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ORIGINAL RESEARCH

Explicit solutions for distributed, boundary and distributed-boundary elliptic optimal control problems

Julieta Bollati[1] · Claudia M. Gariboldi[2] · Domingo A. Tarzia[1]

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Abstract
We consider a steady-state heat conduction problem in a multidimensional bounded domain Ω for the Poisson equation with constant internal energy g and mixed boundary conditions given by a constant temperature b in the portion Γ_1 of the boundary and a constant heat flux q in the remaining portion Γ_2 of the boundary. Moreover, we consider a family of steady-state heat conduction problems with a convective condition on the boundary Γ_1 with heat transfer coefficient α and external temperature b. We obtain explicitly, for a rectangular domain in $\mathbb{R}^2$, an annulus in $\mathbb{R}^2$ and a spherical shell in $\mathbb{R}^3$, the optimal controls, the system states and adjoint states for the following optimal control problems: a *distributed* control problem on the internal energy g, a *boundary* optimal control problem on the heat flux q, a *boundary* optimal control problem on the external temperature b and a *distributed-boundary* simultaneous optimal control problem on the source g and the flux q. These explicit solutions can be used for testing new numerical methods as a benchmark test. In agreement with theory, it is proved that the system state, adjoint state, optimal controls and optimal values corresponding to the problem with a convective condition on Γ_1 converge, when $\alpha \to \infty$, to the corresponding system state, adjoint state, optimal controls and optimal values that arise from the problem with a temperature condition on Γ_1. Also, we analyze the order of convergence in each case, which turns out to be $1/\alpha$ being new for these kind of elliptic optimal control problems.

Keywords Elliptic variational equalities · Distributed and boundary optimal control problems · Mixed boundary conditions · Explicit solutions · Optimality conditions

Mathematics Subject Classification 35C05 · 35J25 · 35J86 · 35R35 · 49J20 · 49K20

✉ Domingo A. Tarzia
DTarzia@austral.edu.ar

Extended author information available on the last page of the article

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1 Introduction

The goal of this paper is to show the explicit solution for eight elliptic optimal control problems in two and three dimensional cases.

We consider a bounded domain Ω in $\mathbb{R}^n$ $(n = 2, 3)$, whose regular boundary Γ consist of the union of three disjoint portions Γ_1, Γ_2 and Γ_3 with $meas(\Gamma_1) > 0$, $meas(\Gamma_2) > 0$ and $meas(\Gamma_3) \geq 0$. We present the following steady-state heat conduction problems S and S_α (for each parameter $\alpha > 0$) respectively, with mixed boundary conditions

$$\Delta u = g, \quad \text{in } \Omega \qquad u\big|_{\Gamma_1} = b, \qquad -\frac{\partial u}{\partial n}\Big|_{\Gamma_2} = q, \qquad \frac{\partial u}{\partial n}\Big|_{\Gamma_3} = 0, \tag{1}$$

$$-\Delta u_\alpha = g \quad \text{in } \Omega \qquad -\frac{\partial u_\alpha}{\partial n}\Big|_{\Gamma_1} = \alpha(u - b), \qquad -\frac{\partial u_\alpha}{\partial n}\Big|_{\Gamma_2} = q, \qquad \frac{\partial u_\alpha}{\partial n}\Big|_{\Gamma_3} = 0, \tag{2}$$

where g is the internal energy in Ω, b is the temperature on Γ_1 for (1) and the temperature of the external neighborhood of Γ_1 for (2), q is the heat flux on Γ_2 and $\alpha > 0$ is the heat transfer coefficient on Γ_1. The above problems can be considered as the steady-state Stefan problems, [11,25–27]. Note that mixed boundary conditions play an important role in various applications, e.g. heat conduction and electric potential problems [16]. In general, the solution of a mixed elliptic boundary problems is not so regular [15] but there exist some examples which solutions are regular [1,20,24].

Let u and u_α the unique solutions of the elliptic problems (1) and (2), respectively. In relation with these state systems, we present the particular eight following optimal control problems [2,21,23,29].

1.1 Distributed optimal control on the constant internal energy *g*

Following [12], we consider the distributed optimal control problems:

$$\text{find} \quad g_{op} \in \mathbb{R} \quad \text{such that} \quad J_1(g_{op}) = \min_{g\in\mathbb{R}} J_1(g) \tag{3}$$

$$\text{find} \quad g_{\alpha_{op}} \in \mathbb{R} \quad \text{such that} \quad J_{1\alpha}(g_{\alpha_{op}}) = \min_{g\in\mathbb{R}} J_{1\alpha}(g) \tag{4}$$

with $J_1 : \mathbb{R}\to\mathbb{R}_0^+$ and $J_{1\alpha} : \mathbb{R}\to\mathbb{R}_0^+$, given by

$$J_1(g) = \frac{1}{2}\left\|u_g - z_d\right\|_H^2 + \frac{M_1}{2}\left\|g\right\|_H^2 \quad \text{and} \quad J_{1\alpha}(g) = \frac{1}{2}\left\|u_{\alpha g} - z_d\right\|_H^2 + \frac{M_1}{2}\left\|g\right\|_H^2$$

with $H = L^2(\Omega)$, and where u_g and $u_{\alpha g}$ denote the unique solutions of the problems (1) and (2) respectively, for data $q \in \mathbb{R}$, $b \in \mathbb{R}$, $z_d \in \mathbb{R}$ and M_1 a positive constant.

1.2 Boundary optimal control on the constant heat flux *q* on Γ_2

Following [13], we formulate the boundary optimal control problems:

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$$\text{find} \quad q_{op} \in \mathbb{R} \quad \text{such that} \quad J_2(q_{op}) = \min_{q \in \mathbb{R}} J_2(q) \tag{5}$$

$$\text{find} \quad q_{\alpha_{op}} \in \mathbb{R} \quad \text{such that} \quad J_{2\alpha}(q_{\alpha_{op}}) = \min_{q \in \mathbb{R}} J_{2\alpha}(q) \tag{6}$$

where $J_2 : \mathbb{R} \to \mathbb{R}_0^+$ and $J_{2\alpha} : \mathbb{R} \to \mathbb{R}_0^+$ given by

$$J_2(q) = \frac{1}{2} \left\| u_q - z_d \right\|_H^2 + \frac{M_2}{2} \left\| q \right\|_Q^2 \quad \text{and} \quad J_{2\alpha}(q) = \frac{1}{2} \left\| u_{\alpha q} - z_d \right\|_H^2 + \frac{M_2}{2} \left\| q \right\|_Q^2$$

with $Q = L^2(\Gamma_2)$ where u_q y $u_{\alpha q}$ are the unique solutions of the problems (1) and (2) respectively, for data $g \in \mathbb{R}$, $b \in \mathbb{R}$, $z_d \in \mathbb{R}$ and M_2 a positive constant.

1.3 Boundary optimal control on the constant temperature *b* in an external neighborhood of Γ_1

Following [3], we consider the boundary optimal control problems:

$$\text{find} \quad b_{op} \in \mathbb{R} \quad \text{such that} \quad J_3(b_{op}) = \min_{b \in \mathbb{R}} J_3(b) \tag{7}$$

$$\text{find} \quad b_{\alpha_{op}} \in \mathbb{R} \quad \text{such that} \quad J_{3\alpha}(b_{\alpha_{op}}) = \min_{b \in \mathbb{R}} J_{3\alpha}(q) \tag{8}$$

with $J_3 : \mathbb{R} \to \mathbb{R}_0^+$ and $J_{3\alpha} : \mathbb{R} \to \mathbb{R}_0^+$, given by

$$J_3(b) = \frac{1}{2} \left\| u_b - z_d \right\|_H^2 + \frac{M_3}{2} \left\| b \right\|_B^2 \quad \text{and} \quad J_{3\alpha}(b) = \frac{1}{2} \left\| u_{\alpha b} - z_d \right\|_H^2 + \frac{M_3}{2} \left\| b \right\|_B^2$$

with $B = L^2(\Gamma_1)$, where u_b y $u_{\alpha b}$ are the unique solutions of the problems (1) and (2) respectively, for data $g \in \mathbb{R}$, $q \in \mathbb{R}$, $z_d \in \mathbb{R}$ and M_3 a positive constant.

1.4 Simultaneous distributed-boundary optimal control on the constant source *g* and the constant flux *q*

Following [14], we formulate the simultaneous distributed-boundary optimal control problems:

$$\text{find} \quad (g,q)_{op} \in \mathbb{R} \times \mathbb{R} \quad \text{such that} \quad J_4((g,q)_{op}) = \min_{g \in \mathbb{R}, q \in \mathbb{R}} J_4(g,q) \tag{9}$$

$$\text{find} \quad (g,q)_{\alpha_{op}} \in \mathbb{R} \times \mathbb{R} \quad \text{such that} \quad J_{4\alpha}((g,q)_{\alpha_{op}}) = \min_{g \in \mathbb{R}, q \in \mathbb{R}} J_{4\alpha}(g,q) \tag{10}$$

with the cost functional $J_4 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}_0^+$ and $J_{4\alpha} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}_0^+$ given by

$$J_4(g,q) = \frac{1}{2} \left\| u_{(g,q)} - z_d \right\|_H^2 + \frac{M_4}{2} \left\| g \right\|_H^2 + \frac{M_5}{2} \left\| q \right\|_Q^2$$
$$J_{4\alpha}(g,q) = \frac{1}{2} \left\| u_{\alpha(g,q)} - z_d \right\|_H^2 + \frac{M_4}{2} \left\| g \right\|_H^2 + \frac{M_5}{2} \left\| q \right\|_Q^2$$

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where $u_{(g,q)}$ and $u_{\alpha(g,q)}$ are the unique solutions of the problems (1) and (2) respectively, for data $b \in \mathbb{R}$, $z_d \in \mathbb{R}$, M_4 and M_5 positive constants.

1.5 Adjoint states

We define the adjoint state corresponding to problems S and S_α as the unique solution of the following mixed elliptic problems, respectively.

$$-\Delta p = u - z_d, \quad \text{in } \Omega \qquad p\big|_{\Gamma_1} = 0, \qquad \frac{\partial p}{\partial n}\Big|_{\Gamma_2} = 0, \qquad \frac{\partial p}{\partial n}\Big|_{\Gamma_3} = 0, \quad (11)$$

and

$$-\Delta p_\alpha = u_\alpha - z_d, \quad \text{in } \Omega \quad -\frac{\partial p_\alpha}{\partial n}\Big|_{\Gamma_1} = \alpha p_\alpha, \quad \frac{\partial p_\alpha}{\partial n}\Big|_{\Gamma_2} = 0, \quad \frac{\partial p_\alpha}{\partial n}\Big|_{\Gamma_3} = 0 \quad (12)$$

with u and u_α given by the unique solution of (1) and (2), respectively. Other theoretical optimal control problems in the subject was done in [4,5,7–10,17–19,22,30].

In [3,12–14] were obtained results of existence and uniqueness of the optimal controls, as well also convergence results, when the heat transfer coefficient α goes to infinity, of the optimal controls, the system states and the adjoint states, in suitable Sobolev spaces.

In Sect. 2, we calculate explicitly the optimal controls, the system states and the adjoint states, for the optimal control problems previously formulated, related to S and S_α respectively, in a rectangular domain in $\mathbb{R}^2$. In Sects. 3 and 4, similar results are obtained in an annulus in $\mathbb{R}^2$ and a spherical shell in $\mathbb{R}^3$, respectively. In all cases, we obtain, in agreement with theory, the convergence of the optimal controls and values when $\alpha \to \infty$ as it was obtained in [3,12–14] and for numerical analysis in [28]. Also, the corresponding rates of convergence are studied in Theorem 2.5 for the first domain and in the Appendix A of our arXiv version [6] (page 23–26 for the second domain and page 27–29 for the third domain), obtaining that the order of convergence in each case is $1/\alpha$ which is new for these elliptic optimal control problems.

We remark that the expressions for the system states u, u_α, the adjoint states p, p_α, the functional cost J_i, $J_{i\alpha}$, $i = 1, \ldots, 4$, and the optimal controls are defined for each particular domain, using the same notation.

2 Optimal solutions for a rectangle in $\mathbb{R}^2$

In this Section, we consider a rectangular domain in the plane, that is

$$\Omega_1 = \{(x, y) \in \mathbb{R}^2 : 0 < x < x_0, \; 0 < y < y_0\}$$

whose boundaries Γ_1, Γ_2 and Γ_3 are given by (see Fig. 1):

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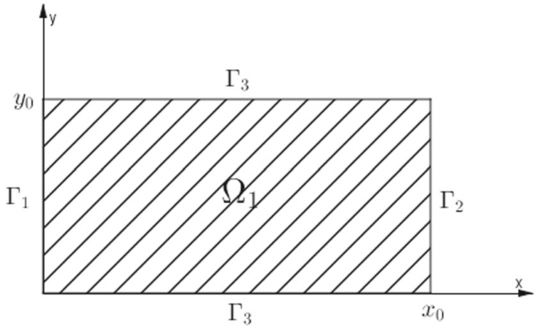

Fig. 1 Rectangular domain in the plane ($n = 2$)

$$\Gamma_1 = \{(x, y) \in \mathbb{R}^2 : x = 0,\ 0 \le y \le y_0\}, \quad \Gamma_2 = \{(x, y) \in \mathbb{R}^2 : x = x_0,\ 0 \le y \le y_0\},$$
$$\Gamma_3 = \{(x, y) \in \mathbb{R}^2 : y = 0,\ 0 < x < x_0\} \cup \{(x, y) \in \mathbb{R}^2 : y = y_0,\ 0 < x < x_0\}$$

If we consider constant data g, b, α, q and the desired system state $z_d \in \mathbb{R}$, we obtain the following result, which proof is omitted:

Lemma 2.1 *(i) The system state and adjoint state for the problem* (1) *and* (11) *respectively are given by:*

$$u(x, y) = u(x) = -g\frac{x^2}{2} + (gx_0 - q)x + b$$
$$p(x, y) = p(x) = g\frac{x^4}{24} - (gx_0 - q)\frac{x^3}{6} - (b - z_d)\frac{x^2}{2} + Ax$$

where $A = x_0\Big[g\frac{x_0^2}{3} - q\frac{x_0}{2} + (b - z_d)\Big]$.

(ii) The system state and adjoint state for the problem (2) *and* (12) *respectively take the expressions:*

$$u_\alpha(x, y) = u_\alpha(x) = -g\frac{x^2}{2} + (gx_0 - q)x + \frac{gx_0 - q}{\alpha} + b$$
$$p_\alpha(x, y) = p_\alpha(x) = g\frac{x^4}{24} - (gx_0 - q)\frac{x^3}{6}$$
$$- \left(\frac{gx_0 - q}{\alpha} + (b - z_d)\right)\frac{x^2}{2} + A_\alpha x + \frac{A_\alpha}{\alpha}$$

where $A_\alpha = x_0\left[gx_0^2\left(\frac{1}{3} + \frac{1}{\alpha x_0}\right) - qx_0\left(\frac{1}{2} - \frac{1}{\alpha x_0}\right) + (b - z_d)\right]$.

Remark 2.2 It is immediate that u_α converges to u and p_α to p, when $\alpha \to \infty$. Moreover, we can prove that there exists a positive constant $K_1 = K_1(x_0, y_0, g, q)$ such that:

$$||u_\alpha - u||_{H^1(\Omega_1)} = ||u_\alpha - u||_{L^2(\Omega_1)} = \frac{K_1}{\alpha}, \qquad K_1 = (x_0 y_0)^{1/2}|q - gx_0|.$$

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In the same way, a similar estimate can be obtained for the adjoint states p_α and p. It can be proved that there exists a positive constant $L_1 = L_1(x_0, y_0, g, q, b, z_d)$ such that:

$$\lim_{\alpha\to\infty} \alpha||p_\alpha - p||_{L^2(\Omega_1)} = L_1$$

where

$$L_1 = \left\{\frac{x_0^3 y_0}{180}\Big|180(b+z_d)^2 + 129q^2x_0^2 - 208gqx_0^3 + 84g^2x_0^4 - \right.$$
$$\left. -60(b-z_d)(5qx_0 - 4gx_0^2)\Big|\right\}^{1/2}.$$

Next, we present the following lemma that will allow us to find the solution of the optimal control problems:

Lemma 2.3 *(i) For the problem (1), it can be obtained that:*

$$\frac{1}{2}\|u - z_d\|^2_{L^2(\Omega_1)} = \frac{y_0}{2}\Big[C_1 g^2 x_0^5 + C_2 q^2 x_0^3 + C_3 x_0 (b - z_d)^2 + C_4 gqx_0^4$$
$$+ C_5 gx_0^3(b - z_d) + C_6 qx_0^2(b - z_d)\Big]$$

with:

$$C_1 = \frac{2}{15},\quad C_2 = \frac{1}{3},\quad C_3 = 1,\quad C_4 = -\frac{5}{12},\quad C_5 = \frac{2}{3},\quad C_6 = -1.$$

(ii) For the problem (2), we have:

$$\frac{1}{2}\|u_\alpha - z_d\|^2_{L^2(\Omega_1)} = \frac{y_0}{2}\Big[C_{1\alpha} g^2 x_0^5 + C_{2\alpha} q^2 x_0^3 + C_{3\alpha} x_0 (b - z_d)^2 + C_{4\alpha} gqx_0^4$$
$$+ C_{5\alpha} gx_0^3(b - z_d) + C_{6\alpha} qx_0^2(b - z_d)\Big]$$

with:

$$C_{1\alpha} = \frac{2}{15} + \frac{2}{3\alpha x_0} + \frac{1}{\alpha^2 x_0^2}\quad C_{2\alpha} = \frac{1}{3} + \frac{1}{\alpha x_0} + \frac{1}{\alpha^2 x_0^2}\quad C_{3\alpha} = 1 = C_3$$
$$C_{4\alpha} = -\frac{5}{12} - \frac{5}{3\alpha x_0} - \frac{2}{\alpha^2 x_0^2}\quad C_{5\alpha} = \frac{2}{3} + \frac{2}{\alpha x_0} C_{6\alpha} = -1 - \frac{2}{\alpha x_0}.$$

Remark 2.4 It is clear that $C_{i\alpha}$ converges to C_i, when $\alpha \to \infty$ for $i = 1, 2, \ldots, 6$.

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Theorem 2.5 *(i) For the distributed optimal control problems (3) and (4), the optimal controls are given by:*

$$g_{op} = -\frac{C_4 q x_0 + C_5(b - z_d)}{2x_0^2\left(C_1 + \frac{M_1}{x_0^4}\right)}, \qquad g_{\alpha_{op}} = -\frac{C_{4\alpha} q x_0 + C_{5\alpha}(b - z_d)}{2x_0^2\left(C_{1\alpha} + \frac{M_1}{x_0^4}\right)} \tag{13}$$

and the optimal values are given by:

$$J_1\left(g_{op}\right) = \frac{x_0 y_0}{8\left(C_1 + \frac{M_1}{x_0^4}\right)}\left[4\left(C_1 + \frac{M_1}{x_0^4}\right)\left(C_2 q^2 x_0^2 + C_3(b - z_d)^2 + C_6 q x_0 (b - z_d)\right) - \left(C_4 q x_0 + C_5(b - z_d)\right)^2\right] \tag{14}$$

and

$$J_{1\alpha}\left(g_{\alpha_{op}}\right) = \frac{x_0 y_0}{8\left(C_{1\alpha} + \frac{M_1}{x_0^4}\right)}\left[4\left(C_{1\alpha} + \frac{M_1}{x_0^4}\right)\left(C_{2\alpha} q^2 x_0^2 + C_{3\alpha}(b - z_d)^2 + C_{6\alpha} q x_0 (b - z_d)\right) - \left(C_{4\alpha} q x_0 + C_{5\alpha}(b - z_d)\right)^2\right] \tag{15}$$

(ii) For the boundary optimal control problems (5) and (6), the optimal controls are given by:

$$q_{op} = -\frac{C_4 g x_0^2 + C_6(b - z_d)}{2x_0\left(C_2 + \frac{M_2}{x_0^3}\right)}, \qquad q_{\alpha_{op}} = -\frac{C_{4\alpha} g x_0^2 + C_{6\alpha}(b - z_d)}{2x_0\left(C_{2\alpha} + \frac{M_2}{x_0^3}\right)} \tag{16}$$

and the optimal values can be expressed as:

$$J_2\left(q_{op}\right) = \frac{x_0 y_0}{8\left(C_2 + \frac{M_2}{x_0^3}\right)}\left[4\left(C_2 + \frac{M_2}{x_0^3}\right)\left(C_1 g^2 x_0^4 + C_3(b - z_d)^2 + C_5 g x_0^2 (b - z_d)\right) - \left(C_4 g x_0^2 + C_6(b - z_d)\right)^2\right] \tag{17}$$

and

$$J_{2\alpha}\left(q_{\alpha_{op}}\right) = \frac{x_0 y_0}{8\left(C_{2\alpha} + \frac{M_2}{x_0^3}\right)}\left[4\left(C_{2\alpha} + \frac{M_2}{x_0^3}\right)\left(C_{1\alpha} g^2 x_0^4 + C_{3\alpha}(b - z_d)^2 + C_{5\alpha} g x_0^2 (b - z_d)\right) - \left(C_{4\alpha} g x_0^2 + C_{6\alpha}(b - z_d)\right)^2\right] \tag{18}$$

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(iii) *For the boundary optimal control problems (7) and (8), the optimal controls are given by:*

$$b_{op} = -\frac{C_5 g x_0^2 + C_6 q x_0 - 2C_3 z_d}{2\left(C_3 + \frac{M_3}{x_0}\right)}, \qquad b_{\alpha_{op}} = -\frac{C_{5\alpha} g x_0^2 + C_{6\alpha} q x_0 - 2C_{3\alpha} z_d}{2\left(C_{3\alpha} + \frac{M_3}{x_0}\right)} \tag{19}$$

and the optimal values are:

$$\begin{aligned} J_3\left(b_{op}\right) = \frac{x_0 y_0}{8\left(C_3 + \frac{M_3}{x_0}\right)} \Big[& 4\Big(C_3 + \frac{M_3}{x_0}\Big)\Big(C_1 g^2 x_0^4 + C_2 q^2 x_0^2 + C_3 z_d^2 \\ & + C_4 g q x_0^3 - C_5 g x_0^2 z_d - C_6 q x_0 z_d\Big) \\ & - \Big(- 2C_3 z_d + C_5 g x_0^2 + C_6 q x_0\Big)^2\Big] \end{aligned} \tag{20}$$

and

$$\begin{aligned} J_{3\alpha}\left(b_{\alpha_{op}}\right) = \frac{x_0 y_0}{8\left(C_{3\alpha} + \frac{M_3}{x_0}\right)} \Big[& 4\Big(C_{3\alpha} + \frac{M_3}{x_0}\Big)\Big(C_{1\alpha} g^2 x_0^4 + C_{2\alpha} q^2 x_0^2 \\ & + C_{3\alpha} z_d^2 + C_{4\alpha} g q x_0^3 - C_{5\alpha} g x_0^2 z_d \\ & - C_{6\alpha} q x_0 z_d\Big) - \Big(- 2C_{3\alpha} z_d + C_{5\alpha} g x_0^2 + C_{6\alpha} q x_0\Big)^2\Big]. \end{aligned} \tag{21}$$

(iv) *For the distributed-boundary optimal control problem (9) and (10), the optimal solutions are given by:*

$$(g, q)_{op} = (g^{op}, q^{op}) = \left(\frac{(b - z_d)}{x_0^2}\Delta_1, \frac{(b - z_d)}{x_0}\Pi_1\right) \tag{22}$$

where

$$\Delta_1 = \frac{C_4 C_6 - 2C_5\left(C_2 + \frac{M_5}{x_0^3}\right)}{4\left(C_1 + \frac{M_4}{x_0^4}\right)\left(C_2 + \frac{M_5}{x_0^3}\right) - C_4^2}, \qquad \Pi_1 = \frac{C_4 C_5 - 2C_6\left(C_1 + \frac{M_4}{x_0^4}\right)}{4\left(C_1 + \frac{M_4}{x_0^4}\right)\left(C_2 + \frac{M_5}{x_0^3}\right) - C_4^2}$$

and

$$(g, q)_{\alpha_{op}} = (g_\alpha^{op}, q_\alpha^{op}) = \left(\frac{(b - z_d)}{x_0^2}\Delta_{1_\alpha}, \frac{(b - z_d)}{x_0}\Pi_{1_\alpha}\right) \tag{23}$$

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with

$$\Delta_{1_\alpha} = \frac{C_{4\alpha}C_{6\alpha} - 2C_{5\alpha}\left(C_{2\alpha} + \frac{M_5}{x_0^3}\right)}{4\left(C_{1\alpha} + \frac{M_4}{x_0^4}\right)\left(C_{2\alpha} + \frac{M_5}{x_0^3}\right) - C_{4\alpha}^2},$$

$$\Pi_{1_\alpha} = \frac{C_{4\alpha}C_{5\alpha} - 2C_{6\alpha}\left(C_{1\alpha} + \frac{M_4}{x_0^4}\right)}{4\left(C_{1\alpha} + \frac{M_4}{x_0^4}\right)\left(C_{2\alpha} + \frac{M_5}{x_0^3}\right) - C_{4\alpha}^2}$$

obtaining the following optimal values:

$$J_4\left(g^{op}, q^{op}\right) = \frac{x_0 y_0 (b - z_d)^2}{2\left(C_4^2 - 4\left(C_1 + \frac{M_4}{x_0^4}\right)\left(C_2 + \frac{M_5}{x_0^3}\right)\right)}\Big[-4C_3\left(C_1 + \frac{M_4}{x_0^4}\right)\left(C_2 + \frac{M_5}{x_0^3}\right) + C_6^2\left(C_1 + \frac{M_4}{x_0^4}\right) + C_5^2\left(C_2 + \frac{M_5}{x_0^3}\right) + C_3C_4^2 - C_4C_5C_6\Big] \tag{24}$$

and

$$J_{4\alpha}\left(g_\alpha^{op}, q_\alpha^{op}\right) = \frac{x_0 y_0 (b - z_d)^2}{2\left(C_{4\alpha}^2 - 4\left(C_{1\alpha} + \frac{M_4}{x_0^4}\right)\left(C_{2\alpha} + \frac{M_5}{x_0^3}\right)\right)} \Big[-4C_{3\alpha}\left(C_{1\alpha} + \frac{M_4}{x_0^4}\right)\left(C_{2\alpha} + \frac{M_5}{x_0^3}\right) + C_{6\alpha}^2\left(C_{1\alpha} + \frac{M_4}{x_0^4}\right) + C_{5\alpha}^2\left(C_{2\alpha} + \frac{M_5}{x_0^3}\right) + C_{3\alpha}C_{4\alpha}^2 - C_{4\alpha}C_{5\alpha}C_{6\alpha}\Big] \tag{25}$$

(v) *When $\alpha \to \infty$ the following convergences and estimates hold:*

(a) $g_{\alpha_{op}} \to g_{op}$ with $|g_{\alpha_{op}} - g_{op}| = \mathcal{O}\left(\frac{1}{\alpha}\right)$
(b) $q_{\alpha_{op}} \to q_{op}$ with $|q_{\alpha_{op}} - q_{op}| = \mathcal{O}\left(\frac{1}{\alpha}\right)$
(c) $b_{\alpha_{op}} \to b_{op}$ with $|b_{\alpha_{op}} - b_{op}| = \mathcal{O}\left(\frac{1}{\alpha}\right)$
(d) $(g,q)_{\alpha_{op}} \to (g,q)_{op}$ with $|g_\alpha^{op} - g^{op}| = \mathcal{O}\left(\frac{1}{\alpha}\right)$ and $|q_\alpha^{op} - q^{op}| = \mathcal{O}\left(\frac{1}{\alpha}\right)$

Moreover, when $\alpha \to \infty$, we have:

(a') $J_{1\alpha}\left(g_{\alpha_{op}}\right) \to J_1(g_{op})$ with $\left|J_{1\alpha}\left(g_{\alpha_{op}}\right) - J_1(g_{op})\right| = \mathcal{O}\left(\frac{1}{\alpha}\right)$
(b') $J_{2\alpha}\left(q_{\alpha_{op}}\right) \to J_2(q_{op})$ with $\left|J_{2\alpha}\left(q_{\alpha_{op}}\right) - J_2(q_{op})\right| = \mathcal{O}\left(\frac{1}{\alpha}\right)$
(c') $J_{3\alpha}\left(b_{\alpha_{op}}\right) \to J_3(b_{op})$ with $\left|J_{3\alpha}\left(b_{\alpha_{op}}\right) - J_3(b_{op})\right| = \mathcal{O}\left(\frac{1}{\alpha}\right)$
(d') $J_{4\alpha}\left((g,q)_{\alpha_{op}}\right) \to J_4((g,q)_{op})$ with $\left|J_{4\alpha}\left((g,q)_{\alpha_{op}}\right) - J_4((g,q)_{op})\right| = \mathcal{O}\left(\frac{1}{\alpha}\right)$.

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Proof (i) Taking into account that the functional J_1 and $J_{1\alpha}$ are given by the following quadratic forms

$$J_1(g) = \frac{y_0}{2}\Big[g^2\Big(C_1x_0^5 + M_1x_0\Big) + g\Big(C_4qx_0^4 + C_5x_0^3(b - z_d)\Big) + C_2q^2x_0^3 + C_3x_0(b - z_d)^2 + C_6qx_0^2(b - z_d)\Big]$$

and

$$J_{1\alpha}(g) = \frac{y_0}{2}\Big[g^2\Big(C_{1\alpha}x_0^5 + M_1x_0\Big) + g\Big(C_{4\alpha}qx_0^4 + C_{5\alpha}x_0^3(b - z_d)\Big) + C_{2\alpha}q^2x_0^3 + C_{3\alpha}x_0(b - z_d)^2 + C_{6\alpha}qx_0^2(b - z_d)\Big]$$

we obtain that the optimal solutions g_{op} and $g_{\alpha_{op}}$ for the problems (3) and (4) are given by (13) since the second derivative is positive in both cases.
In addition, if we evaluate the functional J_1 at g_{op} it is obtained formula (14). In a similar way, computing $J_{1\alpha}$ at $g_{\alpha_{op}}$ it can be derived the closed form (15).

(ii) The functional J_2 and $J_{2\alpha}$ are given by the expressions:

$$J_2(q) = \frac{y_0}{2}\left[q^2\left(C_2x_0^3 + M_2\right) + q\left(C_4gx_0^4 + C_6x_0^2(b - z_d)\right) + C_1g^2x_0^5 + C_3x_0(b - z_d)^2 + C_5gx_0^3(b - z_d)\right]$$

and

$$J_{2\alpha}(q) = \frac{y_0}{2}\left[q^2\left(C_{2\alpha}x_0^3 + M_2\right) + q\left(C_{4\alpha}gx_0^4 + C_{6\alpha}x_0^2(b - z_d)\right) + C_{1\alpha}g^2x_0^5 + C_{3\alpha}x_0(b - z_d)^2 + C_{5\alpha}gx_0^3(b - z_d)\right]$$

and then the corresponding minimum are given by (16) since the second derivative is positive in both cases. Evaluating J_2 and $J_{2\alpha}$ at q_{op} and $q_{\alpha_{op}}$ respectively, and through computations, the formulas (17) and (18) can be obtained.

(iii) For the problems (7) and (8), the functional J_3 and $J_{3\alpha}$ can be expressed as

$$J_3(b) = \frac{y_0}{2}\left[b^2\left(C_3x_0 + M_3\right) + b\left(-2C_3x_0z_d + C_5gx_0^3 + C_6qx_0^2\right) + C_1g^2x_0^5 + C_2q^2x_0^3 + C_3x_0z_d^2 + C_4gqx_0^4 - C_5gx_0^3z_d - C_6qx_0^2z_d\right]$$

and

$$J_{3\alpha}(b) = \frac{y_0}{2}\left[b^2\left(C_{3\alpha}x_0 + M_3\right) + b\left(-2C_{3\alpha}x_0z_d + C_{5\alpha}gx_0^3 + C_{6\alpha}qx_0^2\right) + C_{1\alpha}g^2x_0^5 + C_{2\alpha}q^2x_0^3 + C_{3\alpha}x_0z_d^2 + C_{4\alpha}gqx_0^4 - C_{5\alpha}gx_0^3z_d - C_{6\alpha}qx_0^2z_d\right]$$

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and therefore the optimal controls are given by (19) since the second derivative is positive in both cases. The formulas (20) and (21) are derived from evaluating J_3 and $J_{3\alpha}$ at b_{op} and $b_{\alpha_{op}}$.

(iv) For the distributed-boundary optimal control problems (9) and (10), the functional J_4 and $J_{4\alpha}$ can be written as:

$$J_4(g,q) = \frac{y_0}{2}\Big[g^2\left(C_1x_0^5 + M_4x_0\right) + q^2\left(C_2x_0^3 + M_5\right) + C_4gqx_0^4 + C_5gx_0^3(b-z_d) + C_6qx_0^2(b-z_d) + C_3x_0(b-z_d)^2\Big]$$

and

$$J_{4\alpha}(g,q) = \frac{y_0}{2}\Big[g^2\left(C_{1\alpha}x_0^5 + M_4x_0\right) + q^2\left(C_{2\alpha}x_0^3 + M_5\right) + C_{4\alpha}gqx_0^4 + C_{5\alpha}gx_0^3(b-z_d) + C_{6\alpha}qx_0^2(b-z_d) + C_{3\alpha}x_0(b-z_d)^2\Big].$$

Therefore, the optimal solutions of the problems (9) and (10), take the form (22) and (23), respectively, due to the second partial derivative test. In addition, the optimal values given by formulas (24) and (25) are deduced by evaluating J_4 at $(g,q)_{op}$ and $J_{4\alpha}$ at $(g,q)_{\alpha_{op}}$.

(v) The convergences can be easily proved by taking into account Remark 2.4 and the closed forms of the optimal controls and optimal values given by the preceding items (i)–(iv). Moreover, the following limits can be computed for the optimal controls:

$$\lim_{\alpha\to\infty} \alpha|g_{\alpha_{op}} - g_{op}| = \frac{5x_0\left|-150M_1qx_0 + 4(45M_1-2)(b-z_d)x_0^4 + 5qx_0^5\right|}{4\left(15M_1 + 2x_0^4\right)^2}$$

$$\lim_{\alpha\to\infty} \alpha|q_{\alpha_{op}} - q_{op}| = \frac{x_0\left|60gM_2x_0^2 + 5gx_0^5 + 12(6M_2 - x_0^3)(b-z_d)\right|}{8\left(3M_2 + x_0^3\right)^2}$$

$$\lim_{\alpha\to\infty} \alpha|b_{\alpha_{op}} - b_{op}| = \frac{x_0|q - gx_0|}{M_3 + x_0}$$

and for the simultaneous control we have:

$$\lim_{\alpha\to\infty} \alpha|g_\alpha^{op} - g^{op}| = \frac{40x_0(b-z_d)}{\mathcal{P}_1}\Big|-207360M_4M_5^2 - 8640M_4M_5x_0^3 - 1440M_4x_0^6 + 18432M_5^2x_0^4 + 168M_5x_0^7 + 3x_0^{10}\Big|$$

$$\lim_{\alpha\to\infty} \alpha|q_\alpha^{op} - q^{op}| = \frac{8x_0(b-z_d)}{\mathcal{P}_1}\Big|1036800M_4^2M_5 - 172800M_4^2x_0^3$$

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$$- 227520 M_4 M_5 x_0^4 - 7080 M_4 x_0^7 - 768 M_5 x_0^8 + 3 x_0^{11} \Big|$$

with

$$\mathcal{P}_1 = 3\left(2880 M_4 M_5 + 960 M_4 x_0^3 + 384 M_5 x_0^4 + 3x_0^7\right) \left(320 M_4\left(3M_5 + x_0^3\right) + 128 M_5 x_0^4 + x_0^7\right).$$

In the case of the optimal values, we have:

$$\lim_{\alpha\to\infty} \alpha\Big|J_{1\alpha}(g_{\alpha_{op}}) - J_1(g_{op})\Big| = \frac{x_0 y_0}{192\left(15M_1 + 2x_0^4\right)^2}\Big|\left(40(b-z_d)x_0^3 + 3q(40M_1 + 3x_0^4)\right) \left(8(b-z_d)(45M_1 + x_0^4) + q x_0(x_0^4 - 180M_1)\right)\Big|$$

$$\lim_{\alpha\to\infty} \alpha\Big|J_{2\alpha}(q_{\alpha_{op}}) - J_2(q_{op})\Big| = \frac{x_0^2 y_0}{128\left(3M_2 + x_0^3\right)^2}\Big|\left(-4(b-z_d)x_0 + g(8M_2 + x_0^3)\right) \left(12(b-z_d)(x_0^3 + 12M_2) + g x_0^2(48M_2 + x_0^3)\right)\Big|$$

$$\Big|J_{3\alpha}(b_{\alpha_{op}}) - J_3(b_{op})\Big| = \frac{1}{\alpha}\frac{\Big|M_3 x_0 y_0 (g x_0 - q)\left(2g x_0^2 - 3q x_0 - 6z_d\right)\Big|}{6(M_3 + x_0)}$$

$$\lim_{\alpha\to\infty} \alpha\Big|J_{4\alpha}(g_\alpha^{op}, q_\alpha^{op}) - J_4(g^{op}, q^{op})\Big| = \frac{64 x_0^3 y_0 (b-z_d)^2\left(120M_4 + 80M_5 x_0 + x_0^4\right)}{3\left(960 M_4 M_5 + 320 M_4 x_0^3 + 128 M_5 x_0^4 + x_0^7\right)^2} \left(180 M_4 M_5 + 15 M_4 x_0^3 + 4 M_5 x_0^4 + x_0^7\right).$$

□

3 Optimal solutions for an annulus in $\mathbb{R}^2$

We consider the following particular domain

$$\Omega_2 = \{(r,\theta) \in \mathbb{R}^2 : r_1 < r < r_2,\;\; 0 \le \theta < 2\pi\}$$

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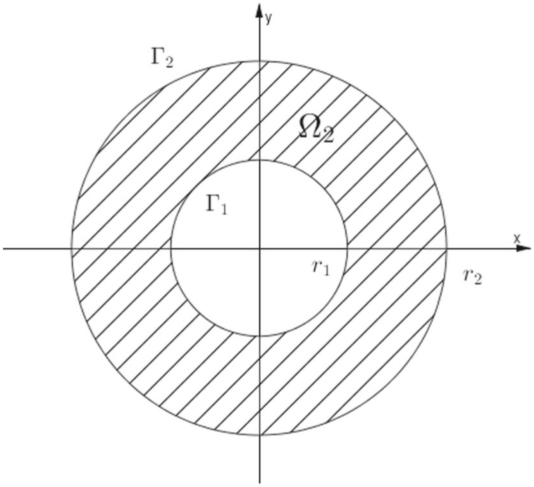

Fig. 2 Annulus in the plane ($n = 2$) and spherical shell in the space ($n = 3$)

with boundary Γ_1 and Γ_2 given by (see Fig. 2):

$$\Gamma_1 = \{(r,\theta) \in \mathbb{R}^2 : r = r_1,\ 0 \le \theta < 2\pi\}, \quad \Gamma_2 = \{(r,\theta) \in \mathbb{R}^2 : r = r_2,\ 0 \le \theta < 2\pi\}.$$

In similar way to previous Section, if we take constant data g, b, α, q and the desired system state $z_d \in \mathbb{R}$, we obtain the following result:

Lemma 3.1 *(i) The system state and the adjoint state for the problem* (1) *are given by*

$$u(r,\theta) = u(r) = g\frac{r_1^2}{2}\left(\left(\frac{r_2}{r_1}\right)^2 \log\left(\frac{r}{r_1}\right) - \frac{1}{2}\left(\frac{r}{r_1}\right)^2 + \frac{1}{2}\right) - qr_2 \log\left(\frac{r}{r_1}\right) + b$$

$$p(r,\theta) = p(r) = g\frac{r_1^2 r^2}{8}\left(\frac{1}{8}\left(\frac{r}{r_1}\right)^2 - \frac{1}{2} - \left(\frac{r_2}{r_1}\right)^2\left(\log\left(\frac{r}{r_1}\right) - 1\right)\right)$$
$$+ q\frac{r_2 r^2}{4}\left(\log\left(\frac{r}{r_1}\right) - 1\right) - (b - z_d)\frac{r^2}{4} + D_1 \log\left(\frac{r}{r_1}\right) + D_2$$

where

$$D_1 = \frac{r_2^2}{2}\left[g\frac{r_1^2}{2}\left(\left(\frac{r_2}{r_1}\right)^2\left(\log\left(\frac{r_2}{r_1}\right) - \frac{3}{4}\right) + \frac{1}{2}\right) - qr_2\left(\log\left(\frac{r_2}{r_1}\right) - \frac{1}{2}\right) + (b - z_d)\right]$$

$$D_2 = \frac{r_1^2}{4}\left[g\frac{r_1^2}{2}\left(\frac{3}{8} - \left(\frac{r_2}{r_1}\right)^2\right) + qr_2 + (b - z_d)\right].$$

(ii) The system state and the adjoint state for the problem (2) *are given by*

$$u_\alpha(r,\theta) = u_\alpha(r) = g\frac{r_1^2}{2}\left[\left(\frac{r_2}{r_1}\right)^2\left(\log\left(\frac{r}{r_1}\right) + \frac{1}{\alpha r_1}\right) - \frac{1}{2}\left(\frac{r}{r_1}\right)^2 + \frac{1}{2} - \frac{1}{\alpha r_1}\right]$$
$$- qr_2\left(\log\left(\frac{r}{r_1}\right) + \frac{1}{\alpha r_1}\right) + b$$

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$$p_\alpha(r,\theta) = p_\alpha(r) = g\frac{r_1^2r^2}{8}\left[\frac{1}{8}\left(\frac{r}{r_1}\right)^2 - \frac{1}{2} - \left(\frac{r_2}{r_1}\right)^2\left(\log\left(\frac{r}{r_1}\right) - 1 - \frac{r_1}{\alpha r_2^2} + \frac{1}{\alpha r_1}\right)\right]$$
$$+ q\frac{r_2r^2}{4}\left(\log\left(\frac{r}{r_1}\right) - 1 + \frac{1}{\alpha r_1}\right) - (b - z_d)\frac{r^2}{4}$$
$$+ D_{1\alpha}\log\left(\frac{r}{r_1}\right) + D_{2\alpha}$$

where

$$D_{1\alpha} = \frac{r_2^2}{2}\left[g\frac{r_1^2}{2}\left(\left(\frac{r_2}{r_1}\right)^2\left(\log\left(\frac{r_2}{r_1}\right) - \frac{3}{4} + \frac{1}{\alpha r_1}\right) + \frac{1}{2} - \frac{1}{\alpha r_1}\right)\right.$$
$$\left. - qr_2\left(\log\left(\frac{r_2}{r_1}\right) - \frac{1}{2} + \frac{1}{\alpha r_1}\right) + (b - z_d)\right]$$
$$D_{2\alpha} = \frac{r_1^2}{4}\left[g\frac{r_1^2}{2}\left(\frac{3}{8} - \left(\frac{r_2}{r_1}\right)^2\left(1 + \frac{r_1}{\alpha r_2^2} - \frac{2}{\alpha r_1} + \frac{2}{\alpha^2 r_1^2}\right) + \frac{2}{\alpha^2 r_1^2} - \frac{1}{2\alpha r_1}\right)\right.$$
$$\left. + qr_2\left(1 - \frac{2}{\alpha r_1} + \frac{2}{\alpha^2 r_1^2}\right) + (b - z_d)\left(1 - \frac{2}{\alpha r_1}\right)\right] + \frac{D_{1\alpha}}{\alpha r_1}.$$

Remark 3.2 From the formulas given above, it is clear that u_α converges to u and p_α to p, when $\alpha \to \infty$. Furthermore, we can prove that there exists a positive constant $K_2 = K_2(r_1, r_2, g, q)$ such that:

$$||u_\alpha - u||_{H^1(\Omega_2)} = ||u_\alpha - u||_{L^2(\Omega_2)} = \frac{K_2}{\alpha}$$

where

$$K_2 = \frac{\sqrt{\pi}\left(r_2^2 - r_1^2\right)^{1/2}|2qr_2 - g\left(r_2^2 - r_1^2\right)|}{2r_1}.$$

In the same way, a similar estimate can be obtained for the adjoint states p_α and p. In Appendix A of our arXiv version [6], it is proved that there exists a positive constant $L_2 = L_2(r_1, r_2, g, q, b, z_d)$ such that:

$$\lim_{\alpha\to\infty}\alpha||p_\alpha - p||_{L^2(\Omega_2)} = L_2$$

Now, we present the following lemma that will allow us to obtain the explicit solutions for the optimal control problems on the annulus in $\mathbb{R}^2$.

Lemma 3.3 *(i) For the problem* (1)*, it can be obtained that:*

$$\frac{1}{2}\|u - z_d\|^2_{L^2(\Omega_2)} = \pi\left[E_1g^2r_1^6 + E_2q^2r_1^4 + E_3r_1^2(b - z_d)^2 + E_4gqr_1^5\right.$$
$$\left. + E_5gr_1^4(b - z_d) + E_6qr_1^3(b - z_d)\right]$$

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with:

$$E_1 = \frac{1}{8}\left[-\frac{1}{12} + \frac{5}{8}\left(\frac{r_2}{r_1}\right)^2 + \left(\frac{r_2}{r_1}\right)^4\left(\log\left(\frac{r_2}{r_1}\right) - \frac{5}{4}\right) + \left(\frac{r_2}{r_1}\right)^6\left(\log^2\left(\frac{r_2}{r_1}\right) - \frac{3}{2}\log\left(\frac{r_2}{r_1}\right) + \frac{17}{24}\right)\right]$$
$$E_2 = \frac{1}{4}\left(-\left(\frac{r_2}{r_1}\right)^2 + \left(\frac{r_2}{r_1}\right)^4\left(2\log^2\left(\frac{r_2}{r_1}\right) - 2\log\left(\frac{r_2}{r_1}\right) + 1\right)\right)$$
$$E_3 = \frac{1}{2}\left(-1 + \left(\frac{r_2}{r_1}\right)^2\right)$$
$$E_4 = \frac{1}{4}\left[-\frac{3}{8}\left(\frac{r_2}{r_1}\right) + \left(\frac{r_2}{r_1}\right)^3\left(\frac{3}{2} - \log\left(\frac{r_2}{r_1}\right)\right) + \left(\frac{r_2}{r_1}\right)^5\left(-2\log^2\left(\frac{r_2}{r_1}\right) + \frac{5}{2}\log\left(\frac{r_2}{r_1}\right) - \frac{9}{8}\right)\right]$$
$$E_5 = \frac{1}{2}\left(-\frac{1}{4} + \left(\frac{r_2}{r_1}\right)^2 + \left(\frac{r_2}{r_1}\right)^4\left(\log\left(\frac{r_2}{r_1}\right) - \frac{3}{4}\right)\right)$$
$$E_6 = -\left(\frac{1}{2}\left(\frac{r_2}{r_1}\right) + \left(\frac{r_2}{r_1}\right)^3\left(\log\left(\frac{r_2}{r_1}\right) - \frac{1}{2}\right)\right)$$

(ii) For the problem (2), we have:

$$\frac{1}{2}\|u - z_d\|^2_{L^2(\Omega_2)} = \pi\Big[E_{1\alpha}g^2r_1^6 + E_{2\alpha}q^2r_1^4 + E_{3\alpha}r_1^2(b - z_d)^2 + \\ + E_{4\alpha}gqr_1^5 + E_{5\alpha}gr_1^4(b - z_d) + E_{6\alpha}qr_1^3(b - z_d)\Big]$$

with

$$E_{1\alpha} = \frac{1}{8}\Bigg[-\frac{1}{12} + \frac{1}{2\alpha r_1} - \frac{1}{\alpha^2 r_1^2} + \left(\frac{r_2}{r_1}\right)^2\left(\frac{5}{8} - \frac{5}{2\alpha r_1} + \frac{3}{\alpha^2 r_1^2}\right) + \left(\frac{r_2}{r_1}\right)^4\left(\log\left(\frac{r_2}{r_1}\right) - \frac{5}{4} + \frac{1}{\alpha r_1}\left(\frac{7}{2} - 2\log\left(\frac{r_2}{r_1}\right)\right) - \frac{3}{\alpha^2 r_1^2}\right) \\ + \left(\frac{r_2}{r_1}\right)^6\left(\log^2\left(\frac{r_2}{r_1}\right) - \frac{3}{2}\log\left(\frac{r_2}{r_1}\right) + \frac{17}{24} - \frac{3}{2\alpha r_1} + \frac{2}{\alpha r_1}\log\left(\frac{r_2}{r_1}\right) + \frac{1}{\alpha^2 r_1^2}\right)\Bigg]$$
$$E_{2\alpha} = \frac{1}{4}\Bigg[-\left(\frac{r_2}{r_1}\right)^2\left(1 + \frac{2}{\alpha^2 r_1^2} - \frac{2}{\alpha r_1}\right) \\ + \left(\frac{r_2}{r_1}\right)^4\left(2\log^2\left(\frac{r_2}{r_1}\right) - 2\log\left(\frac{r_2}{r_1}\right) + 1 + \frac{2}{\alpha^2 r_1^2} + \frac{1}{\alpha r_1}\left(4\log\left(\frac{r_2}{r_1}\right) - 2\right)\right)\Bigg]$$
$$E_{3\alpha} = E_3 = \frac{1}{2}\left(-1 + \left(\frac{r_2}{r_1}\right)^2\right)$$
$$E_{4\alpha} = \frac{1}{4}\Bigg[\frac{r_2}{r_1}\left(-\frac{3}{8} + \frac{3}{2\alpha r_1} - \frac{2}{\alpha^2 r_1^2}\right) \\ + \left(\frac{r_2}{r_1}\right)^3\left(\frac{3}{2} - \log\left(\frac{r_2}{r_1}\right) + \frac{1}{\alpha r_1}\left(2\log\left(\frac{r_2}{r_1}\right) - 4\right) + \frac{4}{\alpha^2 r_1^2}\right) \\ + \left(\frac{r_2}{r_1}\right)^5\left(-2\log^2\left(\frac{r_2}{r_1}\right) + \frac{5}{2}\log\left(\frac{r_2}{r_1}\right) - \frac{9}{8} - \frac{4}{\alpha r_1}\log\left(\frac{r_2}{r_1}\right) + \frac{5}{2\alpha r_1} - \frac{2}{\alpha^2 r_1^2}\right)\Bigg]$$

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$$E_{5\alpha} = \frac{1}{2}\left[-\frac{1}{4} + \frac{1}{\alpha r_1} + \left(\frac{r_2}{r_1}\right)^2\left(1 - \frac{2}{\alpha r_1}\right) + \left(\frac{r_2}{r_1}\right)^4\left(\log\left(\frac{r_2}{r_1}\right) - \frac{3}{4} + \frac{1}{\alpha r_1}\right)\right]$$

$$E_{6\alpha} = -\left(\frac{r_2}{r_1}\left(\frac{1}{2} - \frac{1}{\alpha r_1}\right) + \left(\frac{r_2}{r_1}\right)^3\left(\log\left(\frac{r_2}{r_1}\right) - \frac{1}{2} + \frac{1}{\alpha r_1}\right)\right).$$

Remark 3.4 It is immediate that $E_{i\alpha}$ converges to E_i, when $\alpha \to \infty$ for $i = 1, 2, \ldots, 6$.

Theorem 3.5 *(i) For the distributed optimal control problems* (3) *and* (4)*, the optimal solutions are given by:*

$$g_{op} = -\frac{E_4 q r_1 + E_5(b - z_d)}{2r_1^2\left(E_1 + E_3\frac{M_1}{r_1^4}\right)}, \qquad g_{\alpha_{op}} = -\frac{E_{4\alpha} q r_1 + E_{5\alpha}(b - z_d)}{2r_1^2\left(E_{1\alpha} + E_{3\alpha}\frac{M_1}{r_1^4}\right)} \tag{26}$$

and the optimal values can be expressed as:

$$J_1(g_{op}) = \frac{\pi r_1^2\left[4\left(E_1 + E_3\frac{M_1}{r_1^4}\right)\left(E_2 q^2 r_1^2 + E_3(b - z_d)^2 + E_6 q r_1(b - z_d)\right) - \left(E_4 q r_1 + E_5(b - z_d)\right)^2\right]}{4\left(E_1 + E_3\frac{M_1}{r_1^4}\right)} \tag{27}$$

and

$$J_{1\alpha}(g_{\alpha_{op}}) = \frac{\pi r_1^2\left[4\left(E_{1\alpha} + E_{3\alpha}\frac{M_1}{r_1^4}\right)\left(E_{2\alpha} q^2 r_1^2 + E_{3\alpha}(b - z_d)^2 + E_{6\alpha} q r_1(b - z_d)\right) - \left(E_{4\alpha} q r_1 + E_{5\alpha}(b - z_d)\right)^2\right]}{4\left(E_{1\alpha} + E_{3\alpha}\frac{M_1}{r_1^4}\right)} \tag{28}$$

(ii) For the boundary optimal control problems (5) *and* (6)*, the optimal solutions are given by:*

$$q_{op} = -\frac{E_4 g r_1^2 + E_6(b - z_d)}{2r_1\left(E_2 + \frac{M_2 r_2}{r_1^4}\right)}, \qquad q_{\alpha_{op}} = -\frac{E_{4\alpha} g r_1^2 + E_{6\alpha}(b - z_d)}{2r_1\left(E_{2\alpha} + \frac{M_2 r_2}{r_1^4}\right)} \tag{29}$$

where the optimal values are given by:

$$J_2(q_{op}) = \frac{\pi r_1^2\left[4\left(E_2 + \frac{M_2 r_2}{r_1^4}\right)\left(E_1 g^2 r_1^4 + E_3(b - z_d)^2 + E_5 g r_1^2(b - z_d)\right) - \left(E_4 g r_1^2 + E_6(b - z_d)\right)^2\right]}{4\left(E_2 + \frac{M_2 r_2}{r_1^4}\right)} \tag{30}$$

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and

$$J_{2\alpha}(q_{\alpha_{op}}) = \frac{\pi r_1^2\left[4\left(E_{2\alpha}+\frac{M_2 r_2}{r_1^4}\right)\left(E_{1\alpha}g^2 r_1^4+E_{3\alpha}(b-z_d)^2+E_{5\alpha}g r_1^2(b-z_d)\right)-\left(E_{4\alpha}g r_1^2+E_{6\alpha}(b-z_d)\right)^2\right]}{4\left(E_{2\alpha}+\frac{M_2 r_2}{r_1^4}\right)}. \tag{31}$$

(iii) *For the boundary optimal control problems* (7) *and* (8), *the optimal controls are given by*

$$b_{op} = -\frac{E_5 g r_1^2 + E_6 q r_1 - 2E_3 z_d}{2\left(E_3 + \frac{M_3}{r_1}\right)} \qquad b_{\alpha_{op}} = -\frac{E_{5\alpha} g r_1^2 + E_{6\alpha} q r_1 - 2E_{3\alpha} z_d}{2\left(E_{3\alpha} + \frac{M_3}{r_1}\right)} \tag{32}$$

respectively. In addition, the optimal values are given by:

$$\begin{aligned} J_3(b_{op}) = \Big[4\left(E_3 + \tfrac{M_3}{r_1}\right)\Big(&E_1 g^2 r_1^4 + E_2 q^2 r_1^2 + E_3 z_d^2 \\ &+E_4 g q r_1^3 - E_5 g r_1^2 z_d - E_6 q r_1 z_d\Big) \\ &-\left(-2E_3 z_d + E_5 g r_1^2 + E_6 q r_1\right)^2\Big] \frac{\pi r_1^2}{4\left(E_3+\frac{M_3}{r_1}\right)} \end{aligned} \tag{33}$$

and

$$\begin{aligned} J_{3\alpha}(b_{\alpha_{op}}) = \Big[4\left(E_{3\alpha} + \tfrac{M_3}{r_1}\right)\Big(&E_{1\alpha} g^2 r_1^4 + E_{2\alpha} q^2 r_1^2 + E_{3\alpha} z_d^2 \\ &+E_{4\alpha} g q r_1^3 - E_{5\alpha} g r_1^2 z_d - E_{6\alpha} q r_1 z_d\Big) \\ &-\left(-2E_{3\alpha} z_d + E_{5\alpha} g r_1^2 + E_{6\alpha} q r_1\right)^2\Big] \frac{\pi r_1^2}{4\left(E_{3\alpha}+\frac{M_3}{r_1}\right)}. \end{aligned} \tag{34}$$

(iv) *For the distributed-boundary optimal control problem* (9) *and* (10), *the optimal solutions are given by*

$$(g, q)_{op} = (g^{op}, q^{op}) = \left(\frac{(b - z_d)}{r_1^2}\Delta_2, \frac{(b - z_d)}{r_1}\Pi_2\right) \tag{35}$$

with

$$\Delta_2 = \frac{E_4 E_6 - 2E_5\left(E_2 + \frac{M_5 r_2}{r_1^4}\right)}{4\left(E_1 + E_3\frac{M_4}{r_1^4}\right)\left(E_2 + \frac{M_5 r_2}{r_1^4}\right) - E_4^2}, \qquad \Pi_2 = \frac{E_4 E_5 - 2E_6\left(E_1 + E_3\frac{M_4}{r_1^4}\right)}{4\left(E_1 + E_3\frac{M_4}{r_1^4}\right)\left(E_2 + \frac{M_5 r_2}{r_1^4}\right) - E_4^2}$$

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and

$$(g,q)_{\alpha_{op}}=(g_\alpha^{op},q_\alpha^{op})=\left(\frac{(b-z_d)}{r_1^2}\Delta_{2_\alpha},\frac{(b-z_d)}{r_1}\Pi_{2_\alpha}\right) \tag{36}$$

where

$$\Delta_{2_\alpha}=\frac{E_{4\alpha}E_{6\alpha}-2E_{5\alpha}\left(E_{2\alpha}+\frac{M_5r_2}{r_1^4}\right)}{4\left(E_{1\alpha}+E_{3\alpha}\frac{M_4}{r_1^4}\right)\left(E_{2\alpha}+\frac{M_5r_2}{r_1^4}\right)-E_{4\alpha}^2},\quad \Pi_{2_\alpha}=\frac{E_{4\alpha}E_{5\alpha}-2E_{6\alpha}\left(E_{1\alpha}+E_{3\alpha}\frac{M_4}{r_1^4}\right)}{4\left(E_{1\alpha}+E_{3\alpha}\frac{M_4}{r_1^4}\right)\left(E_{2\alpha}+\frac{M_5r_2}{r_1^4}\right)-E_{4\alpha}^2}$$

Moreover, the optimal values are given by

$$\begin{aligned}J_4(g^{op},q^{op})=&\frac{\pi r_1^2(b-z_d)^2}{\left(4\left(E_1+E_3\frac{M_4}{r_1^4}\right)\left(E_2+\frac{M_5r_2}{r_1^4}\right)-E_4^2\right)}\\&\times\left[4E_3\left(E_1+E_3\frac{M_4}{r_1^4}\right)\left(E_2+\frac{M_5r_2}{r_1^4}\right)\right.\\&\left.-E_6^2\left(E_1+E_3\frac{M_4}{r_1^4}\right)-E_5^2\left(E_2+\frac{M_5r_2}{r_1^4}\right)-E_3E_4^2+E_4E_5E_6\right]\end{aligned} \tag{37}$$

and

$$\begin{aligned}J_{4\alpha}(g_\alpha^{op},q_\alpha^{op})=&\frac{\pi r_1^2(b-z_d)^2}{\left(4\left(E_{1\alpha}+E_{3\alpha}\frac{M_4}{r_1^4}\right)\left(E_{2\alpha}+\frac{M_5r_2}{r_1^4}\right)-E_{4\alpha}^2\right)}\\&\times\left[4E_{3\alpha}\left(E_{1\alpha}+E_{3\alpha}\frac{M_4}{r_1^4}\right)\left(E_{2\alpha}+\frac{M_5r_2}{r_1^4}\right)\right.\\&-E_{6\alpha}^2\left(E_{1\alpha}+E_{3\alpha}\frac{M_4}{r_1^4}\right)-E_{5\alpha}^2\left(E_{2\alpha}+\frac{M_5r_2}{r_1^4}\right)\\&\left.-E_{3\alpha}E_{4\alpha}^2+E_{4\alpha}E_{5\alpha}E_{6\alpha}\right]\end{aligned} \tag{38}$$

(v) *The convergences and estimates obtained in (v) of Theorem 2.5 also hold for the annulus in* $\mathbb{R}^2$.

Proof (i) Taking into account that the functional J_1 and $J_{1\alpha}$ can be expressed in the following quadratic forms:

$$\begin{aligned}J_1(g)=\pi&\left[g^2(E_1r_1^6+M_1E_3r_1^2)+g\left(E_4qr_1^5+E_5(b-z_d)r_1^4\right)\right.\\&\left.+\left(E_2q^2r_1^4+E_3(b-z_d)^2r_1^2+E_6qr_1^2(b-z_d)\right)\right]\end{aligned}$$

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and

$$J_{1\alpha}(g) = \pi \left[g^2 \left(E_{1\alpha} r_1^6 + M_1 E_{3\alpha} r_1^2 \right) + g \left(E_{4\alpha} q r_1^5 + E_{5\alpha}(b - z_d) r_1^4 \right) + \left(E_{2\alpha} q^2 r_1^4 + E_{3\alpha}(b - z_d)^2 r_1^2 + E_{6\alpha} q r_1^2 (b - z_d) \right) \right]$$

it can be obtained that the optimal solutions g_{op} and $g_{\alpha_{op}}$ for the problems (3) and (4) are given by (26) since the second derivative is positive in both cases. The optimal values formulas (27) and (28) are deduced by evaluating J_1 and $J_{1\alpha}$ at g_{op} and $g_{\alpha_{op}}$, respectively.

(ii) The functional J_2 and $J_{2\alpha}$ are given by the expressions:

$$J_2(q) = \pi \left[q^2 \left(E_2 r_1^4 + M_2 r_2 \right) + q \left(E_4 r_1^5 g + E_6 r_1^3 (b - z_d) \right) + \left(E_1 r_1^6 g^2 + E_3 r_1^2 (b - z_d)^2 + E_5 r_1^4 g (b - z_d) \right) \right]$$

and

$$J_{2\alpha}(q) = \pi \left[q^2 \left(E_{2\alpha} r_1^4 + M_2 r_2 \right) + q \left(E_{4\alpha} r_1^5 g + E_{6\alpha} r_1^3 (b - z_d) \right) + \left(E_{1\alpha} r_1^6 g^2 + E_{3\alpha} r_1^2 (b - z_d)^2 + E_{5\alpha} r_1^4 g (b - z_d) \right) \right].$$

Therefore it is immediate that the optimal controls for problems (5) and (6) are given by (29) since the second derivative is positive in both cases.
The computation of $J_2(q_{op})$ and $J_{2\alpha}(q_{\alpha_{op}})$ leads to the closed formulas (30) and (31) for the optimal values of the control problems.

(iii) For the problems (7) and (8), the functional J_3 and $J_{3\alpha}$ are given by

$$J_3(b) = \pi \left[\left(E_3 r_1^2 + M_3 r_1 \right) b^2 + \left(-2 z_d E_3 r_1^2 + E_5 r_1^4 g + E_6 r_1^3 q \right) b + \left(E_1 r_1^6 g^2 + E_2 r_1^4 q^2 + E_3 r_1^2 z_d^2 + E_4 r_1^5 g q - E_5 r_1^4 g z_d - E_6 r_1^3 q z_d \right) \right]$$

and

$$J_{3\alpha}(b) = \pi \left[\left(E_{3\alpha} r_1^2 + M_3 r_1 \right) b^2 + \left(-2 z_d E_{3\alpha} r_1^2 + E_{5\alpha} r_1^4 g + E_{6\alpha} r_1^3 q \right) b + \left(E_{1\alpha} r_1^6 g^2 + E_{2\alpha} r_1^4 q^2 + E_{3\alpha} r_1^2 z_d^2 + E_{4\alpha} r_1^5 g q - E_{5\alpha} r_1^4 g z_d - E_{6\alpha} r_1^3 q z_d \right) \right].$$

Therefore the optimal controls are given by (32) since the second derivative is positive in both cases.
The optimal values given by expressions (33) and (34) are obtained by computing J_3 and $J_{3\alpha}$ at b_{op} and $b_{\alpha_{op}}$, respectively.

(iv) For the distributed-boundary optimal control problems (9) and (10), the functional J_4 can be expressed as

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$$J_4(g,q) = \pi\Big[(E_1 r_1^6 + M_4 E_3 r_1^2)g^2 + (E_2 r_1^4 + M_5 r_2)q^2 + E_4 r_1^5 gq$$
$$+ E_5 r_1^4 g(b - z_d) + E_6 r_1^3 q(b - z_d) + E_3 r_1^2 (b - z_d)^2\Big]$$

and the functional $J_{4\alpha}$ is given by:

$$J_{4\alpha}(g,q) = \pi\Big[(E_{1\alpha} r_1^6 + M_4 E_{3\alpha} r_1^2)g^2 + (E_{2\alpha} r_1^4 + M_5 r_2)q^2 + E_{4\alpha} r_1^5 gq$$
$$+ E_{5\alpha} r_1^4 g(b - z_d) + E_{6\alpha} r_1^3 q(b - z_d) + E_{3\alpha} r_1^2 (b - z_d)^2\Big]$$

from where it can be obtained that the optimal solutions are given by (35) and (36), respectively, due to the second partial derivative test. Formulas (37) and (38) are deduced by evaluating J_4 at $(g,q)_{op}$ and $J_{4\alpha}$ at $(g,q)_{\alpha_{op}}$.

(v) The convergences and estimates of the optimal controls and the optimal values when $\alpha \to \infty$ are obtained by taking into account the closed formulas given in (i)–(iv) and the Remark 3.4. As the computations become cumbersome, they can be found in the Appendix A (pages 23–26) of our arXiv version [6].

□

4 Optimal solutions for a spherical shell in $\mathbb{R}^3$

We consider the particular domain

$$\Omega_3 = \{(r,\theta,\phi) : r_1 < r < r_2;\ 0 \le \theta < 2\pi;\ 0 \le \phi \le \pi\}$$

with boundary $\Gamma = \cup_{i=1}^{2}\Gamma_i$, where

$$\Gamma_1 = \{(r_1,\theta,\phi) \in \mathbb{R}^3 :\ 0 \le \theta < 2\pi\ ,\ 0 \le \phi \le \pi\}$$
$$\Gamma_2 = \{(r_2,\theta,\phi) \in \mathbb{R}^3 :\ 0 \le \theta < 2\pi\ ,\ 0 \le \phi \le \pi\}.$$

In similar way to previous Sections, if we take constant data g, b, α, q and the desired system state $z_d \in \mathbb{R}$, we obtain the following result:

Lemma 4.1 *(i) The system state and the adjoint state for the problem* (1) *are given by*

$$u(r,\theta,\phi) = u(r) = g\frac{r_1^2}{3}\left[\frac{1}{2} - \frac{1}{2}\left(\frac{r}{r_1}\right)^2 + \left(\frac{r_2}{r_1}\right)^3 - \frac{r_2}{r}\left(\frac{r_2}{r_1}\right)^2\right] + q\frac{r_2^2}{r_1}\left(\frac{r_1}{r} - 1\right) + b$$

$$p(r,\theta,\phi) = p(r) = g r_1^2 \frac{r^2}{6}\left(\frac{1}{20}\left(\frac{r}{r_1}\right)^2 + \left(\frac{r_2}{r}\right)\left(\frac{r_2}{r_1}\right)^2 - \frac{1}{3}\left(\frac{r_2}{r_1}\right)^3 - \frac{1}{6}\right)$$
$$+ q r_2^2 \frac{r}{2}\left(\frac{r}{3r_1} - 1\right) - \frac{r^2}{6}(b - z_d) + \frac{F_1}{r} + F_2$$

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where

$$F_1 = gr_1^2 r_2^3\left(-\frac{1}{9}\left(\tfrac{r_2}{r_1}\right)^3 + \frac{1}{5}\left(\tfrac{r_2}{r_1}\right)^2 - \frac{1}{18}\right) + qr_2^4\left(\frac{1}{3}\left(\tfrac{r_2}{r_1}\right) - \frac{1}{2}\right) - \frac{r_2^3}{3}(b - z_d)$$

$$F_2 = g\frac{r_1^4}{9}\left(\frac{7}{40} - \left(\tfrac{r_2}{r_1}\right)^3\right) + qr_1\frac{r_2^2}{3} + \frac{r_1^2}{6}(b - z_d) - \frac{F_1}{r_1}.$$

(ii) *The system state and the adjoint state for the problem* (2) *are given by*

$$u_\alpha(r,\theta,\phi) = u_\alpha(r) = g\frac{r_1^2}{3}\left[\frac{1}{2} - \frac{1}{\alpha r_1} - \frac{1}{2}\left(\frac{r}{r_1}\right)^2 + \left(\tfrac{r_2}{r_1}\right)^3\left(1 + \frac{1}{\alpha r_1}\right) - \frac{r_2}{r}\left(\tfrac{r_2}{r_1}\right)^2\right]$$
$$+ q\frac{r_2^2}{r_1}\left(\frac{r_1}{r} - 1 - \frac{1}{\alpha r_1}\right) + b$$

$$p_\alpha(r,\theta,\phi) = p_\alpha(r) = gr_1^2\frac{r^2}{6}\left(\frac{1}{20}\left(\frac{r}{r_1}\right)^2 + \left(\frac{r_2}{r}\right)\left(\tfrac{r_2}{r_1}\right)^2\right.$$
$$\left. - \frac{1}{3}\left(\tfrac{r_2}{r_1}\right)^3\left(1 + \frac{1}{\alpha r_1}\right) - \frac{1}{6} + \frac{1}{3\alpha r_1}\right)$$
$$+ qr_2^2\frac{r}{2}\left(\frac{r}{3r_1} - 1 + \frac{r}{3\alpha r_1^2}\right) - \frac{r^2}{6}(b - z_d) + \frac{F_{1\alpha}}{r} + F_{2\alpha}$$

where

$$F_{1\alpha} = gr_1^2 r_2^3\left(-\frac{1}{9}\left(\tfrac{r_2}{r_1}\right)^3\left(1 + \frac{1}{\alpha r_1}\right) + \frac{1}{5}\left(\tfrac{r_2}{r_1}\right)^2 - \frac{1}{18}\left(1 - \frac{2}{\alpha r_1}\right)\right)$$
$$+ qr_2^4\left(\frac{1}{3}\left(\tfrac{r_2}{r_1}\right)\left(1 + \frac{1}{\alpha r_1}\right) - \frac{1}{2}\right) - \frac{r_2^3}{3}(b - z_d)$$
$$F_{2\alpha} = g\frac{r_1^4}{9}\left[\frac{7}{40} - \frac{7}{10\alpha r_1} + \frac{1}{\alpha^2 r_1^2} - \left(\tfrac{r_2}{r_1}\right)^3\left(1 - \frac{1}{\alpha r_1} + \frac{1}{\alpha^2 r_1^2}\right)\right]$$
$$+ qr_1\frac{r_2^2}{3}\left(1 - \frac{1}{\alpha r_1} + \frac{1}{\alpha^2 r_1^2}\right) + \frac{r_1^2}{6}(b - z_d)\left(1 - \frac{2}{\alpha r_1}\right) - \frac{F_{1\alpha}}{r_1}\left(1 + \frac{1}{\alpha r_1}\right).$$

Remark 4.2 The convergences of u_α to u, and p_α to p, when $\alpha \to \infty$ can be immediately verified.

In addition, there exists a positive constant $K_3 = K_3(r_1, r_2, g, q)$ such that:

$$||u_\alpha - u||_{H^1(\Omega_3)} = \frac{K_3}{\alpha}, \qquad K_3 = \left(\frac{4\pi(r_2^3 - r_1^3)(3qr_2^2 + g(r_1^3 - r_2^3))^2}{27r_1^4}\right)^{1/2}.$$

Analogously, a similar estimate can be proved for the adjoint states p_α and p (see Appendix A in our arXiv version (page 23) [6]).

Now, we present the following lemma that will allow us to obtain the explicit solutions for the optimal control problems on the spherical shell in $\mathbb{R}^3$.

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Lemma 4.3 *(i) For the problem (1), it can be obtained that:*

$$\frac{1}{2}\|u-z_d\|^2_{L^2(\Omega_3)} = \pi\Big[G_1 r_1^7 g^2 + G_2 r_1 r_2^4 q^2 + G_3 r_1^3 (b-z_d)^2 + G_4 r_1^4 r_2^2 gq + G_5 r_1^5 g(b-z_d) + G_6 r_1^2 r_2^2 q(b-z_d)\Big]$$

with:

$$G_1 = -\tfrac{2}{945} + \tfrac{1}{45}\left(\tfrac{r_2}{r_1}\right)^3 - \tfrac{1}{15}\left(\tfrac{r_2}{r_1}\right)^5 + \tfrac{1}{7}\left(\tfrac{r_2}{r_1}\right)^7 - \tfrac{2}{15}\left(\tfrac{r_2}{r_1}\right)^8 + \tfrac{1}{27}\left(\tfrac{r_2}{r_1}\right)^9$$

$$G_2 = -\tfrac{1}{3} + \tfrac{r_2}{r_1} - \left(\tfrac{r_2}{r_1}\right)^2 + \tfrac{1}{3}\left(\tfrac{r_2}{r_1}\right)^3$$

$$G_3 = \tfrac{1}{3}\left(-1 + \left(\tfrac{r_2}{r_1}\right)^3\right)$$

$$G_4 = -\tfrac{7}{180} + \tfrac{1}{6}\left(\tfrac{r_2}{r_1}\right)^2 + \tfrac{1}{9}\left(\tfrac{r_2}{r_1}\right)^3 - \tfrac{3}{4}\left(\tfrac{r_2}{r_1}\right)^4 + \tfrac{11}{15}\left(\tfrac{r_2}{r_1}\right)^5 - \tfrac{2}{9}\left(\tfrac{r_2}{r_1}\right)^6$$

$$G_5 = -\tfrac{2}{45} + \tfrac{2}{9}\left(\tfrac{r_2}{r_1}\right)^3 - \tfrac{2}{5}\left(\tfrac{r_2}{r_1}\right)^5 + \tfrac{2}{9}\left(\tfrac{r_2}{r_1}\right)^6$$

$$G_6 = -\tfrac{1}{3} + \left(\tfrac{r_2}{r_1}\right)^2 - \tfrac{2}{3}\left(\tfrac{r_2}{r_1}\right)^3$$

(ii) For the problem (2), we have:

$$\frac{1}{2}\|u_\alpha-z_d\|^2_{L^2(\Omega_3)} = \pi\Big[G_{1\alpha} r_1^7 g^2 + G_{2\alpha} r_1 r_2^4 q^2 + G_{3\alpha} r_1^3 (b-z_d)^2 + G_{4\alpha} r_1^4 r_2^2 gq + G_{5\alpha} r_1^5 g(b-z_d) + G_{6\alpha} r_1^2 r_2^2 q(b-z_d)\Big]$$

with

$$G_{1\alpha} = -\tfrac{2}{945} + \tfrac{2}{135\alpha r_1} - \tfrac{1}{27\alpha^2 r_1^2} + \left(\tfrac{r_2}{r_1}\right)^3\left(\tfrac{1}{45} - \tfrac{4}{45\alpha r_1} + \tfrac{1}{9\alpha^2 r_1^2}\right) - \tfrac{1}{15}\left(\tfrac{r_2}{r_1}\right)^5\left(1 - \tfrac{2}{\alpha r_1}\right) - \tfrac{1}{9\alpha^2 r_1^2}\left(\tfrac{r_2}{r_1}\right)^6 + \tfrac{1}{7}\left(\tfrac{r_2}{r_1}\right)^7 - \tfrac{2}{15}\left(\tfrac{r_2}{r_1}\right)^8\left(1 + \tfrac{1}{\alpha r_1}\right) + \tfrac{1}{27}\left(\tfrac{r_2}{r_1}\right)^9\left(1 + \tfrac{2}{\alpha r_1} + \tfrac{1}{\alpha^2 r_1^2}\right)$$

$$G_{2\alpha} = -\tfrac{1}{3}\left(1 - \tfrac{1}{\alpha r_1} + \tfrac{1}{\alpha^2 r_1^2}\right) + \tfrac{r_2}{r_1} - \left(\tfrac{r_2}{r_1}\right)^2\left(1 + \tfrac{1}{\alpha r_1}\right) + \tfrac{1}{3}\left(\tfrac{r_2}{r_1}\right)^3\left(1 + \tfrac{2}{\alpha r_1} + \tfrac{1}{\alpha^2 r_1^2}\right)$$

$$G_{3\alpha} = G_3 = \tfrac{1}{3}\left(-1 + \left(\tfrac{r_2}{r_1}\right)^3\right)$$

$$G_{4\alpha} = -\tfrac{7}{180} + \tfrac{7}{45\alpha r_1} - \tfrac{2}{\alpha^2 r_1^2} + \tfrac{1}{6}\left(\tfrac{r_2}{r_1}\right)^2\left(1 - \tfrac{2}{\alpha r_1}\right) + \tfrac{1}{9}\left(\tfrac{r_2}{r_1}\right)^3\left(1 - \tfrac{1}{\alpha r_1} + \tfrac{4}{\alpha^2 r_1^2}\right) - \tfrac{3}{4}\left(\tfrac{r_2}{r_1}\right)^4 + \tfrac{11}{15}\left(\tfrac{r_2}{r_1}\right)^5\left(1 + \tfrac{1}{\alpha r_1}\right) - \tfrac{2}{9}\left(\tfrac{r_2}{r_1}\right)^6\left(1 + \tfrac{2}{\alpha r_1} + \tfrac{1}{\alpha^2 r_1^2}\right)$$

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$$G_{5\alpha} = -\tfrac{2}{45} + \tfrac{2}{9\alpha r_1} + \tfrac{2}{9}\left(\tfrac{r_2}{r_1}\right)^3\left(1 - \tfrac{2}{\alpha r_1}\right) - \tfrac{2}{5}\left(\tfrac{r_2}{r_1}\right)^5 + \tfrac{2}{9}\left(\tfrac{r_2}{r_1}\right)^6\left(1 + \tfrac{1}{\alpha r_1}\right)$$
$$G_{6\alpha} = -\tfrac{1}{3}\left(1 - \tfrac{2}{\alpha r_1}\right) + \left(\tfrac{r_2}{r_1}\right)^2 - \tfrac{2}{3}\left(\tfrac{r_2}{r_1}\right)^3\left(1 + \tfrac{1}{\alpha r_1}\right)$$

Remark 4.4 It is clear that $G_{i\alpha}$ converges to G_i, when $\alpha \to \infty$ for $i = 1, 2, \ldots, 6$.

Theorem 4.5 *(i) For the distributed optimal control problems* (3) *and* (4)*, the optimal solutions are given by:*

$$g_{op} = -\frac{G_4 q \frac{r_2^2}{r_1} + G_5(b - z_d)}{2r_1^2\left(G_1 + G_3\frac{M_1}{r_1^4}\right)}, \qquad g_{\alpha_{op}} = -\frac{G_{4\alpha} q \frac{r_2^2}{r_1} + G_{5\alpha}(b - z_d)}{2r_1^2\left(G_{1\alpha} + G_{3\alpha}\frac{M_1}{r_1^4}\right)}. \tag{39}$$

The optimal values corresponding to those optimal controls are given by the following formulas:

$$J_1(g_{op}) = \left[4\left(G_1 + G_3\frac{M_1}{r_1^4}\right)\left(G_2 q^2\frac{r_2^4}{r_1^2} + G_3(b - z_d)^2 + G_6 q\frac{r_2^2}{r_1}(b - z_d)\right) - \left(G_4 q\frac{r_2^2}{r_1} + G_5(b - z_d)\right)^2\right]\frac{\pi r_1^3}{2\left(G_1 + G_3\frac{M_1}{r_1^4}\right)} \tag{40}$$

and

$$J_{1\alpha}(g_{\alpha_{op}}) = \left[4\left(G_{1\alpha} + G_{3\alpha}\frac{M_1}{r_1^4}\right)\left(G_{2\alpha} q^2\frac{r_2^4}{r_1^2} + G_{3\alpha}(b - z_d)^2 + G_{6\alpha} q\frac{r_2^2}{r_1}(b - z_d)\right) - \left(G_{4\alpha} q\frac{r_2^2}{r_1} + G_{5\alpha}(b - z_d)\right)^2\right]\frac{\pi r_1^3}{2\left(G_{1\alpha} + G_{3\alpha}\frac{M_1}{r_1^4}\right)} \tag{41}$$

(ii) For the boundary optimal control problems (5) *and* (6)*, the optimal solutions are given by:*

$$q_{op} = -\frac{r_1}{2r_2^2}\frac{\left(G_4 g r_1^2 + G_6(b - z_d)\right)}{\left(G_2 + \frac{M_2}{r_1 r_2^2}\right)}, \quad q_{\alpha_{op}} = -\frac{r_1}{2r_2^2}\frac{\left(G_{4\alpha} g r_1^2 + G_{6\alpha}(b - z_d)\right)}{\left(G_{2\alpha} + \frac{M_2}{r_1 r_2^2}\right)}. \tag{42}$$

The corresponding optimal values can be expressed by:

$$J_2(q_{op}) = \left[4\left(G_2 + \frac{M_2}{r_1 r_2^2}\right)\left(G_1 g^2 r_1^4 + G_3(b - z_d)^2 + G_5 g r_1^2(b - z_d)\right) - \left(G_4 g r_1^2 + G_6(b - z_d)\right)^2\right]\frac{\pi r_1^3}{2\left(G_2 + \frac{M_2}{r_1 r_2^2}\right)} \tag{43}$$

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and

$$J_{2\alpha}(q_{\alpha_{op}}) = \Bigg[4\left(G_{2\alpha} + \tfrac{M_2}{r_1 r_2^2}\right)\left(G_{1\alpha} g^2 r_1^4 + G_{3\alpha}(b - z_d)^2 + G_{5\alpha} g r_1^2 (b - z_d)\right) - \left(G_{4\alpha} g r_1^2 + G_{6\alpha}(b - z_d)\right)^2\Bigg] \frac{\pi r_1^3}{2\left(G_{2\alpha} + \frac{M_2}{r_1 r_2^2}\right)}. \quad (44)$$

(iii) *For the boundary optimal control problems* (7) *and* (8)*, the optimal controls are given by*

$$b_{op} = -\frac{G_5 g r_1^2 + G_6 q \frac{r_2^2}{r_1} - 2G_3 z_d}{2\left(G_3 + \frac{M_3}{r_1}\right)}, \quad b_{\alpha_{op}} = -\frac{G_{5\alpha} g r_1^2 + G_{6\alpha} q \frac{r_2^2}{r_1} - 2G_{3\alpha} z_d}{2\left(G_{3\alpha} + \frac{M_3}{r_1}\right)}. \quad (45)$$

Moreover, $J_3(b_{op})$ *and* $J_{3\alpha}(b_{\alpha_{op}})$ *can be obtained by the following formulas:*

$$J_3(b_{op}) = \Bigg[4\left(G_3 + \tfrac{M_3}{r_1}\right)\Big(G_1 g^2 r_1^4 + G_2 q^2 \frac{r_2^4}{r_1^2} + G_3 z_d^2 + G_4 g q r_1 r_2^2 - G_5 g r_1^2 z_d + G_6 q \frac{r_2^2}{r_1} z_d\Big) - \left(-2G_3 z_d + G_5 g r_1^2 + G_6 q \frac{r_2^2}{r_1}\right)^2\Bigg] \frac{\pi r_1^3}{2\left(G_3 + \frac{M_3}{r_1}\right)} \quad (46)$$

and

$$J_{3\alpha}(b_{\alpha_{op}}) = \Bigg[4\left(G_{3\alpha} + \tfrac{M_3}{r_1}\right)\Big(G_{1\alpha} g^2 r_1^4 + G_{2\alpha} q^2 \frac{r_2^4}{r_1^2} + G_{3\alpha} z_d^2 + G_{4\alpha} g q r_1 r_2^2 - G_{5\alpha} g r_1^2 z_d + G_{6\alpha} q \frac{r_2^2}{r_1} z_d\Big) - \left(-2G_{3\alpha} z_d + G_{5\alpha} g r_1^2 + G_{6\alpha} q \frac{r_2^2}{r_1}\right)^2\Bigg] \frac{\pi r_1^3}{2\left(G_{3\alpha} + \frac{M_3}{r_1}\right)} \quad (47)$$

(iv) *For the distributed-boundary optimal control problem* (9) *and* (10)*, the optimal solutions are given by*

$$(g, q)_{op} = (g^{op}, q^{op}) = \left(\frac{(b - z_d)}{r_1^2}\Delta_3, \frac{(b - z_d) r_1}{r_2^2}\Pi_3\right) \quad (48)$$

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with

$$\Delta_3 = \frac{\left(G_4G_6-2G_5\left(G_2+\frac{M_5}{r_1r_2^2}\right)\right)}{\left(4\left(G_1+G_3\frac{M_4}{r_1^4}\right)\left(G_2+\frac{M_5}{r_1r_2^2}\right)-G_4^2\right)}, \quad \Pi_3 = \frac{\left(G_4G_5-2G_6\left(G_1+G_3\frac{M_4}{r_1^4}\right)\right)}{\left(4\left(G_1+G_3\frac{M_4}{r_1^4}\right)\left(G_2+\frac{M_5}{r_1r_2^2}\right)-G_4^2\right)}$$

and

$$(g,q)_{\alpha_{op}} = (g_\alpha^{op}, q_\alpha^{op}) = \left(\frac{(b-z_d)}{r_1^2}\Delta_{3_\alpha}, \frac{(b-z_d)r_1}{r_2^2}\Pi_{3_\alpha}\right) \tag{49}$$

with

$$\Delta_{3_\alpha} = \frac{G_{4\alpha}G_{6\alpha}-2G_{5\alpha}\left(G_{2\alpha}+\frac{M_5}{r_1r_2^2}\right)}{4\left(G_{1\alpha}+G_{3\alpha}\frac{M_4}{r_1^4}\right)\left(G_{2\alpha}+\frac{M_5}{r_1r_2^2}\right)-G_{4\alpha}^2}, \quad \Pi_{3_\alpha} = \frac{G_{4\alpha}G_{5\alpha}-2G_{6\alpha}\left(G_{1\alpha}+G_{3\alpha}\frac{M_4}{r_1^4}\right)}{4\left(G_{1\alpha}+G_{3\alpha}\frac{M_4}{r_1^4}\right)\left(G_{2\alpha}+\frac{M_5}{r_1r_2^2}\right)-G_{4\alpha}^2}$$

Furthermore, J_4 at $(g,q)_{op}$ and $J_{4\alpha}$ at $(g,q)_{\alpha_{op}}$ can be computed by the following expressions:

$$J_4(g^{op}, q^{op}) = \left[G_4G_5G_6 + 4\left(G_1+G_3\frac{M_4}{r_1^4}\right)\left(G_2+\frac{M_5}{r_1r_2^2}\right)G_3 - \left(G_1+G_3\frac{M_4}{r_1^4}\right)G_6^2 - \left(G_2+\frac{M_5}{r_1r_2^2}\right)G_5^2 - G_3G_4^2\right]\frac{2\pi(b-z_d)^2r_1^3}{\left(4\left(G_1+G_3\frac{M_4}{r_1^4}\right)\left(G_2+\frac{M_5}{r_1r_2^2}\right)-G_4^2\right)} \tag{50}$$

and

$$J_{4\alpha}(g_\alpha^{op}, q_\alpha^{op}) = \left[G_{4\alpha}G_{5\alpha}G_{6\alpha} + 4\left(G_{1\alpha}+G_{3\alpha}\frac{M_4}{r_1^4}\right)\left(G_{2\alpha}+\frac{M_5}{r_1r_2^2}\right)G_{3\alpha} - \left(G_{1\alpha}+G_{3\alpha}\frac{M_4}{r_1^4}\right)G_{6\alpha}^2 - \left(G_{2\alpha}+\frac{M_5}{r_1r_2^2}\right)G_{5\alpha}^2 - G_{3\alpha}G_{4\alpha}^2\right]\frac{2\pi(b-z_d)^2r_1^3}{\left(4\left(G_{1\alpha}+G_{3\alpha}\frac{M_4}{r_1^4}\right)\left(G_{2\alpha}+\frac{M_5}{r_1r_2^2}\right)-G_{4\alpha}^2\right)} \tag{51}$$

(v) *The estimates and convergences obtained in (v) of Theorem 2.5 are also verified for the spherical shell in $\mathbb{R}^3$.*

Proof (i) Taking into account that the functional J_1 and $J_{1\alpha}$ can be expressed in the following quadratic forms:

$$J_1(g) = 2\pi\left[(G_1r_1^7 + M_1G_3r_1^3)g^2 + \left(G_4qr_1^4r_2 + G_5r_1^5(b-z_d)\right)g + \left(G_2q^2r_1r_2^4 + G_3r_1^3(b-z_d)^2 + G_6qr_1^2r_2^2(b-z_d)\right)\right]$$

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and

$$J_{1\alpha}(g) = 2\pi \Big[(G_{1\alpha} r_1^7 + M_1 G_{3\alpha} r_1^3) g^2 + \Big(G_{4\alpha} q r_1^4 r_2 + G_{5\alpha} r_1^5 (b - z_d) \Big) g + \Big(G_{2\alpha} q^2 r_1 r_2^4 + G_{3\alpha} r_1^3 (b - z_d)^2 + G_{6\alpha} q r_1^2 r_2^2 (b - z_d) \Big) \Big]$$

it can be obtained that the optimal solutions g_{op} and $g_{\alpha_{op}}$ for the problems (3) and (4) are given by (39) since the second derivative is positive in both cases. The optimal values formulas (40) and (41) are deduced by evaluating J_1 and $J_{1\alpha}$ at g_{op} and $g_{\alpha_{op}}$, respectively.

(ii) The functional J_2 and $J_{2\alpha}$ are given by the expressions:

$$J_2(q) = 2\pi \Big[\Big(G_2 r_1 r_2^4 + M_2 r_2^2 \Big) q^2 + \Big(G_4 r_1^4 r_2^2 g + G_6 r_1^2 r_2^2 (b - z_d) \Big) q + \Big(G_1 r_1^7 g^2 + G_3 r_1^3 (b - z_d)^2 + G_5 r_1^5 g (b - z_d) \Big) \Big]$$

and

$$J_{2\alpha}(q) = 2\pi \Big[\Big(G_{2\alpha} r_1 r_2^4 + M_2 r_2^2 \Big) q^2 + \Big(G_{4\alpha} r_1^4 r_2^2 g + G_{6\alpha} r_1^2 r_2^2 (b - z_d) \Big) q + \Big(G_{1\alpha} r_1^7 g^2 + G_{3\alpha} r_1^3 (b - z_d)^2 + G_{5\alpha} r_1^5 g (b - z_d) \Big) \Big].$$

Therefore it is immediate that the optimal controls for problems (5) and (6) are given by (42) since the second derivative is positive in both cases.
The computation of $J_2(q_{op})$ and $J_{2\alpha}(q_{\alpha_{op}})$ leads to the closed formulas (43) and (44) for the optimal values of the control problems considered.

(iii) For the problems (7) and (8), the functional J_3 and $J_{3\alpha}$ are given by

$$J_3(b) = 2\pi \Big[\big(G_3 r_1^3 + M_3 r_1^2 \big) b^2 + \Big(-2G_3 r_1^3 z_d + G_5 r_1^5 g + G_6 r_1^2 r_2^2 q \Big) b + G_1 r_1^7 g^2 + G_2 r_1 r_2^4 q^2 + G_3 r_1^3 z_d^2 + G_4 r_1^4 r_2^2 g q - G_5 r_1^5 g z_d + G_6 r_1^2 r_2^2 q z_d \Big]$$

and

$$J_{3\alpha}(b) = 2\pi \Big[\big(G_{3\alpha} r_1^3 + M_3 r_1^2 \big) b^2 + \Big(-2G_{3\alpha} r_1^3 z_d + G_{5\alpha} r_1^5 g + G_{6\alpha} r_1^2 r_2^2 q \Big) b + G_{1\alpha} r_1^7 g^2 + G_{2\alpha} r_1 r_2^4 q^2 + G_{3\alpha} r_1^3 z_d^2 + G_{4\alpha} r_1^4 r_2^2 g q - G_{5\alpha} r_1^5 g z_d + G_{6\alpha} r_1^2 r_2^2 q z_d \Big].$$

Therefore the optimal controls are given by (45) since the second derivative is positive in both cases.
The optimal values given by expressions (46) and (47) are obtained by computing J_3 and $J_{3\alpha}$ at b_{op} and $b_{\alpha_{op}}$ respectively.

(iv) For the distributed-boundary optimal control problems (9) and (10), the functional J_4 can be expressed as

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$$J_4(g,q) = 2\pi \Big[(G_1 r_1^7 + M_4 G_3 r_1^3) g^2 + (G_2 r_1 r_2^4 + M_5 r_2^2) q^2 + G_4 r_1^4 r_2^2 g q + G_5 r_1^5 g (b - z_d) + G_6 r_1^2 r_2^2 q (b - z_d) + G_3 r_1^3 (b - z_d)^2 \Big]$$

and the functional $J_{4\alpha}$ is given by:

$$J_{4\alpha}(g,q) = 2\pi \Big[(G_{1\alpha} r_1^7 + M_4 G_{3\alpha} r_1^3) g^2 + (G_{2\alpha} r_1 r_2^4 + M_5 r_2^2) q^2 + G_{4\alpha} r_1^4 r_2^2 g q + G_{5\alpha} r_1^5 g (b - z_d) + G_{6\alpha} r_1^2 r_2^2 q (b - z_d) + G_{3\alpha} r_1^3 (b - z_d)^2 \Big]$$

from where it can be obtained that the optimal solutions are given by (48) and (49), respectively, due to the second partial derivative test. Formulas (50) and (51) are deduced by evaluating J_4 at $(g,q)_{op}$ and $J_{4\alpha}$ at $(g,q)_{\alpha_{op}}$.

(v) The convergences and estimates of the optimal controls and the optimal values, when $\alpha \to \infty$ are obtained by taking into account the formulas given in (i)–(iv) and the Remark 4.4. The corresponding computations can be found in Appendix A of our arXiv version (pages 27–29) [6]. They are omitted here due to the fact that they become cumbersome.

□

5 Conclusions

In this paper, two different steady-state heat conduction problems S and S_α, for the Poisson equation with constant internal energy g and mixed boundary conditions have been considered. The problem S corresponds to the case when a constant temperature b is prescribed in the portion Γ_1 of the boundary and a constant flux q on Γ_2, while in the problem S_α, a convective condition is imposed at Γ_1 with a heat transfer coefficient α and external temperature b. Different optimal control problems can be also considered: a *distributed* control problem on the internal energy g, a *boundary* optimal control problem on the heat flux q, a *boundary* optimal control problem on the external temperature b and a *distributed-boundary* simultaneous optimal control problem on the source g and the flux q have been defined. We have obtained explicitly the optimal values of these optimal control problems, already study theoretically in literature in a general framework, for the particular domains: a rectangle in $\mathbb{R}^2$, an annulus in $\mathbb{R}^2$ and a spherical shell in $\mathbb{R}^3$. We point out that this solutions provide a benchmark for testing the accuracy of numerical methods. Also, the limit behaviour of the system state, adjoint state, optimal controls and optimal values for the optimal control problems defined from S_α, when $\alpha \to \infty$ have been analysed; concluding that they converge to the corresponding system state, adjoint state, optimal controls and optimal values for the optimal control problems defined from S. All these limits have been proved to present an order of convergence of $1/\alpha$ which can be considered as new results for these kind of elliptic optimal control problems. This estimate, obtained for this particular domains, make us to believe that it also holds for a more general domain, encouraging to prove it analytically.

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Affiliations

Julieta Bollati[1] · Claudia M. Gariboldi[2] · Domingo A. Tarzia[1]

Julieta Bollati
JBollati@austral.edu.ar

Claudia M. Gariboldi
cgariboldi@exa.unrc.edu.ar

1 Departamento de Matemática-CONICET, FCE, Univ. Austral, Paraguay 1950, S2000FZF Rosario, Argentina

2 Departamento de Matemática, FCEFQyN, Univ. Nac. de Río Cuarto, Ruta 36 Km 601, 5800 Río Cuarto, Argentina