A Two-Phase Stefan Problem with Power-Type Temperature-Dependent Thermal Conductivity

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Abstract A one-dimensional two-phase Stefan problem is examined for the melting of a semi-infinite material with thermal conductivity that depends on temperature following a power-law relationship. The choice to model thermal parameters as functions of temperature is rooted in both physical and industrial considerations, providing a more accurate and realistic representation of phase change processes. By applying a Dirichlet type boundary condition at the fixed face, a theoretical analysis is conducted. A similarity transformation is used to convert the problem into an equivalent ordinary differential problem, from which an integral problem is derived. The existence of at least one analytic solution is assured through the application of the Banach fixed-point theorem.

Keywords Stefan problem · Variable thermal conductivity · Fixed point · Analytic solution

1 Mathematical Formulation

In the classical formulation of Stefan's problem, certain assumptions are made about the physical factors involved in the phase change process to simplify the model description. One of these assumptions is that the thermal conductivity, the specific heat, the latent heat and the density of the material are considered positive constants. However, various arguments from thermodynamics motivate the solution of Stefan's problems with variable thermal coefficients.

In engineering calculations, it is common to measure thermal coefficients phenomenologically, but for some materials, the values obtained in tables are not adequate to predict those thermal coefficients with satisfactory precision since they

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depend on other factors. In the last years, several studies have shown that modeling thermal parameters as functions of temperature can describe phase change processes with greater precision and realism and can be more useful for physical and industrial applications [3–5].

The following previous works motivate the idea of taking a variable thermal conductivity. Some generalizations considering power-type thermal coefficients are available in the literature [1, 2].

Based on the bibliography mentioned above it is quite natural from a mathematical point of view to define a one-dimensional two-phase Stefan problem with a power type temperature-dependent thermal conductivity. The problem models the melting of a semi-infinite material $x \geq 0$ imposing a Dirichlet type condition $T_0 > 0$ at the fixed face $x = 0$ which is initially at temperature $-T_r < 0$ with a null phase-change temperature. This free boundary problem consists of finding the temperature $T_1 = T_1(x, t)$ of the liquid region, the temperature $T_2 = T_2(x, t)$ of the solid region, and the position of the interface $x = s(t)$, $t \geq 0$. This problem can be formulated mathematically in the following way:

$$\frac{\partial}{\partial x}\left(k_1(T_1)\frac{\partial T_1}{\partial x}\right) = \rho c_1 \frac{\partial T_1}{\partial t}, \quad 0 < x < s(t),\ t > 0, \tag{1}$$

$$\frac{\partial}{\partial x}\left(k_2(T_2)\frac{\partial T_2}{\partial x}\right) = \rho c_2 \frac{\partial T_2}{\partial t}, \quad x > s(t),\ t > 0, \tag{2}$$

$$T_1(0, t) = T_0, \quad t > 0, \tag{3}$$

$$T_1(s(t), t) = T_2(s(t), t) = 0, \quad t > 0, \tag{4}$$

$$T_2(+\infty, t) = T_2(x, 0) = -T_r, \quad x > s(t),\ t > 0, \tag{5}$$

$$k_2(T_2(s(t), t))\frac{\partial T_2}{\partial x}(s(t), t) - k_1(T_1(s(t), t))\frac{\partial T_1}{\partial x}(s(t), t) = \rho \ell \dot{s}(t), \quad t > 0, \tag{6}$$

$$s(0) = 0, \tag{7}$$

where $\ell > 0$ is the latent heat of fusion by unit of mass, $\rho > 0$ is the mass density, $c_i > 0$ are the specific heat of each region $i = 1, 2$. The thermal conductivity coefficients depend on temperature by the expressions:

$$k_1(T_1(x, t)) = k_1^0 \left[1 + \delta_1 \left(\tfrac{T_1(x,t)}{T_0}\right)^{p_1}\right], \tag{8}$$

$$k_2(T_2(x, t)) = k_2^0 \left[1 + \delta_2 \left(-\tfrac{T_2(x,t)}{T_r}\right)^{p_2}\right], \tag{9}$$

with $\delta_i \geq 0$ and $p_i \geq 1$, where $k_i^0 > 0$ are the reference thermal conductivity and $a_i^2 = \frac{k_i^0}{\rho c_i}$, are the thermal diffusivity at the liquid phase $i = 1$ and solid phase $i = 2$.

2 Existence of Solution to the Two-Phase Free Boundary Problem with a Dirichlet Type Condition

In this section we will prove existence of solution of similarity type to the two-phase free boundary problem with a Dirichlet type condition defined by (1)-(7) using the Banach fixed point theorem. First, we obtain an equivalence between the Stefan problem and two coupled ordinary differential problems through a suitable convenient change of variables. Then, we show that these problems are equivalent to finding the fixed point of certain contracting operators.

The temperatures $T_1 = T_1(x,t)$ and $T_2 = T_2(x,t)$ depends on a similarity variable given by

$$\eta = \frac{x}{2a_1\sqrt{t}}. \tag{10}$$

Through the following change of variables:

$$y_1(\eta) = \frac{T_1(x,t)}{T_0} \geq 0 \qquad \text{and} \qquad y_2(\eta) = \frac{T_2(x,t)}{T_r} \leq 0, \tag{11}$$

the phase front moves as

$$s(t) = 2a_1\lambda\sqrt{t}, \tag{12}$$

where $\lambda > 0$ must be determined and thus we have the following result:

Theorem 1 *Let $p_i \geq 1$, $\delta_i \geq 0$ for $i = 1, 2$. The Stefan problem (1)–(7) has a similarity-type solution (T_1, T_2, s) given by:*

$$T_1(x,t) = T_0\ y_1(\eta), \qquad 0 < x < s(t), \quad t > 0, \tag{13}$$

$$T_2(x,t) = T_r\ y_2(\eta), \qquad x > s(t), \quad t > 0, \tag{14}$$

$$s(t) = 2a_1\lambda\sqrt{t}, \qquad t > 0, \tag{15}$$

if and only if the functions $y_1 = y_1(\eta) \in C^2(0,\lambda)$, $y_2 = y_2(\eta) \in C^2(\lambda, +\infty)$ and the parameter $\lambda > 0$ satisfy the following ordinary differential problems:

$$\Big((1 + \delta_1 y_1^{p_1}(\eta))y_1'(\eta)\Big)' + 2\eta\ y_1'(\eta) = 0, \qquad 0 < \eta < \lambda, \tag{16}$$

$$y_1(0) = 1, \tag{17}$$

$$y_1(\lambda) = 0, \tag{18}$$

and

$$\frac{a_2^2}{a_1^2}\Big((1 + \delta_1(-y_2)^{p_2}(\eta))y_2'(\eta)\Big)' + 2\eta\ y_2'(\eta) = 0, \qquad \eta > \lambda, \tag{19}$$

$$y_2(\lambda) = 0, \tag{20}$$

$$y_2(+\infty) = -1, \tag{21}$$

coupled through the following condition

$$\tfrac{a_2^2}{a_1^2}\mathrm{Ste}_2\ y_2'(\lambda) - \mathrm{Ste}_1\ y_1'(\lambda) = 2\lambda, \quad \lambda > 0, \tag{22}$$

where $\mathrm{Ste}_1 = \frac{c_1 T_0}{\ell} > 0$, $\mathrm{Ste}_2 = \frac{c_2 T_r}{\ell} > 0$ *are the Stefan numbers.*

First, we will analyze for a fixed $\lambda > 0$ the existence and uniqueness of solution to the ordinary differential problems (16)–(18) and (19)–(21). For this purpose, we will show that solving these problems is equivalent to finding the fixed point of certain operators that will be defined later in the following theorems.

Theorem 2 *Let us assume* $\delta_1 \geq 0$ *and* $p_1 \geq 1$.

(a) For each $\lambda > 0$*, the function* $y_1 \in K_1$ *is a solution to the problem* (16)–(18) *if and only if* y_1 *is a fixed point of the operator* $F_1 : K_1 \to K_1$ *given by:*

$$F_1(y_1)(\eta) = 1 - \frac{P_{y_1}(\eta)}{P_{y_1}(\lambda)}, \tag{23}$$

where

$$K_1 = \{y_1 \in C[0,\lambda] : 0 \leq y_1 \leq 1,\ y_1(0) = 1,\ y_1(\lambda) = 0\}, \tag{24}$$

$$P_{y_1}(\eta) = \int_0^\eta \frac{f_{y_1}(w)}{\Psi_{y_1}(w)}\, dw, \qquad 0 \leq \eta \leq \lambda, \tag{25}$$

$$f_{y_1}(\eta) = \exp\left(-2\int_0^\eta \frac{\xi}{\Psi_{y_1}(\xi)} d\xi\right), \qquad 0 \leq \eta \leq \lambda, \tag{26}$$

$$\Psi_{y_1}(\eta) = 1 + \delta_1 y_1^{p_1}(\eta), \qquad 0 \leq \eta \leq \lambda. \tag{27}$$

(b) For each $\lambda > 0$, $y_1, \overline{y}_1 \in K_1$ *we have that*

$$|F_1(y_1)(\eta) - F_1(\overline{y}_1)(\eta)| \leq \varepsilon_1(\lambda)||y_1 - \overline{y}_1||_*, \qquad 0 \leq \eta \leq \lambda, \tag{28}$$

where $\varepsilon_1(\lambda)$ *is given by*

$$\varepsilon_1(\lambda) = \tfrac{p_1}{\sqrt{\pi}}\delta_1\,(1+\delta_1)^{3/2}\left[-2\,(1+\delta_1)\,H\left(\tfrac{\lambda}{\sqrt{1+\delta_1}}\right) + \sqrt{\pi}\,(3+\delta_1)\right], \tag{29}$$

with

$$\mathrm{erf}(\eta) = \tfrac{2}{\sqrt{\pi}}\int_0^\eta \exp(-\xi^2)\, d\xi, \qquad H(\eta) = \tfrac{\eta\exp(-\eta^2)}{\mathrm{erf}(\eta)}, \quad \eta \geq 0, \tag{30}$$

and $||y||_* = \sup\limits_{0\leq\eta\leq\lambda} |y(\eta)|$.

(c) For each $\lambda > 0$, *if we assume*

$$(1+\delta_1)^{3/2} p_1 \delta_1 (3+\delta_1) < 1, \tag{31}$$

then there exists a unique fixed point $\widehat{y_1} \in K_1$ *of the operator* F_1 *defined by* (23).

Proof *(a)* Let $y_1 \in K_1$ be a solution to the problem (16)–(18). It is easy to see that F_1 given by (23) is well-defined. If we define $v_1 = y_1'$, taking into account that Ψ_{y_1} is given by (27), the ordinary differential equation (16) becomes

$$\left[\Psi_{y_1}(\eta) v_1(\eta)\right]' + 2\eta v_1(\eta) = 0,$$

Integrating the previous equation and taking into account (17) and (18) it follows that:

$$y_1(\eta) = 1 - \frac{P_{y_1}(\eta)}{P_{y_1}(\lambda)}, \qquad 0 \le \eta \le \lambda. \tag{32}$$

In a similar manner, we can prove the reciprocal.

(b) Let us consider $y_1, \overline{y}_1 \in K_1$. Notice that for each $w \in [0, \lambda]$:

$$1 \le \Psi_{y_1}(w) \le 1+\delta_1, \qquad \exp(-w^2) \le f_{y_1}(w) \le \exp\left(-\tfrac{w^2}{1+\delta_1}\right), \tag{33}$$

and then

$$\frac{\sqrt{\pi}}{2(1+\delta_1)} \operatorname{erf}(w) \le P_{y_1}(w) \le \frac{\sqrt{\pi}}{2}\sqrt{1+\delta_1}\operatorname{erf}\left(\tfrac{w}{\sqrt{1+\delta_1}}\right). \tag{34}$$

Applying the mean value theorem we deduce the following inequality:

$$|\Psi_{\overline{y}_1}(w) - \Psi_{y_1}(w)| \le \delta_1 p_1 ||y_1 - \overline{y}_1||_*. \tag{35}$$

Moreover, if we apply the mean value theorem again and taking into account (33) and (35), we have that

$$|f_{y_1}(w) - f_{\overline{y}_1}(w)| \le \exp\left(-\tfrac{w^2}{1+\delta_1}\right) \delta_1 p_1 w^2 \|y_1 - \overline{y}_1\|_*. \tag{36}$$

In addition, inequalities (33), (35) and (36) yields to

$$|P_{y_1}(\eta) - P_{\overline{y}_1}(\eta)| \le \tfrac{\delta_1 p_1 \sqrt{1+\delta_1}}{4} \operatorname{erf}\left(\tfrac{\eta}{\sqrt{1+\delta_1}}\right) \Big[-2\,(1+\delta_1)\, H\left(\tfrac{\eta}{\sqrt{1+\delta_1}}\right) + \sqrt{\pi}\,(3+\delta_1)\Big] \|y_1 - \overline{y}_1\|_*, \tag{37}$$

where H is given by (30). Therefore, from (34) and (37), we have that

$$|F_1(y_1)(\eta) - F_1(\overline{y}_1)(\eta)| = \left| \frac{P_{\overline{y}_1}(\eta)}{P_{\overline{y}_1}(\lambda)} - \frac{P_{y_1}(\eta)}{P_{y_1}(\lambda)} \right| \le \varepsilon_1(\lambda) \|y_1 - \overline{y}_1\|_*, \tag{38}$$

with $\varepsilon_1(\lambda)$ given by (29).

(c) First, notice that K_1 is a closed subset of the Banach space of the continuous functions $C^0[0, \lambda]$ endowed with the supremum norm $\|\cdot\|_*$. In addition, H given by (30) is a decreasing function that satisfies $H(0) = \frac{\sqrt{\pi}}{2}$ and $H(+\infty) = 0$. Then, $\varepsilon_1(\lambda)$ defined by (29) is an increasing function that satisfies

$$\varepsilon_1(0) = 2(1+\delta_1)^{3/2} p_1 \delta_1, \qquad \varepsilon_1(+\infty) = (1+\delta_1)^{3/2} p_1 \delta_1 (3+\delta_1).$$

Assuming (31), yields to $\varepsilon_1(+\infty) < 1$, and therefore the operator F_1 becomes a contraction. The fixed point Banach theorem assures the existence and uniqueness of a fixed point $\widehat{y}_1 \in K_1$ of the operator F_1 for each $\lambda > 0$.

Theorem 3 *Let us assume $\delta_2 \ge 0$ and $p_2 \ge 1$.*

(a) For each $\lambda > 0$, the function $y_2 \in K_2$ is a solution to the problem (19)–(21) if and only if y_2 is a fixed point of the operator $F_2 : K_2 \to K_2$ given by:

$$F_2(y_2)(\eta) = -\frac{Q_{y_2}(\eta)}{Q_{y_2}(+\infty)}, \tag{39}$$

where

$$K_2 = \{y_2 \in C_b[\lambda, +\infty) : -1 \le y_2 \le 0,\ y_2(\lambda) = 0,\ y_2(+\infty) = -1\}, \tag{40}$$

$$Q_{y_2}(\eta) = \int_\lambda^\eta \frac{g_{y_2}(w)}{\Phi_{y_2}(w)}\, dw, \quad \eta \ge \lambda, \tag{41}$$

$$g_{y_2}(\eta) = \exp\left(-2\tfrac{a_1^2}{a_2^2} \int_\lambda^\eta \frac{\xi}{\Phi_{y_2}(\xi)} d\xi\right), \quad \eta \ge \lambda, \tag{42}$$

$$\Phi_{y_2}(\eta) = 1 + \delta_2 (-y_2)^{p_2}(\eta), \quad \eta \ge \lambda. \tag{43}$$

(b) For each $\lambda > 0$, $y_2, \overline{y}_2 \in K_2$ we have that

$$|F_2(y_2)(\eta) - F_2(\overline{y}_2)(\eta)| \le \varepsilon_2(\lambda) ||y_2 - \overline{y}_2||_{**}, \qquad \eta \ge \lambda, \tag{44}$$

where $\varepsilon_2(\lambda)$ is given by

$$\varepsilon_2(\lambda) = \tfrac{\delta_2 p_2 (1+\delta_2)}{\sqrt{\pi}} G\left(\tfrac{a_1 \lambda}{a_2}\right) \left[\tfrac{2(1+\delta_2) a_1 \lambda}{a_2} + \sqrt{\pi(1+\delta_2)}\, (3+\delta_2) \exp\left(\tfrac{a_1^2 \lambda^2}{a_2^2 (1+\delta_2)}\right)\right], \tag{45}$$

with

$$\operatorname{erfc}(\eta) = 1 - \tfrac{2}{\sqrt{\pi}} \int_0^\eta \exp(-\xi^2)\, d\xi, \qquad G(\eta) = \frac{\exp(-\eta^2)}{\operatorname{erfc}(\eta)}, \qquad \eta \geq 0, \tag{46}$$

and $||y||_{**} = \sup\limits_{\eta \geq \lambda} |y(\eta)|$.

(c) For each $0 < \lambda < L$*, with*

$$L = \varepsilon_2^{-1}(1), \tag{47}$$

if we assume

$$\delta_2 p_2 (1 + \delta_2)^{3/2} (3 + \delta_2) < 1, \tag{48}$$

then there exists a unique fixed point $\widehat{y_2} \in K_2$ *of the operator* F_2 *defined by* (39).

Proof It is proven in a similar way to Theorem 2.

The following result is a direct consequence of the Theorems 2 and 3.

Corollary 1 *Suppose that* (31) *and* (48) *holds, then for each* $0 < \lambda < L$ *there exists a unique solution* $\widehat{y_1} \in K_1$, $\widehat{y_2} \in K_2$ *to the ordinary differential problems* (16)–(18) *and* (19)–(21)*, respectively where* L *is given by* (47)*,* K_1 *by* (24) *and* K_2 *by* (40)*.*

Now we will provide the existence of solution to the ordinary differential problems (16)–(18) and (19)–(21) coupled with (22). Taking into account Corollary 1 it remains to analyse the existence of solution to equation (22), which can be rewritten as

$$\mathcal{V}(\lambda) = 2\lambda, \qquad 0 < \lambda < L, \tag{49}$$

where

$$\mathcal{V}(\lambda) = \mathrm{Ste}_1 \frac{f_{y_1^*}(\lambda)}{\Psi_{y_1^*}(\lambda) P_{y_1^*}(\lambda)} - \frac{a_2^2}{a_1^2} \mathrm{Ste}_2 \frac{g_{y_2^*}(\lambda)}{\Phi_{y_2^*}(\lambda) Q_{y_2^*}(+\infty)}. \tag{50}$$

Lemma 1 *Suppose that* (31) *and* (48) *holds. For each* $0 < \lambda < L$*, where* L *is given by* (47)*, the following inequalities hold*

$$\mathcal{V}_1(\lambda) \leq \mathcal{V}(\lambda) \leq \mathcal{V}_2(\lambda), \tag{51}$$

where

$$\mathcal{V}_1(\lambda) = \tfrac{2}{\sqrt{\pi}} \left(\tfrac{\mathrm{Ste}_1}{(1+\delta_1)^{3/2}} \frac{\exp(-\lambda^2)}{\operatorname{erf}\left(\frac{\lambda}{\sqrt{1+\delta_1}}\right)} - \tfrac{a_2}{a_1} \mathrm{Ste}_2 (1 + \delta_2) G\left(\tfrac{a_1 \lambda}{a_2}\right) \right), \tag{52}$$

$$\mathcal{V}_2(\lambda) = \tfrac{2}{\sqrt{\pi}} \left(\mathrm{Ste}_1 (1 + \delta_1) \frac{\exp\left(-\frac{\lambda^2}{1+\delta_1}\right)}{\operatorname{erf}(\lambda)} - \tfrac{a_2}{a_1} \tfrac{\mathrm{Ste}_2}{(1+\delta_2)^{3/2}} G\left(\tfrac{a_1 \lambda}{a_2 \sqrt{1+\delta_2}}\right) \right), \tag{53}$$

with G *given by* (46)*.*

Proof The proof is straightforward.

Remark 1 Notice that the functions $\mathcal{V}_1$ and $\mathcal{V}_2$ do not depend on $\widehat{y}_1$, $\widehat{y}_2$.

Lemma 2 *If we assume* (31), (48) *and*

$$\mathrm{Ste}_1 < \left(\sqrt{\pi}L + \tfrac{a_2}{a_1}\tfrac{\mathrm{Ste}_2}{(1+\delta_2)^{3/2}}G\left(\tfrac{a_1 L}{a_2\sqrt{1+\delta_2}}\right)\right)\frac{\exp\left(\frac{L^2}{1+\delta_1}\right)\mathrm{erf}(L)}{1+\delta_1}, \tag{54}$$

where L is given by (47) *and G is defined by* (46)*, then there exists at least one solution $\widehat{\lambda} \in (0, L)$ to the equation* (49).

Proof Let us define $\mathcal{W}(\lambda) = \mathcal{V}(\lambda) - 2\lambda$, $\lambda \in (0, L)$ where $\mathcal{V}$ is defined by (50). According to Lemma 1, we have that

$$\mathcal{W}(0) \geq \mathcal{V}_1(0) = +\infty, \qquad \mathcal{W}(L) \leq \mathcal{V}_2(L) - 2L, \tag{55}$$

where $\mathcal{V}_1$ and $\mathcal{V}_2$ are given by (52) and (53), respectively.

Notice that assumption (54) implies $\mathcal{W}(L) < 0$. Then there exists at least one root $\widehat{\lambda} \in (0, L)$ of the function $\mathcal{W}$, i.e. $\widehat{\lambda}$ is a solution to equation (49).

From Corollary 1 and Lemma 2, we can conclude the following main result:

Theorem 4 *If* (31), (48) *and* (54) *hold, then there exists at least one solution $\widehat{y}_1 \in K_1$, $\widehat{y}_2 \in K_2$ and $\widehat{\lambda} \in (0, L)$ to the ordinary differential problems* (16)–(18) *and* (19)–(21) *coupled with* (22)*, where L is given by* (47)*, K_1 by* (24)*, K_2 is defined by* (40) *and $\widehat{\lambda}$ is one solution to equation* (49).

As a consequence, and using Theorem 1, we establish the existence of a solution to the two-phase Stefan problem, which is a fundamental result in the theory, as it guarantees the solvability of this physical model.

Theorem 5 *If* (31), (48) *and* (54) *hold, then there exists at least one solution to the two-phase Stefan problem* (1)–(7) *given by*

$$T_1(x,t) = T_0\,\widehat{y}_1\left(\tfrac{x}{2a_1\sqrt{t}}\right), \qquad 0 < x < s(t), \quad t > 0, \tag{56}$$

$$T_2(x,t) = T_r\,\widehat{y}_2\left(\tfrac{x}{2a_1\sqrt{t}}\right), \qquad x > s(t), \quad t > 0, \tag{57}$$

$$s(t) = 2a_1\widehat{\lambda}\sqrt{t}, \qquad t > 0, \tag{58}$$

where $\widehat{y}_1$, $\widehat{y}_2$ and $\widehat{\lambda}$ are defined in Theorem 4.

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