Computational and Applied Mathematics (2022) 41:375
https://doi.org/10.1007/s40314-022-02095-8

Exact solution for non-classical one-phase Stefan problem with variable thermal coefficients and two different heat source terms

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Received: 6 July 2022 / Revised: 14 October 2022 / Accepted: 23 October 2022

Abstract
A one-phase Stefan problem for a semi-infinite material is studied for special functional forms of the thermal conductivity and specific heat depending on the temperature of the phase-change material. Using the similarity transformation technique, an exact solution for these situations are shown. The mathematical analysis is made for two different kinds of heat source terms, and the existence and uniqueness of the solutions are proved.

Keywords Stefan problem · Temperature-dependent thermal coefficients · Phase-change material · Non-classical heat equation · Heat source terms · Explicit solution

Mathematics Subject Classification 35R35 · 35C06 · 80A22 · 35K05

1 Introduction

The phase change problems that contain one or more moving boundaries have attracted growing attention in the last decades due to their wide range of engineering, industrial applications and natural sciences. Stefan problems can be modelled as basic phase-change processes where

Communicated by Wei GONG.

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Published online: 03 November 2022

the location of the interface is a priori an unknown function (Alexiades and Solomon 1993; Carslaw and Jaeger 1959; Crank 1984; Gupta 2018; Tarzia 2011).

The present study provides the existence and uniqueness of solution of the similarity type to a one-phase Stefan fusion problem for a semi-infinite material where it is assumed a Dirichlet condition at the fixed face $x = 0$ and it is governed by a non-classical and nonlinear heat equation with temperature-dependent thermal conductivity and specific heat coefficients and two different kinds of heat source terms.

Non-classical heat conduction problems are considered when the source term is linear or nonlinear depending on the heat flux or the temperature on the boundary of the domain according to the corresponding boundary condition imposed. The non-classical problems are motivated by the modelling of a system of temperature regulation in isotropic media and the source term describes a cooling or a heating effect depending on different types of sources which are related to the evolution of the unknown boundary condition on the boundary of the domain. Problems of this type are related to the thermostat problem (Cannon and Yin 1989; Friedman and Jiang 1988; Furuya et al. 1986; Glashoff and Sprekels 1981, 1982; Kenmochi 1990; Kenmochi and Primicerio 1988). For example, we will use mathematical ideas developed for the one-dimensional case in Berrone et al. (2000), Ceretani et al. (2015), Tarzia and Villa (1998), Villa (1986) and for the n-dimensional case in Boukrouche and Tarzia (2017, 2020). The first paper connecting the non-classical heat equation with a phase-change process (i.e. the Stefan problem) was Briozzo and Tarzia (2006) and after this some other works on the subject were published, for example (Bougoffa and Khanfer 2021; Bougoffa et al. 2021; Briozzo and Natale 2019). Moreover, in Briozzo and Tarzia (2010), explicit solutions for the non-classical one-phase Stefan problem were given for cases corresponding to different boundary conditions on the fixed face $x = 0$: temperature, heat flux and convective boundary condition.

The mathematical model of the governing phase-change process is described as follows:

$$\rho c(\theta)\frac{\partial\theta}{\partial t} = \frac{\partial}{\partial x}\left(k(\theta)\frac{\partial\theta}{\partial x}\right) - F, \qquad 0 < x < s(t), \quad t > 0, \tag{1.1}$$

$$\theta(0,t) = \theta_0 > \theta_f \qquad t > 0, \tag{1.2}$$

$$\theta(s(t),t) = \theta_f, \qquad t > 0, \tag{1.3}$$

$$k_0\frac{\partial\theta}{\partial x}(s(t),t) = -\rho l\dot{s}(t), \qquad t > 0, \tag{1.4}$$

$$s(0) = 0, \tag{1.5}$$

where the unknown functions are the temperature $\theta = \theta(x,t)$ and the free boundary $x = s(t)$ separating both phases (the liquid phase at temperature $\theta(x,t)$ and the solid phase at constant temperature θ_f). The parameters $\rho > 0$ (density), $l > 0$ (latent heat per unit mass), $\theta_0 > 0$ (temperature imposed at the fixed face $x = 0$) and θ_f (phase change temperature at the free boundary $x = s(t)$) are all known constants.

If the thermal coefficients of the material are temperature-dependent, we have a doubly non-linear free boundary problem. The functions k and c are defined as:

$$k(\theta) = k_0\left(1 + \delta\left(\tfrac{\theta-\theta_f}{\theta_0-\theta_f}\right)^p\right), \tag{1.6}$$

$$c(\theta) = c_0\left(1 + \delta\left(\tfrac{\theta-\theta_f}{\theta_0-\theta_f}\right)^p\right), \tag{1.7}$$

where δ and p are given non-negative constants, $k_0 = k\left(\theta_f\right)$ and $c_0 = c\left(\theta_f\right)$ are the reference coefficients of the thermal conductivity and the specific heat, respectively.

Some other models involving temperature-dependent thermal conductivity can also be found in Ceretani et al. (2018, 2020), Kumar and Singh (2020), Makinde et al. (2018), Natale and Tarzia (2003), Oliver and Sunderland (1987), Rogers (1985, 2015, 2019).

Existence and uniqueness to the problem (1.1)–(1.5) with null source term, $F = 0$, was developed in Bollati et al. (2020).

The control function F represents a heat source term for the nonlinear heat equation. Several applied papers give us the significance of the source term in the interior of the material which can undergo a change of phase (Scott 1994; Briozzo et al. 2007). In this paper we considered two different control functions F. The first one is defined as in Briozzo et al. (2007) and the second one depends on the evolution of the heat flux at the fixed face $x = 0$ like in Briozzo and Natale (2019). In this last case, we have a non-classical heat equation as in Tarzia and Villa (1998), Villa (1986).

We are interested in obtaining a similarity solution to problem (1.1)–(1.5) in which the temperature $\theta = \theta(x,t)$ can be written as a function of a single variable. Through the following change of variables:

$$y(\eta) = \tfrac{\theta(x,t)-\theta_f}{\theta_0-\theta_f} \geq 0, \tag{1.8}$$

where the similarity variable η is defined by:

$$\eta = \tfrac{x}{2a\sqrt{t}}, \quad 0 < x < s(t), \quad t > 0, \tag{1.9}$$

the phase front moves as

$$s(t) = 2a\lambda\sqrt{t}, \tag{1.10}$$

where $a^2 = \frac{k_0}{\rho c_0}$ (thermal diffusivity) and $\lambda > 0$ is a parameter to be determined.

The plan of this paper is the following. In Sect. 2, we prove the existence and uniqueness of solution to the problem (1.1)–(1.5) considering the control function given by Scott (1994):

$$F = F_1(x,t) = \frac{\rho l}{t}\beta\left(\frac{x}{2a\sqrt{t}}\right), \tag{1.11}$$

where $\beta = \beta(\eta)$ in a function with appropriate regularity properties (Scott 1994; Briozzo et al. 2007). Moreover, a particular case where β is of exponential type given by

$$\beta(\eta) = \frac{1}{2}\exp(-\eta^2), \tag{1.12}$$

is also studied in detail. This type of heat source term is important through the use of microwave energy following (Scott 1994).

Finally, in Sect. 3, we prove existence and uniqueness of solution to the problem (1.1)–(1.5) considering the control function given by

$$F = F_2(t) = \frac{\lambda_0}{\sqrt{t}}\frac{\partial T}{\partial x}(0,t), \tag{1.13}$$

that can be thought of by modelling of a system of temperature regulation in isotropic mediums (Briozzo and Natale 2019) with nonuniform source term, which provides a cooling or heating effect depending upon the properties of F_2 related to the heat flux (or the temperature in other cases) at the fixed face boundary $x = 0$.

2 Free boundary problem when the heat source term is of a similarity type

We consider now the control function F given by (1.11).

2.1 General case

Throughout this section, we will assume the following hypothesis on the function β:

$$H_\beta : \beta = \beta(\eta) \in C^1(\mathbb{R}_+) \text{ is such that } \beta(\cdot)exp(\cdot^2) \in L^1(\mathbb{R}_+).$$

Following the classical Neumann method, we propose a similarity type solution (θ, s) to the non-classical Stefan problem (1.1)–(1.5) given by:

$$\theta(x,t) = \left(\theta_0 - \theta_f\right) y\left(\tfrac{x}{2a\sqrt{t}}\right) + \theta_f, \quad 0 < x < s(t), \quad t > 0, \tag{2.1}$$

$$s(t) = 2a\lambda\sqrt{t}, \qquad t > 0. \tag{2.2}$$

Then, recalling that the similarity variable η is given by (1.9) we have:

$$\frac{\partial\theta}{\partial t}(x,t) = -\frac{1}{2t}(\theta_0 - \theta_f)\eta y'(\eta), \qquad \frac{\partial\theta}{\partial x}(x,t) = \frac{1}{2a\sqrt{t}}(\theta_0 - \theta_f)y'(\eta).$$

Replacing these expressions in equation (1.1), we obtain that the function y should satisfy:

$$2\eta(1+\delta y^p(\eta))y'(\eta) + [(1+\delta y^p(\eta))y'(\eta)]' = \tfrac{4}{\text{Ste}}\beta(\eta), \qquad 0 < \eta < \lambda, \tag{2.3}$$

where $\text{Ste} = \frac{c_0(\theta_0-\theta_f)}{l} > 0$ is the Stefan number.

Moreover, condition (1.2) implies that $\theta(0,t) = (\theta_0 - \theta_f)y(0) + \theta_f = \theta_0$ resulting in the following condition on the function y:

$$y(0) = 1. \tag{2.4}$$

In a similar way, taking into account that s is given by (2.2), we can obtain that condition (1.3) yields to $\theta(s(t),t) = (\theta_0 - \theta_f)y(\lambda) + \theta_f = \theta_f$ and then

$$y(\lambda) = 0. \tag{2.5}$$

Finally, the Stefan condition (1.4) is equivalent to $\frac{k_0}{2a\sqrt{t}}(\theta_0 - \theta_f)y'(\lambda) = -\rho l \frac{a\lambda}{\sqrt{t}}$. Taking into account the definition of the parameters a and Ste, we get

$$y'(\lambda) = -\tfrac{2\lambda}{\text{Ste}}. \tag{2.6}$$

Furthermore, it can be easily seen that if (y, λ) is a solution to the problem (2.3)–(2.6), then (θ, s) given by (2.1)–(2.2) verify the problem (1.1)–(1.5).

In conclusion, the Stefan problem (1.1)–(1.5) has a similarity solution (θ, s) given by (2.1)–(2.2) if and only if the pair (y, λ) satisfies the problem (2.3)–(2.6).

Lemma 2.1 *Assume that $p \geq 0$, $\delta \geq 0$, $\lambda > 0$, $y \in C^2[0,\lambda]$, $y \geq 0$, and $\beta = \beta(\eta)$ verifies the hypothesis H_β.*

Then, (y, λ) is a solution to the ordinary differential equation (2.3)–(2.6) if and only if $\lambda > 0$ is a solution to the equation:

$$\varphi_1(x) = 1 + \tfrac{\delta}{p+1}, \quad x > 0, \tag{2.7}$$

and function $y = y(\eta)$ satisfies the functional equation:

$$\Phi(y(\eta)) = \Psi_1(\eta), \quad 0 \le \eta \le \lambda, \tag{2.8}$$

where

$$\varphi_1(x) = \tfrac{\sqrt{\pi}}{Ste} x \,\mathrm{erf}(x) \exp(x^2) + \tfrac{2\sqrt{\pi}}{Ste} \int_0^x \exp(\xi^2)\,\mathrm{erf}(\xi)\beta(\xi)\,\mathrm{d}\xi, \tag{2.9}$$

$$\Phi(x) = x + \tfrac{\delta}{p+1} x^{p+1}, \quad 0 \le x \le 1, \tag{2.10}$$

$$\begin{aligned}\Psi_1(x) = 1 + \tfrac{\delta}{p+1} - \tfrac{\sqrt{\pi}\,\mathrm{erf}(x)}{\mathrm{Ste}} \left(2\int_0^\lambda \beta(\xi)\exp(\xi^2)\,\mathrm{d}\xi + \lambda \exp(\lambda^2)\right)\\ + \tfrac{2\sqrt{\pi}}{Ste}\left(\int_0^x \beta(\xi)\exp(\xi^2)\left(\mathrm{erf}(x) - \mathrm{erf}(\xi)\right)\,\mathrm{d}\xi\right), \quad 0 \le x \le \lambda.\end{aligned} \tag{2.11}$$

Proof Let (y, λ) be a solution to (2.3)–(2.6). As in Bollati et al. (2020), we define the function:

$$v(\eta) = \left(1 + \delta y^p(\eta)\right) y'(\eta). \tag{2.12}$$

Taking into account (2.3) and the condition (2.4), the function v can be rewritten as

$$v(\eta) = \exp\left(-\eta^2\right)\left(\tfrac{4}{\mathrm{Ste}}\int_0^\eta \beta(\xi)\exp(\xi^2)\,\mathrm{d}\xi + (1+\delta)y'(0)\right). \tag{2.13}$$

From (2.12) and (2.13), we get

$$\left(1 + \delta y^p(\eta)\right) y'(\eta) = \exp\left(-\eta^2\right)\left(\tfrac{4}{\mathrm{Ste}}\int_0^\eta \beta(\xi)\exp(\xi^2)\,\mathrm{d}\xi + (1+\delta)y'(0)\right). \tag{2.14}$$

Taking $\eta = \lambda$ in the above equation, using (2.5) and (2.6), we obtain:

$$y'(0) = -\tfrac{2}{\mathrm{Ste}(1+\delta)}\left(2\int_0^\lambda \beta(\xi)\exp(\xi^2)\,\mathrm{d}\xi + \lambda\exp\left(\lambda^2\right)\right). \tag{2.15}$$

Integrating equation (2.14) in the domain $(0, \eta)$ and by virtue of (2.4), it follows that:

$$\begin{aligned}y(\eta)\left(1 + \tfrac{\delta}{p+1} y^p(\eta)\right) = 1 + \tfrac{\delta}{p+1} + (1+\delta)y'(0)\tfrac{\sqrt{\pi}}{2}\,\mathrm{erf}(\eta)\\ + \tfrac{4}{\mathrm{Ste}}\int_0^\eta\int_\xi^\eta \beta(\xi)\exp(-z^2)\exp(\xi^2)\,\mathrm{d}z\,\mathrm{d}\xi.\end{aligned} \tag{2.16}$$

Given that

$$\int_0^\eta\int_\xi^\eta \beta(\xi)\exp(-z^2)\exp(\xi^2)\,\mathrm{d}z\,\mathrm{d}\xi = \tfrac{\sqrt{\pi}}{2}\int_0^\eta \left(\mathrm{erf}(\eta) - \mathrm{erf}(\xi)\right)\beta(\xi)\exp(\xi^2)\,\mathrm{d}\xi,$$

and from (2.15), we obtain that $y = y(\eta)$ is a solution to (2.8).

Taking $\eta = \lambda$ in equation (2.8) and using (2.5), we conclude that $\lambda > 0$ is a solution to equation (2.7).

Reciprocally, if (y, λ) is a solution to (2.7)–(2.8), then

$$\begin{aligned}y(\eta) = 1 + \tfrac{\delta}{p+1} - \tfrac{\delta}{p+1} y^{p+1}(\eta) - \tfrac{\sqrt{\pi}\,\mathrm{erf}(\eta)}{\mathrm{Ste}}\left(2\int_0^\lambda \beta(\xi)\exp(\xi^2)\,\mathrm{d}\xi + \lambda\exp(\lambda^2)\right)\\ + \tfrac{2\sqrt{\pi}}{\mathrm{Ste}}\left(\int_0^\eta \beta(\xi)\exp(\xi^2)\left(\mathrm{erf}(\eta) - \mathrm{erf}(\xi)\right)\,\mathrm{d}\xi\right),\end{aligned}$$

and it follows immediately that (y, λ) is a solution to (2.3)–(2.6). □

Lemma 2.2 *If $p \geq 0$, $\delta \geq 0$ and $\beta = \beta(\eta) \geq 0$ verifies the hypothesis H_β, then there exists a unique solution (y, λ) to the functional problem defined by (2.7)–(2.8) with $\lambda > 0$, $y \in C^2[0, \lambda]$ and $y \geq 0$.*

Proof It is easy to see that $\varphi_1(0) = 0$, $\varphi_1(+\infty) = +\infty$ and φ_1 is an increasing function. Then, there exists a unique $\lambda > 0$ solution to equation (2.7).
On the one hand, $\Phi(0) = 0$, $\Phi(1) = 1+\frac{\delta}{p+1}$ and Φ in an increasing function then, there exists the inverse function $\Phi^{-1} : [0, 1 + \frac{\delta}{p+1}] \to [0, 1]$. On the other hand, $\Psi_1(0) = 1 + \frac{\delta}{p+1}$, $\Psi_1(\lambda) = 0$ and Ψ_1 is a decreasing function. Furthermore, $\Psi_1(x) \in \left[0, 1 + \frac{\delta}{p+1}\right]$ for all $x \in [0, \lambda]$. Therefore, we conclude that there exists a unique function $y \in C^2[0, \lambda]$ solution to the equation (2.8) given by

$$y(\eta) = \Phi^{-1}\left(\Psi_1(\eta)\right), \quad 0 \leq \eta \leq \lambda. \tag{2.17}$$

□

From the above lemmas, we are able to claim the following result:

Theorem 2.3 *The Stefan problem governed by (1.1)–(1.5) has a unique similarity type solution (θ, s) given by (2.1)–(2.2) where (y, λ) is the unique solution to the functional problem (2.7)–(2.8).*

Remark 2.4 On the one hand, we have that Φ is an increasing function with $\Phi(0) = 0$ and $\Phi(1) = 1+\frac{\delta}{p+1}$. On the other hand, Ψ_1 is a decreasing function with $\Psi_1(0) = 1+\frac{\delta}{p+1}$ and $\Psi_1(\lambda) = 0$. Then it follows that $0 \leq y(\eta) \leq 1$, for $0 \leq \eta \leq \lambda$.

In virtue of this and Theorem 2.3, we have that $\theta_f < \theta(x, t) < \theta_0$, for all $0 < x < s(t)$, $t > 0$.

2.2 Particular case

Now, let us consider the particular case where β is of exponential type given by

$$\beta(\eta) = \frac{1}{2}\exp(-\eta^2).$$

Theorem 2.5 *If $p \geq 0$, $\delta \geq 0$, the Stefan problem governed by (1.1)–(1.5) has a unique similarity type solution (θ, s) given by (2.1)–(2.2) where $\lambda > 0$ is the unique solution to the equation:*

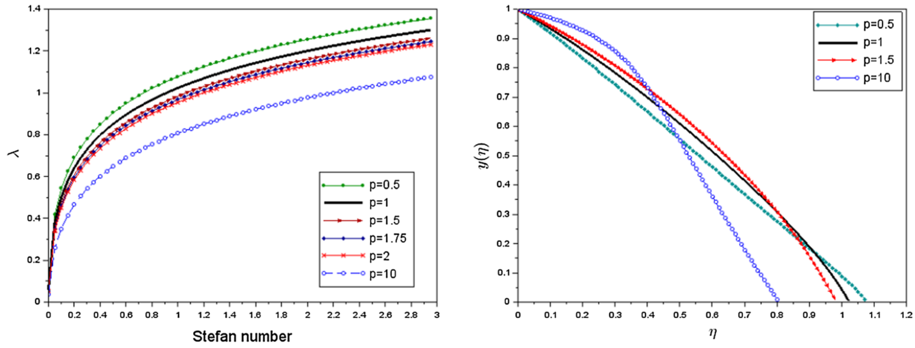
$$\mathcal{Z}(x) = \frac{\delta}{p+1}, \qquad x > 0, \tag{2.18}$$

with

$$\mathcal{Z}(x) = \frac{1}{\text{Ste}}\left(1 - \exp(-x^2)\right) + \frac{\sqrt{\pi}}{\text{Ste}} x \ \text{erf}(x) \left(\exp(x^2) - 1\right) - 1, \tag{2.19}$$

and the function $y \in C^\infty[0, \lambda]$, $y \geq 0$ satisfies the equation

$$\begin{aligned} y(\eta)\left(1 + \frac{\delta}{p+1} y^p(\eta)\right) &= 1 + \frac{\delta}{p+1} + \frac{1}{\text{Ste}}\left(\exp(-\eta^2) - 1\right) \\ &\quad + \frac{\sqrt{\pi}}{\text{Ste}} \lambda \ \text{erf}(\eta)\left(1 - \exp(\lambda^2)\right), \qquad 0 \leq \eta \leq \lambda. \end{aligned} \tag{2.20}$$



(a) Coefficient λ against Ste for $\delta = 5$ **(b)** Function $y = y(\eta)$ for $\text{Ste} = 1$, $\delta = 5$.

Fig. 1 Plot of the coefficient λ and the function $y = y(\eta)$

Remark 2.6 For the particular case $p = 1$, the unique function $y = y(\eta)$ solution to equation (2.20) is given by

$$y(\eta) = \tfrac{1}{\delta}\left[\sqrt{1 + 4\left(1 + \tfrac{\delta}{2}\right) + \tfrac{1}{\text{Ste}}\left(\exp(-\eta^2 - 1\right) + \tfrac{\sqrt{\pi}}{\text{Ste}}\lambda\,\text{erf}(\eta)\left(1 - \exp(\lambda^2)\right)} - 1\right],$$

where $\lambda > 0$ is the unique solution to equation (2.18) for $p = 1$, i.e,

$$\tfrac{1}{\text{Ste}}\left(1 - \exp(-x^2)\right) + \tfrac{\sqrt{\pi}}{\text{Ste}}x\ \text{erf}(x)\left(\exp(x^2) - 1\right) = 1 + \frac{\delta}{2}.$$

In Fig. 1a, we plot the solution λ to the equation (2.18) for different values of Ste and p, assuming $\delta = 5$.

Moreover in Fig. 1b, we plot the solution y to the equation (2.20) for different values or p, assuming $\delta = 5$ and $\text{Ste} = 1$.

Lemma 2.7 *For a fixed $p \geq 0$, let us define λ_p as the unique solution to equation (2.18). Then the following estimates hold:*

$$0 < \lambda_1 - \lambda_p = \mathcal{O}(p-1) \qquad \textit{when } p \to 1^+. \tag{2.21}$$

$$0 < \lambda_p - \lambda_1 = \mathcal{O}(1-p) \qquad \textit{when } p \to 1^-. \tag{2.22}$$

Proof To prove (2.21), let us consider the triangle with vertices $P_0(\lambda_p, \mathcal{Z}(\lambda_p))$, $P_1(\lambda_1, \mathcal{Z}(\lambda_p))$ and $P_2(\lambda_1, \mathcal{Z}(\lambda_1))$ where $\mathcal{Z}$ is the function given by (2.19). Taking into account that $p > 1$, if $\alpha_p = P_1\hat{P_0}P_2$ then

$$0 < \lambda_1 - \lambda_p = \tfrac{\delta(p-1)}{2(1+p)}\tfrac{1}{\tan(\alpha_p)}.$$

Notice that $\mathcal{Z}$ is an increasing convex function that satisfies $\mathcal{Z}(0) = -1$, $\mathcal{Z}(+\infty) = +\infty$. If we denote with $r > 0$, the unique root of $\mathcal{Z}$, then $0 < \mathcal{Z}'(r) < \mathcal{Z}'(\lambda_p) < \tan(\alpha_p)$. As a consequence, we get

$$0 < \lambda_1 - \lambda_p < \frac{\delta}{4\mathcal{Z}'(r)}(p-1).$$

We can prove equation (2.22) in a similar way. □

3 Free boundary problem with a heat source that depends on the evolution of the heat flux at the fixed face $x = 0$

If we consider that the control function F depends on the evolution of the heat flux at the fixed face $x = 0$, that is

$$F = F_2(t) = \frac{\lambda_0}{\sqrt{t}} \frac{\partial T}{\partial x}(0, t),$$

where $\lambda_0 > 0$, it is easy to see that the Stefan problem (1.1)–(1.5) has a similarity solution (θ, s) given by:

$$\theta(x, t) = \left(\theta_0 - \theta_f\right) y\left(\tfrac{x}{2a\sqrt{t}}\right) + \theta_f, \quad 0 < x < s(t), \quad t > 0, \tag{3.1}$$

$$s(t) = 2a\lambda\sqrt{t}, \qquad t > 0, \tag{3.2}$$

if and only if the function y and the parameter $\lambda > 0$ satisfy the following ordinary differential problem:

$$2\eta(1 + \delta y^p(\eta))y'(\eta) + [(1 + \delta y^p(\eta))y'(\eta)]' = Ay'(0), \qquad 0 < \eta < \lambda, \tag{3.3}$$

$$y(0) = 1, \tag{3.4}$$

$$y(\lambda) = 0, \tag{3.5}$$

$$y'(\lambda) = -\tfrac{2\lambda}{\text{Ste}}, \tag{3.6}$$

where $\delta \geq 0$, $p \geq 0$, $A = \frac{2\lambda_0}{\rho c_0 a}$ and $\text{Ste} = \frac{c_0(\theta_0 - \theta_f)}{l} > 0$ is the Stefan number.

Lemma 3.1 *Assume that $p \geq 0$, $\delta \geq 0$, $\lambda > 0$, $y \in C^\infty[0, \lambda]$ and $y \geq 0$. Then, (y, λ) is a solution to the ordinary differential equation (3.3)–(3.6) if and only if λ is a solution to the equation:*

$$\varphi_2(x) = 1 + \tfrac{\delta}{p+1}, \quad x > 0, \tag{3.7}$$

and the function $y = y(\eta)$ satisfies the equation:

$$\Phi\left(y(\eta)\right) = \Psi_2(\eta), \qquad 0 \leq \eta \leq \lambda, \tag{3.8}$$

where Φ is given by (2.10) and

$$\varphi_2(x) = \frac{\sqrt{\pi} x \exp(x^2)}{Ste\left(A\int_0^x \exp(z^2)\,dz + 1 + \delta\right)}\xi(x),$$

$$\Psi_2(x) = 1 + \frac{\delta}{p+1} - \frac{\sqrt{\pi}\lambda \exp(\lambda^2)}{Ste\left(A\int_0^\lambda \exp(z^2)\,dz + 1 + \delta\right)}\xi(x), \quad 0 \leq \eta \leq \lambda,$$

with

$$\xi(x) = A\int_0^x \exp(z^2)\left(\text{erf}(x) - \text{erf}(z)\right)\,dz + (1 + \delta)\,\text{erf}(x).$$

Proof Let (y, λ) be a solution to (3.3)–(3.6). As in Bollati et al. (2020), we define the function

$$v(\eta) = \left(1 + \delta y^p(\eta)\right) y'(\eta). \tag{3.9}$$

Taking into account (3.3) and the condition (3.4), the function v can rewrite as

$$v(\eta) = \exp(-\eta^2) y'(0)\left(A\int_0^\eta \exp(z^2)\,dz + 1 + \delta\right). \tag{3.10}$$

From (3.9) and (3.10), we obtain

$$\left(1+\delta y^{p}(\eta)\right) y'(\eta)=\exp\left(-\eta^{2}\right) y'(0)\left(A \int_{0}^{\eta} \exp(z^{2})\, \mathrm{d} z+1+\delta\right). \tag{3.11}$$

Taking $\eta=\lambda$ in the above equation, using (3.5) and (3.6), we obtain:

$$y'(0)=-\frac{2 \lambda \exp(\lambda^{2})}{\operatorname{Ste}\left(A \int_{0}^{\lambda} \exp(z^{2})\, \mathrm{d} z+1+\delta\right)}. \tag{3.12}$$

Integrating into $(0, \eta)$ equation (3.11) and by virtue of (3.4), we obtain:

$$\begin{aligned} y(\eta)\left(1+\tfrac{\delta}{p+1} y^{p}(\eta)\right)= & \, 1+\tfrac{\delta}{p+1}+(1+\delta) y'(0) \tfrac{\sqrt{\pi}}{2} \operatorname{erf}(\eta) \\ & +A y'(0) \int_{0}^{\eta} \int_{z}^{\eta} \exp(-z^{2}) \exp(\xi^{2})\, \mathrm{d} z\, \mathrm{d} \xi. \end{aligned} \tag{3.13}$$

Given that $\int_{0}^{\eta} \int_{z}^{\eta} \exp(-z^{2}) \exp(\xi^{2})\, \mathrm{d} z\, \mathrm{d} \xi=\frac{\sqrt{\pi}}{2} \int_{0}^{\eta}(\operatorname{erf}(\eta)-\operatorname{erf}(z)) \exp(z^{2})\, \mathrm{d} z$ and from (3.12), we obtain that $y=y(\eta)$ is a solution to (3.8). Taking $\eta=\lambda$ in equation (3.8) and using (3.5), we conclude that $\lambda>0$ is a solution to equation (3.7).

Reciprocally, if (y, λ) is a solution to (3.7)–(3.8),

$$\begin{aligned} y(\eta)= & \, 1+\tfrac{\delta}{p+1}-\tfrac{\delta}{p+1} y^{p+1}(\eta) \\ & -\frac{\sqrt{\pi} \lambda \exp(\lambda^{2})}{\operatorname{Ste}\left(A \int_{0}^{\lambda} \exp(z^{2})\, \mathrm{d} z+1+\delta\right)}\left(A \int_{0}^{\eta} \exp(z^{2})(\operatorname{erf}(\eta)-\operatorname{erf}(z))\, \mathrm{d} z+(1+\delta) \operatorname{erf}(\eta)\right), \end{aligned}$$

and it follows immediately that (y, λ) is a solution to (3.3)–(3.6). □

Lemma 3.2 *If $p \geq 0$, $\delta \geq 0$, then there exists a unique solution (y, λ) to the functional problem defined by (3.7)–(3.8) with $\lambda>0$, $y \in C^{\infty}[0, \lambda]$ and $y \geq 0$.*

Proof It is easy to see that $\varphi_{2}(0)=0$, $\varphi_{2}(+\infty)=+\infty$ and φ_{2} is an increasing function. Then, there exists a unique $\lambda>0$ solution to equation (3.7).

Let Φ be the function given by (2.10). On the one hand, $\Phi(0)=0$, $\Phi(1)=1+\frac{\delta}{p+1}$ and Φ in an increasing function then, there exists the function $\Phi^{-1}:[0,1+\frac{\delta}{p+1}] \rightarrow[0,1]$. On the other hand, $\Psi_{2}(0)=1+\frac{\delta}{p+1}$, $\Psi_{2}(\lambda)=0$ and Ψ_{2} is a decreasing function. Furthermore, $\Psi_{2}(x) \in\left[0,1+\frac{\delta}{p+1}\right]$ for all $x \in[0, \lambda]$.

We conclude that there exists a unique function $y \in C^{\infty}[0, \lambda]$ solution to the equation (3.8) given by

$$y(\eta)=\Phi^{-1}\left(\Psi_{2}(\eta)\right), \quad 0 \leq \eta \leq \lambda. \tag{3.14}$$

□

From the above lemmas, we are able to claim the following result:

Theorem 3.3 *The Stefan problem governed by (1.1)–(1.5) has a unique similarity type solution (θ, s) given by (3.1)–(3.2) where (y, λ) is the unique solution to the functional problem (3.7)–(3.8).*

Remark 3.4 On the one hand, we have that Φ is an increasing function with $\Phi(0)=0$ and $\Phi(1)=1+\frac{\delta}{p+1}$. On the other hand, Ψ_{2} is a decreasing function with $\Psi_{2}(0)=1+\frac{\delta}{p+1}$ and $\Psi_{2}(\lambda)=0$. Then it follows that $0 \leq y(\eta) \leq 1$, for $0 \leq \eta \leq \lambda$. From this and Theorem 3.3, we have that $\theta_{f}<\theta(x, t)<\theta_{0}$, for all $0<x<s(t), \quad t>0$.

4 Conclusion

One-dimensional non-classical Stefan problems with temperature-dependent thermal coefficients and a Dirichlet type condition at fixed face $x = 0$ for a semi-infinite phase-change material were considered. For two different types of heat sources, existence and uniqueness of solution were obtained using the similarity method and exact solutions were found.

Acknowledgements The present work has been partially sponsored by the European Union's Horizon 2020 Research and Innovation Programme under the Marie Sklodowska-Curie grant agreement 823731 CONMECH and the Projects 80020210100002 and 80020210200003 from Austral University, Rosario, Argentina.

Declarations

Conflict of interest The authors declare no potential conflict of interests.

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