression is derived:

$$k_3\frac{\partial u_3}{\partial x}(0,t) = \frac{-(A-B)k_3}{\sqrt{\pi\alpha_3}\,\operatorname{erf}\left(\mu_2\sqrt{\frac{\alpha_1}{\alpha_3}}\right)}\frac{1}{\sqrt{t}},$$

and therefore, the flux condition at the fixed face $x = 0$ given by (2.16) can be computed. Since (3.15) holds, the unique solution to the problem **(P3)** with q_0 defined by (3.16) is given by

$$\theta_3(x,t) = B + \frac{A-B}{\operatorname{erf}\left(\mu_2\sqrt{\frac{\alpha_1}{\alpha_3}}\right)}\left(\operatorname{erf}\left(\lambda_2\sqrt{\frac{\alpha_1}{\alpha_3}}\right) - \operatorname{erf}\left(\frac{x}{2\sqrt{\alpha_3 t}}\right)\right),$$
$$0 < x < s_2(t),\ t > 0, \tag{3.19}$$

and (2.61)–(2.66). In addition, taking into account (2.38), (2.65) and (2.66), the coefficients λ_1 and λ_2 constitute the unique solution to the following system of equations

$$\begin{cases} \frac{\ell_1}{\ell_2}\varphi(z_1)\exp\left(z_1^2\frac{\alpha_1}{\alpha_2}\right) = \frac{A-B}{\ell_2}\sqrt{\frac{c_1 c_3 k_3}{k_1\pi}}\,\frac{\exp\left(-z_2^2\alpha_1\left(\frac{1}{\alpha_3}-\frac{1}{\alpha_2}\right)\right)}{\operatorname{erf}\left(\mu_2\sqrt{\frac{\alpha_1}{\alpha_3}}\right)} \\ \quad -z_2\exp\left(z_2^2\frac{\alpha_1}{\alpha_2}\right), \\ z_2 = \sqrt{\frac{\alpha_2}{\alpha_1}}\operatorname{erf}^{-1}(H(z_1)). \end{cases} \tag{3.20}$$

Given that μ_2 is defined by (2.56) and μ_1 represents the unique solution to (2.57), it can be concluded that μ_1 and μ_2 together form a solution to the system (3.20). Due to the uniqueness property, $\lambda_1 = \mu_1$ and $\lambda_2 = \mu_2$. As a result, it follows that $\theta_i(x,t) = u_i(x,t)$ for $i = 1,2,3$. □

Remark 3.9. A sufficient condition for (3.15) to be satisfied is $A > \frac{q_2\sqrt{\pi\alpha_3}}{k_3} + B$.

The earlier theorem reveals that problems **(P2)** and **(P3)** are equivalent under certain data-related conditions. As a result, the following relationships can be established, leading to a generalization of the findings presented in [31].

Corollary 3.10. *Let $A > B$ be the data of the temperature boundary condition of the problem **(P2)**. The coefficient μ_2 given by (2.56) satisfies the following inequality:*

$$\operatorname{erf}\left(\mu_2\sqrt{\frac{\alpha_1}{\alpha_3}}\right) < \frac{k_3}{k_2}\sqrt{\frac{\alpha_2}{\alpha_3}}\,\frac{A-B}{B-C}\operatorname{erf}\left(z_0\sqrt{\frac{\alpha_1}{\alpha_2}}\right), \tag{3.21}$$

where z_0 is given by (2.39).

Proof. Part (b) of the previous theorem indicates that q_0, as defined by (3.16), must satisfy $q_0 > q_2$, where q_2 is given by (2.59). Consequently, the coefficient μ_2 that characterizes the interface $x = r_2(t)$ of the solution to problem **(P2)** must also comply with the inequality (3.21). □

Corollary 3.11. *The value $A = A(q_0)$ given by (3.14) is an increasing function of q_0.*

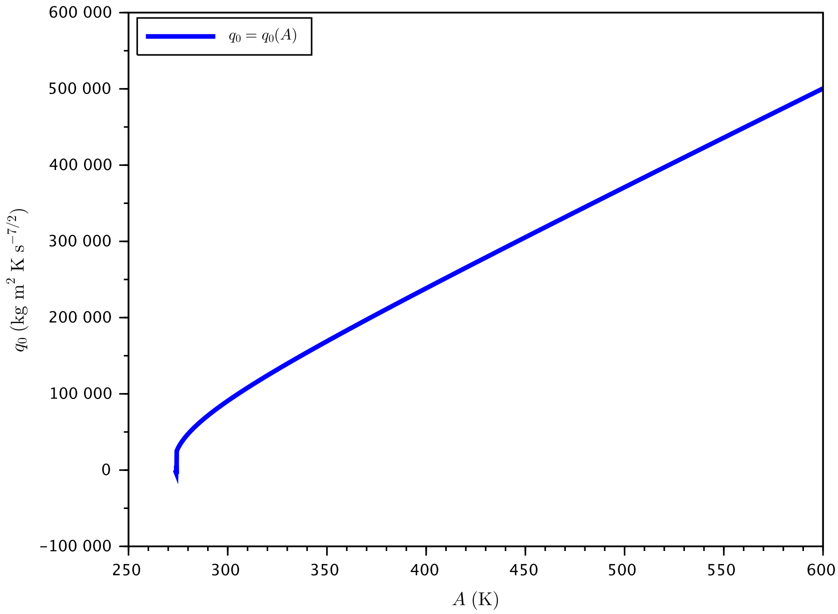

Fig. 9. Coefficient q_0 characterizing the flux condition as a function of A.

Proof. It is important to note that λ_2, as defined by (2.65), is dependent on λ_1. Furthermore, λ_1 represents the unique solution to Eq. (2.66), which itself is influenced by q_0. On one hand, the function Q is independent of q_0. Conversely, P is an increasing function of q_0. Consequently, λ_1 is also an increasing function of q_0, and this behavior extends to λ_2 as well. According to the definition of A, it follows directly that the thesis is validated. $\square$

Remark 3.12. The coefficient q_0 defined by (3.16) can be considered a function of A. By using the parameters provided in Tables 1 and 2, q_0 can be plotted as a function of A. Fig. 9 clearly illustrates that q_0 is a strictly increasing function.

*3.3. Equivalence between problems **(P1)** and **(P3)***

As illustrated in the previous subsections, the following theorem establishes the relationship between the problems involving convective and flux boundary conditions at the fixed face.

Theorem 3.13.

(a) *Let h_0 and A_∞ be the given constants of the convective condition of the problem **(P1)** with $A_\infty > B$ and $h_0 > h_2$ where h_2 is given by (2.48). If the following inequality holds:*

$$\frac{(A_\infty - B)h_0}{1+\frac{h_0\sqrt{\pi\alpha_3}}{k_3}\operatorname{erf}\left(\xi_2\sqrt{\frac{\alpha_1}{\alpha_3}}\right)} > \frac{k_2(B-C)}{\sqrt{\alpha_2\pi}\operatorname{erf}\left(z_0\sqrt{\frac{\alpha_1}{\alpha_2}}\right)}, \tag{3.22}$$

*where ξ_2 is given by (2.40), then the solution to problem **(P3)** with*

$$q_0 = \frac{(A_\infty - B)h_0}{1+\frac{h_0\sqrt{\pi\alpha_3}}{k_3}\operatorname{erf}\left(\xi_2\sqrt{\frac{\alpha_1}{\alpha_3}}\right)}, \tag{3.23}$$

*coincides with the solution to problem **(P1)**.*

(b) *Let $q_0 > q_2$ be the given constant of the flux condition of the problem **(P3)** where q_2 is defined by (2.59). If the following inequality holds:*

$$\frac{q_0}{(A_\infty - B) - q_0\frac{\sqrt{\pi\alpha_3}}{k_3}\operatorname{erf}\left(\lambda_2\sqrt{\frac{\alpha_1}{\alpha_3}}\right)} > \frac{B-C}{A_\infty - B}\sqrt{\frac{k_2k_3c_2}{\pi\alpha_3c_3}}\frac{1}{\operatorname{erf}\left(z_0\sqrt{\frac{\alpha_1}{\alpha_2}}\right)}, \tag{3.24}$$

*where λ_2 is given by (2.65) and $A_\infty > B$, then the solution to problem **(P1)** with*

$$h_0 = \frac{q_0}{(A_\infty - B) - q_0\frac{\sqrt{\pi\alpha_3}}{k_3}\operatorname{erf}\left(\lambda_2\sqrt{\frac{\alpha_1}{\alpha_3}}\right)}, \tag{3.25}$$

*coincides with the solution to problem **(P3)**.*

Proof. The proof is straightforward. $\square$

Remark 3.14. Assuming that

$$A_\infty > B + \sqrt{\frac{\alpha_3}{\alpha_2}}\frac{k_2}{k_3}\frac{B-C}{\operatorname{erf}\left(z_0\sqrt{\frac{\alpha_1}{\alpha_2}}\right)},$$

and

$$h_0 > \max\left\{h_2, h_2^*\right\},$$

where h_2 is given by (2.48) and $h_2^* > 0$ is such that $F(h_2^*) = 0$, with

$$F(z) = \frac{k_3(A_\infty - B)\sqrt{\pi\alpha_2}\operatorname{erf}\left(z_0\sqrt{\frac{\alpha_1}{\alpha_2}}\right)z}{k_2(B-C)\left(k_3 + z\sqrt{\pi\alpha_3}\right)}, \qquad z \geq 0,$$

and z_0 is given by (2.39), then condition (3.22) is automatically satisfied.

In addition, if

$$q_0 > \frac{B-C}{A_\infty - B}\sqrt{\frac{k_2k_3c_2}{\pi\alpha_3c_3}}\frac{1}{\operatorname{erf}\left(z_0\sqrt{\frac{\alpha_1}{\alpha_2}}\right)},$$

then the inequality given by (3.24) holds, for all $A_\infty > B$.

Conclusions

This study provided a unique explicit similarity-type solution for the three-phase Stefan problem in a semi-infinite material with a convective boundary condition at the fixed face. The equivalence among the solutions of three Stefan problems with different boundary conditions (Robin, Dirichlet and Neumann) was demonstrated, provided that a specific relationship between the problem data was satisfied. Additionally, computational examples were performed to illustrate the validity of the obtained results and to explore the system's behavior under various boundary condition configurations. These findings offer a deeper understanding of heat transfer processes in phase-change systems, with significant implications for material science and engineering applications.

CRediT authorship contribution statement

Julieta Bollati: Writing – original draft, Software, Formal analysis. **María F. Natale:** Writing – original draft, Resources, Methodology, Formal analysis. **José A. Semitiel:** Writing – original draft, Software, Methodology, Conceptualization. **Domingo A. Tarzia:** Writing – review & editing, Supervision, Methodology, Formal analysis, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Data availability

No data was used for the research described in the article.

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