Journal of Computational and Applied Mathematics 475 (2026) 117001

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Tau method implementation for approximating the solution to a two-phase change problem with temperature-dependent thermal coefficients

Julieta Bollati [a,c], M. Teresa Cao-Rial [b,d,*], María F. Natale [a], José A. Semitiel [a], Domingo A. Tarzia [a,c]

[a] Depto de Matemática, FCE-Universidad Austral, 2000 Rosario, Argentina
[b] Depto de Matemáticas, Universidade da Coruña, 15001 A Coruña, Spain
[c] CONICET, Argentina
[d] CITMaga, 15782 Santiago de Compostela, Spain

ARTICLE INFO

Keywords:
Stefan problem
Variable thermal conductivity
Variable heat capacity
Similarity type solution
Tau method

ABSTRACT

A one dimensional two-phase Stefan problem is considered to model the solidification process of a semi-infinite material with power-type temperature-dependent thermal coefficients and a Dirichlet boundary condition at the fixed face. Through a similarity transformation, an equivalent system of ordinary differential equations is obtained, which will be shown to have a unique solution. Since the domain is unbounded, a novel condition is imposed to transform it into a finite domain, allowing the application of the Tau approximation method. This method is based on shifted Chebyshev operational matrix of differentiation. Some comparisons between exact and numerical solutions are shown in order to test the accuracy of the method.

1. Introduction

Phase-change Stefan problems have been widely studied through the years due to their applicability to many significant areas of engineering, nature and industry [1,2]. They are essential to understand phase transition phenomena, specially in situations that involve heat transfer and solidification or melting processes. Stefan problems aim not only to describe the solid and liquid phases for a material undergoing a phase change process but also to determine the location of the sharp interface separating both phases, known as the free boundary. Among the multiple applications of Stefan type problems one can mention the solidification of binary alloys [3], the continuous casting of steel [4], the cryopreservation of cells [5] or the shoreline movement in a sedimentary basis [6]. The classical formulation of Stefan problems presented in [7] was obtained assuming certain hypothesis on the various physical factors that influence the phase change process, in order to obtain a simpler model. One of these assumptions is to consider constant thermal coefficients, like thermal conductivity, specific heat, latent heat or mass density. However, some extensions have been developed to describe and elucidate more complex physical scenarios. These generalizations and from various arguments from thermodynamics motivate the solution of Stefan's problems with variable thermal coefficients [8–11], imposing different boundary conditions [12,13] or considering source or convective terms in the heat equation [14,15].

A more recent (and broad) field of research within the study of Stefan problems involves the consideration of thermal coefficients that may vary with temperature or position. An analysis of Stefan problems with linear thermal conductivity and specific heat can

* Corresponding author at: Depto de Matemáticas, Universidade da Coruña, 15001 A Coruña, Spain.
E-mail address: teresa.cao@udc.es (M.T. Cao-Rial).

https://doi.org/10.1016/j.cam.2025.117001
Received 7 January 2025; Received in revised form 10 June 2025
Available online 8 August 2025

be found in [16]. Some extensions taking power-type thermal coefficients were given in [10,14,17,18]. Examples with quadratic polynomials thermophysical properties are exhibited in [19]. In [20], a thermal conductivity that depends not only on the temperature but also on time was considered.

Another branch on the study of Stefan problems is devoted to developing numerical methods in order to obtain approximate solutions. To this aim, there have been classically three approaches available, each one having advantages and disadvantages. One of the most used is the finite difference approach, which implies dividing a continuous domain into a grid of discrete points and approximating function derivatives in time and space by using finite differences between neighboring grid points [21]. Another approach is the finite element method, which consists on partitioning the domain into a tessellation of smaller, simpler subdomains and expressing the solution as a linear combination of basis functions defined on each subdomain [22].

The so called Tau method was developed by Lanczos [23] in the late thirties of last century as a tool to approximate special functions. In 1969, Ortiz [24] gave a systematic account in his paper "The Tau method" and since then, it has become a powerful tool for obtaining numerical solutions to many kinds of problems, from optimal control [25] to integro-differential equations, [26–30], and even fractional differential equations [31]. Later on, in a joint work with Samara [32], Ortiz presented an operational approach to the Tau method, that could be applied to compute a numerical solution for linear and nonlinear IVPs, BVPs or mixed problems for ODEs. In order to obtain an approximate solution with this method, one has to consider a basis comprised of the shifted Chebyshev polynomials which will allow them to approximate the solution of the problem, without it being analytically derived.

Since then, the method has been used to compute approximate solutions to real-life problems whose analytic solution cannot be obtained, and many authors have investigated variations on the kind of Chebyshev functions that are considered to comprise such basis [33].

With the development of the fractional calculus in the last decades, an effort has been made to adapt the Tau method to fractional differential equations (FDEs). This is when the rational Chebyshev Tau method appears. In [34], the rational Chebyshev Tau method is proposed by the authors to solve ordinary differential equations of higher order. The use of shifted Chebyshev Tau technique to approximate the solution of FDEs is described in [35] and [31]. In [36] a generalization of the Tau method and a convergence analysis to the numerical solution of multi-order fractional differential equations were discussed.

Nevertheless, its application to phase change boundary moving problems is quite scarce in the literature. One of the first works would be [37] and more recently, there are several works by Kumar et al. [17,20,38,39] where the authors applied the Tau method to this kind of problem with different particularities, like variable thermal conductivity, convection on the boundary, and so on.

Matlab code packages have been developed to implement the spectral Chebyshev Tau method for solving initial value problems, boundary value problems, eigenproblems, non-local problems for ordinary, fractional or distributed order differential equations [40]. In this work, we develop a computer code in SciLab to compute an approximate solution to the two-phase Stefan problem by applying the Tau method based on shifted Chebyshev polynomials for both phases.

The aim of this work is threefold. The first one is to prove existence and uniqueness of solution to the two-phase solidification Stefan problem when considering power-type temperature-dependent thermal coefficients. The second one is to transform the unbounded domain into a finite one, by introducing a thermal layer, beyond which there is no heat transfer. The third one is to develop a Scilab computer code for obtaining numerical solutions to this problem using the operational Tau method based on shifted Chebyshev polynomials. Furthermore, we will illustrate the accuracy of the method by providing a comparison between the exact and approximate solutions obtained for each problem.

The structure of the paper is the following: First, in Section 2, we present the mathematical formulation of a two-phase Stefan problem taking power-type thermal coefficients. By means of a similarity transformation, the Stefan problem is reduced to an ordinary differential problem. Next, in Section 3, we will prove that there exists a unique solution, through an equivalent functional problem. Section 4 is devoted to presenting the implementation of the Tau method technique based on shifted Chebyshev operational matrix of differentiation. In this step it will be key to simulate the behavior at the infinity, the assumption of a heat penetration depth $r(t) > s(t)$, beyond which there is no heat transfer. Finally, Section 5 will be devoted to the presentation of the obtained numerical results. The exact solutions analytically derived will be compared with the numerical approximations to give an account of the error committed.

2. Mathematical model

Based on the bibliography mentioned in the previous section, it is quite natural from a mathematical standpoint to define a one-dimensional two-phase Stefan problem with a power-type temperature-dependent thermal coefficients. The governing process can be described as in the following two-phase Stefan problem:

Problem (2PSP). Find the temperature $u = u(x,t)$ and the phase-changing interface $x = s(t)$ such that

$$\rho c_1(u_1)\frac{\partial u_1}{\partial t} = \frac{\partial}{\partial x}\left(k_1(u_1)\frac{\partial u_1}{\partial x}\right), \qquad 0 < x < s(t),\ t > 0, \tag{2.1}$$

$$\rho c_2(u_2)\frac{\partial u_2}{\partial t} = \frac{\partial}{\partial x}\left(k_2(u_2)\frac{\partial u_2}{\partial x}\right), \qquad x > s(t),\ t > 0, \tag{2.2}$$

$$u_2(+\infty,t) = u_2(x,0) = u_\infty, \qquad x > 0,\ t > 0, \tag{2.3}$$

$$u_1(s(t),t) = u_2(s(t),t) = u_s, \qquad t > 0, \tag{2.4}$$

$$u_1(0,t) = u_0, \qquad t > 0, \tag{2.5}$$

$$k_1(u_1(s(t),t))\frac{\partial u_1}{\partial x}(s(t),t) - k_2(u_2(s(t),t))\frac{\partial u_2}{\partial x}(s(t),t) = \rho\ell\dot{s}(t), \qquad t > 0, \tag{2.6}$$

$$s(0) = 0, \tag{2.7}$$

where $u = u(x,t)$ represents the temperature and is given by:

$$u(x,t) = \begin{cases} u_1(x,t) & \text{in} \quad 0 < x < s(t),\ t > 0, \\ u_2(x,t) & \text{in} \quad x > s(t),\ t > 0, \end{cases}$$

where $u_i(x,t)$ are the temperatures in the solid and the liquid region i respectively. From now on, and throughout the paper, Latin index i will take values $i = 1,2$ referring to each material phase respectively, unless otherwise explicitly indicated.

The following parameters are known data: $\rho > 0$ is the mass density, $\ell > 0$ is the latent heat per unit mass, u_0 is the temperature imposed at $x = 0$, u_∞ is the initial temperature of the material, u_s is the phase change temperature at $x = s(t)$ such that $u_0 < u_s < u_\infty$.

The thermal conductivity and specific heat in each phase are given by

$$k_i(u_i) = k_i^0\left(1 + \delta\left(\frac{u_\infty - u_i}{u_\infty - u_s}\right)^p\right), \tag{2.8}$$

$$c_i(u_i) = c_i^0\left(1 + \delta\left(\frac{u_\infty - u_i}{u_\infty - u_s}\right)^p\right), \tag{2.9}$$

where $\delta \geq 0$ and $p \geq 0$ are known parameters, $k_i^0 > 0$ represents the reference thermal conductivity and $c_i^0 > 0$ the specific heat. Moreover, we define the thermal diffusivity of each phase as $\alpha_i = \frac{k_i^0}{\rho c_i^0}$.

We look for similarity type solutions to (2PSP). More precisely, the aim is to write the temperature $u = u(x,t)$ as a function of a single variable, η. This will be achieved through the following change of variables:

$$y_i(\eta) = \frac{u_\infty - u_i(x,t)}{u_\infty - u_s} \geq 0, \tag{2.10}$$

and

$$\eta = \frac{x}{2\sqrt{\alpha_2 t}}, \quad x > 0,\ t > 0. \tag{2.11}$$

Taking into account condition (2.4), it follows that the sharp interface takes the form

$$s(t) = 2\lambda\sqrt{\alpha_2 t}, \quad t \geq 0, \tag{2.12}$$

where λ is an unknown positive parameter.

Assuming the hypothesis

$$H : \begin{cases} p, \delta \in \mathbb{R}_0^+, \lambda \in \mathbb{R}^+, \\ y_1 \text{ is twice differentiable on } (0,\lambda), \\ y_2 \text{ is twice differentiable on } (\lambda, +\infty), \end{cases} \tag{2.13}$$

and considering (2.10)–(2.12), we can establish the following immediate result.

Theorem 2.1. *Under the assumption H given by (2.13) the problem (2PSP) has a similarity type solution (u,s) given by*

$$u_1(x,t) = \left(u_s - u_\infty\right) y_1\left(\tfrac{x}{2\sqrt{\alpha_2 t}}\right) + u_\infty, \qquad 0 < x \leq s(t),\ t > 0, \tag{2.14}$$

$$u_2(x,t) = \left(u_s - u_\infty\right) y_2\left(\tfrac{x}{2\sqrt{\alpha_2 t}}\right) + u_\infty, \qquad x > s(t),\ t > 0, \tag{2.15}$$

$$s(t) = 2\lambda\sqrt{\alpha_2 t}, \qquad t \geq 0, \tag{2.16}$$

if and only if (y_1, y_2, λ) is a solution to the following ordinary differential problem (ODPη) given by

$$2\alpha\eta(1 + \delta y_1^p(\eta))y_1'(\eta) + [(1 + \delta y_1^p(\eta))y_1'(\eta)]' = 0, \qquad \eta \in (0,\lambda), \tag{2.17}$$

$$y_1(0) = \beta_1, \tag{2.18}$$

$$y_1(\lambda) = 1, \tag{2.19}$$

$$2\eta(1 + \delta y_2^p(\eta))y_2'(\eta) + [(1 + \delta y_2^p(\eta))y_2'(\eta)]' = 0, \qquad \eta \in (\lambda, +\infty), \tag{2.20}$$

$$y_2(\lambda) = 1, \tag{2.21}$$

$$y_2(+\infty) = 0, \tag{2.22}$$

$$y_1'(\lambda) - \tfrac{k_2^0}{k_1^0} y_2'(\lambda) = \tfrac{-2\alpha\lambda}{(1+\delta)Ste}, \tag{2.23}$$

where the dimensionless parameters β_1*,* α *and the Stefan number Ste are given by*

$$\beta_1 = \frac{u_\infty - u_0}{u_\infty - u_s} > 1, \qquad \alpha = \frac{\alpha_2}{\alpha_1} > 0, \qquad Ste = \frac{c_1^0\left(u_\infty - u_s\right)}{\ell} > 0. \tag{2.24}$$

Proof. If (u, s), given by (2.14)–(2.16), is a solution to (2PSP), then the following expressions hold:

$$\frac{\partial u_i}{\partial t} = -\frac{u_s - u_\infty}{2t}\,\eta\, y_i'(\eta), \qquad \frac{\partial u_i}{\partial x} = \frac{u_s - u_\infty}{2\sqrt{\alpha_2 t}}\, y_i'(\eta),$$

and

$$k_i(u_i) = k_i^0\left(1 + \delta y_i^p(\eta)\right), \qquad c_i(u_i) = c_i^0\left(1 + \delta y_i^p(\eta)\right),$$

for $i = 1, 2$ where η is given by (2.11).

By substituting these expressions into Eqs. (2.1) and (2.2) we obtain Eqs. (2.17) and (2.20). Likewise, conditions (2.3)–(2.5) yield the boundary conditions (2.18), (2.19), (2.21), and (2.22). Finally, applying the Stefan condition (2.6) we derive Eq. (2.23).

Therefore, the triplet (y_1, y_2, λ) satisfies the problem (ODPη). The converse implication follows immediately. □

3. Existence and uniqueness of solution

In this section, we will find the conditions that ensure the existence of a unique solution to (2PSP) through the analysis of the equivalent problem (ODPη).

Lemma 3.1. *Under the assumption H given by (2.13),* (y_1, y_2, λ) *is a solution to problem (ODPη) if and only if* (y_1, y_2, λ) *is a solution to the following functional problem (FPη) given by*

$$y_1(\eta) + \tfrac{\delta}{p+1} y_1^{p+1}(\eta) = \left(\beta_1 + \tfrac{\delta}{p+1}\beta_1^{p+1}\right) - A\frac{\operatorname{erf}(\sqrt{\alpha}\eta)}{\operatorname{erf}(\sqrt{\alpha}\lambda)}, \qquad \eta \in [0, \lambda], \tag{3.1}$$

$$y_2(\eta) + \tfrac{\delta}{p+1} y_2^{p+1}(\eta) = \left(1 + \tfrac{\delta}{p+1}\right)\frac{\operatorname{erfc}(\eta)}{\operatorname{erfc}(\lambda)}, \qquad \eta \in [\lambda, +\infty), \tag{3.2}$$

$$\frac{A\exp(-\alpha\lambda^2)}{\operatorname{erf}(\sqrt{\alpha}\lambda)} - \frac{B\exp(-\lambda^2)}{\operatorname{erfc}(\lambda)} = \frac{\sqrt{\alpha\pi}}{\text{Ste}}\lambda, \tag{3.3}$$

where erf *and* erfc *denote the error function and the complementary error function, respectively, defined as*

$$\operatorname{erf}(\eta) = \frac{2}{\sqrt{\pi}}\int_0^\eta e^{-\xi^2}\,d\xi, \qquad \operatorname{erfc}(\eta) = 1 - \operatorname{erf}(\eta), \quad \eta \geq 0,$$

and

$$A = \beta_1 - 1 + \tfrac{\delta}{p+1}\left(\beta_1^{p+1} - 1\right) > 0, \qquad B = \tfrac{k_2^0}{k_1^0\sqrt{\alpha}}\left(1 + \tfrac{\delta}{p+1}\right) > 0.$$

Proof. Assume that (y_1, y_2, λ) verifies the problem (ODPη). Firstly, let us consider the change of variable

$$\phi_1(\eta) = \left(1 + \delta y_1^p(\eta)\right) y_1'(\eta), \quad \eta \in [0, \lambda], \tag{3.4}$$

in the ordinary differential Eq. (2.17). Taking into account condition (2.18), it follows that

$$\phi_1(\eta) = (1 + \delta\beta_1^p) y_1'(0)\exp(-\alpha\eta^2), \quad \eta \in [0, \lambda]. \tag{3.5}$$

By equating expressions (3.4) and (3.5), integrating from 0 to η and using condition (2.18), we obtain:

$$y_1(\eta) + \tfrac{\delta}{p+1} y_1^{p+1}(\eta) = \beta_1\left(1 + \tfrac{\delta}{p+1}\beta_1^p\right) + \left(1 + \delta\beta_1^p\right) y_1'(0)\tfrac{\sqrt{\pi}}{2\sqrt{\alpha}}\operatorname{erf}\left(\sqrt{\alpha}\eta\right). \tag{3.6}$$

Now, let us evaluate the above equation at $\eta = \lambda$ and from the condition (2.19), it follows the value $y_1'(0)$. Furthermore, we can rewrite (3.6) as (3.1).

Secondly, we can proceed analogously to prove that the function y_2 satisfies the functional Eq. (3.2). Defining:

$$\phi_2(\eta) = \left(1 + \delta y_2^p(\eta)\right) y_2'(\eta), \quad \eta \in [\lambda, +\infty), \tag{3.7}$$

replacing it in the ordinary differential equation (2.20) and taking into account the condition (2.21), we obtain

$$y_2'(\eta) + \delta y_2^p(\eta) y_2'(\eta) = (1 + \delta) y_2'(\lambda)\exp(\lambda^2 - \eta^2). \tag{3.8}$$

Integrating (3.8) in $[\eta, +\infty)$ and using condition (2.22) we can deduce that

$$y_2(\eta) + \frac{\delta}{p+1} y_2^{p+1}(\eta) = -(1 + \delta)\, y_2'(\lambda)\frac{\sqrt{\pi}}{2}\exp(\lambda^2)\operatorname{erfc}(\eta). \tag{3.9}$$

Substituting $\eta = \lambda$ in the above equation and from (2.21), we obtain $y_2'(\lambda)$ and therefore we can rewrite (3.9) as (3.2).

Finally, deriving (3.1) and (3.2) with respect to η, taking $\eta = \lambda$ and using the condition (2.23), we obtain that $\lambda \in \mathbb{R}^+$ must be a solution to Eq. (3.3).

Reciprocally, assuming that (y_1, y_2, λ) is a solution to the functional problem (FPη) we have that:

$$y_1(\eta) = -\tfrac{\delta}{p+1} y_1^{p+1}(\eta) + \left(\beta_1 + \tfrac{\delta}{p+1}\beta_1^{p+1}\right) - A\frac{\mathrm{erf}(\sqrt{\alpha}\eta)}{\mathrm{erf}(\sqrt{\alpha}\lambda)}, \quad \eta \in [0,\lambda], \tag{3.10}$$

$$y_2(\eta) = -\tfrac{\delta}{p+1} y_2^{p+1}(\eta) + \left(1 + \tfrac{\delta}{p+1}\right)\frac{\mathrm{erfc}(\eta)}{\mathrm{erfc}(\lambda)}, \quad \eta \in [\lambda, +\infty), \tag{3.11}$$

and it is straightforward to check that (y_1, y_2, λ) solves the problem (ODPη). □

Next, we will show that the aforementioned solution to the functional problem (FPη) is indeed unique.

Lemma 3.2. *Assuming hypothesis H given by (2.13), the problem (FPη) has a unique solution.*

Proof. Let us define the function

$$g(z) = \frac{A\exp(-\alpha z^2)}{\mathrm{erf}(\sqrt{\alpha} z)} - \frac{B\exp(-z^2)}{\mathrm{erfc}(z)}, \qquad z \in \mathbb{R}^+. \tag{3.12}$$

The derivative of the function is given by

$$g'(z) = -\left[2Ae^{-\alpha z^2}\frac{\alpha z\,\mathrm{erf}(\sqrt{\alpha}z) + \frac{\sqrt{\alpha}}{\sqrt{\pi}}e^{-\alpha z^2}}{\left(\mathrm{erf}(\sqrt{\alpha}z)\right)^2} + \frac{2Be^{-2z^2}}{\sqrt{\pi}}\frac{1-\sqrt{\pi}ze^{z^2}\mathrm{erfc}(z)}{(\mathrm{erfc}(z))^2}\right].$$

Since it has been established in [41] that

$$1 - \sqrt{\pi}\, z\, e^{z^2}\,\mathrm{erfc}(z) > 0 \quad \text{for} \quad z \geq 0,$$

it follows that $g'(z) < 0$ for all $z \geq 0$. From the fact that g is a strictly decreasing function such that $g(0^+) = +\infty$ and $g(+\infty) = -\infty$, one can infer that the solution λ to the equation

$$g(z) = \frac{\sqrt{\alpha\pi}}{\mathrm{Ste}} z, \tag{3.13}$$

is, indeed, unique in $\mathbb{R}^+$. Now, for this λ, let us rewrite (3.1) and (3.2) as

$$F\left(y_1(\eta)\right) = H_1(\eta), \qquad \eta \in [0,\lambda], \tag{3.14}$$

and

$$F\left(y_2(\eta)\right) = H_2(\eta), \qquad \eta \in [\lambda,+\infty), \tag{3.15}$$

respectively, where

$$H_1(\eta) = \beta_1 + \frac{\delta}{p+1}\beta_1^{p+1} - A\frac{\mathrm{erf}(\sqrt{\alpha}\eta)}{\mathrm{erf}(\sqrt{\alpha}\lambda)}, \qquad \eta \in [0,\lambda], \tag{3.16}$$

$$H_2(\eta) = \left(1 + \frac{\delta}{p+1}\right)\frac{\mathrm{erfc}(\eta)}{\mathrm{erfc}(\lambda)}, \qquad \eta \in [\lambda,+\infty), \tag{3.17}$$

and

$$F(z) = z + \frac{\delta}{p+1} z^{p+1}, \qquad z \in \mathbb{R}_0^+. \tag{3.18}$$

It is also quite straightforward to prove that F is a strictly increasing function, so that the inverse function, $F^{-1} : [0,+\infty) \to [0,+\infty)$, is well defined. As H_1 is a strictly decreasing function such that $H_1(0) = \beta_1 + \frac{\delta}{p+1}\beta_1^{p+1} > 1 + \frac{\delta}{p+1} = H_1(\lambda) > 0$, then H_1 is a positive function. In a similar way, one can check that H_2 is also a positive function. Therefore, the functions y_1 and y_2 given by

$$y_1(\eta) = F^{-1}\left(H_1(\eta)\right), \qquad \eta \in [0,\lambda], \tag{3.19}$$

and

$$y_2(\eta) = F^{-1}\left(H_2(\eta)\right), \qquad \eta \in [\lambda,+\infty), \tag{3.20}$$

are the unique solutions to equations (3.1) and (3.2), respectively. □

Remark 3.3. Notice that since $F'(z) > 0$ in $[0,1]$ with $F(0) = 0$ and $F(1) = 1 + \frac{\delta}{p+1}$; $H_1'(\eta) < 0$ in $[0,\lambda]$ with $H_1(0) = \beta_1 + \frac{\delta}{p+1}\beta_1^{p+1}$ and $H_1(\lambda) = 1 + \frac{\delta}{p+1}$; $H_2'(\eta) < 0$ in $[\lambda,+\infty)$ with $H_2(\lambda) = 1 + \frac{\delta}{p+1}$ and $H_2(+\infty) = 0$, it follows that

$$y_1(\lambda) = 1 \leq y_1(\eta) \leq \beta_1 = y_1(0), \qquad \eta \in [0,\lambda],$$

and

$$y_2(+\infty) = 0 \le y_2(\eta) \le 1 = y_2(\lambda), \qquad \eta \in [\lambda, +\infty).$$

Now, the previous lemmas and Theorem 2.1 allow us to postulate this work main result:

Theorem 3.4. *Suppose that hypothesis H given by (2.13) holds and let (y_1, y_2, λ) be the unique solution to problem (FPη). Then, (u, s) given by (2.14)–(2.16) constitutes the unique similarity type solution to problem (2PSP).*

Remark 3.5. Notice that from Remark 3.3 and Theorem 3.4 we have, as expected, the following relations:

$$u_0 \le u_1(x,t) \le u_s, \qquad 0 < x < s(t),\ t > 0,$$

and

$$u_s \le u_2(x,t) \le u_\infty, \qquad x > s(t),\ t > 0.$$

4. Tau method implementation

In this section we are going to obtain an approximate solution to the Stefan (2PSP) applying the Tau method based on shifted Chebyshev operational matrix of differentiation.

Let us present some properties of the Chebyshev polynomials. It is common knowledge [28] that the classical Chebyshev polynomials $\{T_j(x)\}_{j\ge0}$ are defined on the interval $[-1,1]$ by the following recurrent system:

$$T_0(x) = 1, \tag{4.1}$$

$$T_1(x) = x, \tag{4.2}$$

$$T_j(x) = 2xT_{j-1}(x) - T_{j-2}(x), \quad j = 2,3,4,\dots. \tag{4.3}$$

The set $\{T_j(x)\}_{j\ge0}$ is orthogonal with respect to the inner product with a weight function $w(x) = \frac{1}{\sqrt{1-x^2}}$, $x \in [-1,1]$, i.e.,

$$\left(T_k, T_j\right) = \int_{-1}^{1} T_k(x)T_j(x)w(x)dx = \begin{cases} \pi & \text{if} \quad k=j=0 \\ \frac{\pi}{2} & \text{if} \quad k=j\neq 0 \\ 0 & \text{if} \quad k\neq j \end{cases}. \tag{4.4}$$

A function $f \in L^2(-1,1)$ may be formulated in terms of Chebyshev polynomials through the following series

$$f(x) = \sum_{j=0}^{\infty} a_j T_j(x), \tag{4.5}$$

where the coefficients a_j are given by

$$a_0 = \frac{1}{\pi}\int_{-1}^{1} f(x)T_0(x)w(x)dx, \quad a_j = \frac{2}{\pi}\int_{-1}^{1} f(x)T_j(x)w(x)dx, \quad j = 1,2,\dots \tag{4.6}$$

For the purpose of using the Chebyshev polynomials on a general interval $[a,b]$, we introduce the shifted Chebyshev polynomials [27,33], denoted by $T_j^{[a,b]}(x)$, satisfying the following recurrence formula

$$T_j^{[a,b]}(x) = T_j\left(\frac{2}{b-a}x - \frac{b+a}{b-a}\right), \quad j = 2,3,4,\dots. \tag{4.7}$$

where $T_0^{[a,b]}(x) = 1$ and $T_1^{[a,b]}(x) = \frac{2}{b-a}x - \frac{b+a}{b-a}$. In a similar manner, a function $f \in L^2[a,b]$ can be written in terms of shifted Chebyshev polynomials $T_j^{[a,b]}(x)$ as in (4.5).

In practice, only the first $(N+1)$-terms shifted Chebyshev polynomials are considered. In approximate theory, the series given by (4.5) can be approximated by taking the first $N+1$ terms as follows:

$$f(x) \approx f_N(x) = \sum_{j=0}^{N} a_j T_j^{[a,b]}(x) = A\, T^{[a,b]}(x), \quad x \in [a,b], \tag{4.8}$$

where $A = \left[a_0\ a_1\ \dots\ a_N\right]$ is a row $(N+1)$-vector and

$$T^{[a,b]}(x) = \left[T_0^{[a,b]}(x)\ T_1^{[a,b]}(x)\ \dots\ T_N^{[a,b]}(x)\right]^t$$

is a $(N+1)$-vector.

Following [29], we can deduce that

$$T^{[a,b]}(x) = TW_{a,b}X(x), \tag{4.9}$$

where the $(N+1)$-vector X is given by

$$X(x) = \left[1 \ x \ x^2 \ \dots \ x^N\right]^t, \tag{4.10}$$

$T = [t_{kj}]$ is the matrix $(N+1)\times(N+1)$ defined by

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & \dots. & 0 \\ -1 & t_{21} & 2 & 0 & 0 & 0 & \dots & 0 \\ 0 & t_{31} & t_{32} & 2^2 & 0 & 0 & \dots & 0 \\ 1 & t_{41} & t_{42} & t_{43} & 2^3 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \dots & \vdots \\ \cos\left(\frac{N\pi}{2}\right) & t_{N1} & t_{N2} & t_{N3} & t_{N4} & t_{N5} & \dots & 2^{N-1} \end{bmatrix}, \tag{4.11}$$

with

$$t_{kj} = \begin{cases} \cos\left(\frac{k\pi}{2}\right) & \text{if} \quad k = 0,1,\dots,N, \ j = 0, \\ 2^{k-1} & \text{if} \quad k = j = 1,2,\dots,N, \\ \operatorname{sgn}\left(t_{k-1,\,j-1}\right)\left(2|t_{k-1,\,j-1}| + |t_{k-2,\,j}|\right) & \text{if} \quad k = 2,3,\dots,N, \\ & \qquad j = 1,\dots,k-1. \\ 0 & \qquad \text{otherwise,} \end{cases}$$

and

$$W_{a,b} = [w_{kj}], \tag{4.12}$$

is the matrix $(N+1)\times(N+1)$ where

$$w_{kj} = \begin{cases} \binom{k}{j}\left(-\frac{b+a}{b-a}\right)^{k-j}\left(\frac{2}{b-a}\right)^j & \text{if} \quad k = 0,1,\dots,N, \ j = 0,1,\dots,k, \\ 0 & \text{if} \quad k < j. \end{cases}$$

Taking into account [28], the nth order derivative of the vector $T^{[a,b]}(x)$ given by (4.9) can be written as

$$\left(T^{[a,b]}\right)^{(n)}(x) = T\ W_{a,b}\ B^n X(x), \tag{4.13}$$

where

$$B = [b_{kj}], \tag{4.14}$$

is the $(N+1)\times(N+1)$ matrix defined by:

$$b_{kj} = \begin{cases} j+1 & \text{if} \quad k = j+1, \ j = 0,1,\dots,N, \\ 0 & \qquad \text{otherwise,} \end{cases}$$

and $n \in \mathbb{N}$ denotes nth powers of matrix B. Then, from (4.8), the nth order derivative of the function f_N is given by

$$f_N^{(n)}(x) = A\ T\ W_{a,b}\ B^n X(x). \tag{4.15}$$

4.1. Approximate solution to problem (2PSP)

In order to obtain approximate solutions to the Stefan problem (2PSP), following [9], let us assume in the liquid phase the existence of a heat penetration depth $\widetilde{r}(t) > s(t)$, called thermal layer, beyond which there is no heat transfer. This is equivalent to assume that for each $x > \widetilde{r}(t)$, the slab, is at an equilibrium temperature and so

$$\frac{\partial u_2}{\partial x}(\widetilde{r}(t),t) = 0, \qquad u_2(\widetilde{r}(t),t) = u_\infty, \qquad t > 0. \tag{4.16}$$

For this reason, we consider a new two-phase Stefan problem that takes into account the above assumptions and can be stated as:

Problem (2PSPN). Find $(\widetilde{u}_1, \widetilde{u}_2, \widetilde{s}, \widetilde{r})$ such that:

$$\rho c_1(\widetilde{u}_1)\frac{\partial \widetilde{u}_1}{\partial t} = \frac{\partial}{\partial x}\left(k_1(\widetilde{u}_1)\frac{\partial \widetilde{u}_1}{\partial x}\right), \qquad 0 < x < \widetilde{s}(t),\ t > 0, \tag{4.17}$$

$$\rho c_2(\widetilde{u}_2)\frac{\partial \widetilde{u}_2}{\partial t} = \frac{\partial}{\partial x}\left(k_2(\widetilde{u}_2)\frac{\partial \widetilde{u}_2}{\partial x}\right), \qquad \widetilde{s}(t) < x < \widetilde{r}(t),\ t > 0, \tag{4.18}$$

$$\widetilde{u}_2(\widetilde{r}(t),t) = u_\infty, \qquad t > 0, \tag{4.19}$$

$$\widetilde{u}_1(\widetilde{s}(t),t) = \widetilde{u}_2(\widetilde{s}(t),t) = u_s, \qquad t > 0, \tag{4.20}$$

$$\widetilde{u}_1(0,t) = u_0, \qquad t>0, \tag{4.21}$$

$$k_1(\widetilde{u}_1(\widetilde{s}(t),t))\frac{\partial \widetilde{u}_1}{\partial x}(\widetilde{s}(t),t) - k_2(\widetilde{u}_2(\widetilde{s}(t),t))\frac{\partial \widetilde{u}_2}{\partial x}(\widetilde{s}(t),t) = \rho\ell\dot{\widetilde{s}}(t), \qquad t>0, \tag{4.22}$$

$$\frac{\partial \widetilde{u}_2}{\partial x}(\widetilde{r}(t),t) = 0, \qquad t>0, \tag{4.23}$$

$$\widetilde{s}(0) = 0, \tag{4.24}$$

$$\widetilde{r}(0) = 0. \tag{4.25}$$

Through the change of variables given by (2.10) and (2.11), we can establish the following direct result:

Theorem 4.1. *Under the assumption:*

$$\widetilde{H}: \begin{cases} p,\delta \in \mathbb{R}_0^+, \quad \widetilde{\lambda}\in\mathbb{R}^+, \quad \widetilde{\mu}\in(\widetilde{\lambda},+\infty), \\ \widetilde{y}_1 \text{ is twice differentiable on } (0,\widetilde{\lambda}), \\ \widetilde{y}_2 \text{ is twice differentiable on } (\widetilde{\lambda},\widetilde{\mu}), \end{cases} \tag{4.26}$$

the problem (2PSPN) has a similarity solution $(\widetilde{u}_1,\widetilde{u}_2,\widetilde{s},\widetilde{r})$ given by:

$$\widetilde{u}_1(x,t) = \left(u_s - u_\infty\right)\widetilde{y}_1\left(\frac{x}{2\sqrt{\alpha_2 t}}\right) + u_\infty, \qquad 0<x<\widetilde{s}(t),\ t>0, \tag{4.27}$$

$$\widetilde{u}_2(x,t) = \left(u_s - u_\infty\right)\widetilde{y}_2\left(\frac{x}{2\sqrt{\alpha_2 t}}\right) + u_\infty, \qquad \widetilde{s}(t)<x<\widetilde{r}(t),\ t>0, \tag{4.28}$$

$$\widetilde{s}(t) = 2\widetilde{\lambda}\sqrt{\alpha_2 t}, \qquad t>0, \tag{4.29}$$

$$\widetilde{r}(t) = 2\widetilde{\mu}\sqrt{\alpha_2 t}, \qquad t>0, \tag{4.30}$$

if and only if $(\widetilde{y}_1,\widetilde{y}_2,\widetilde{\lambda},\widetilde{\mu})$ satisfies the ordinary differential problem (ODPηN) given by

$$2\alpha\eta(1+\delta\widetilde{y}_1^{\,p}(\eta))\widetilde{y}_1{}'(\eta) + [(1+\delta\widetilde{y}_1^{\,p}(\eta))\widetilde{y}_1{}'(\eta)]' = 0, \qquad \eta\in(0,\widetilde{\lambda}), \tag{4.31}$$

$$\widetilde{y}_1(0) = \beta_1, \tag{4.32}$$

$$\widetilde{y}_1(\widetilde{\lambda}) = 1, \tag{4.33}$$

$$2\eta(1+\delta\widetilde{y}_2^{\,p}(\eta))\widetilde{y}_2{}'(\eta) + [(1+\delta\widetilde{y}_2^{\,p}(\eta))\widetilde{y}_2{}'(\eta)]' = 0, \qquad \eta\in(\widetilde{\lambda},\widetilde{\mu}), \tag{4.34}$$

$$\widetilde{y}_2(\widetilde{\lambda}) = 1, \tag{4.35}$$

$$\widetilde{y}_2(\widetilde{\mu}) = 0, \tag{4.36}$$

$$\widetilde{y}_2{}'(\widetilde{\mu}) = 0, \tag{4.37}$$

$$\widetilde{y}_1{}'(\widetilde{\lambda}) - \frac{k_2^0}{k_1^0}\widetilde{y}_2{}'(\widetilde{\lambda}) = \frac{-2\alpha\widetilde{\lambda}}{(1+\delta)Ste}, \tag{4.38}$$

where the dimensionless parameters β_1, α and Ste are given in Theorem 2.1.

In order to obtain approximate solutions to this ordinary differential problem in the bounded domain $[0,\widetilde{\mu}]$, we apply the Tau method based on shifted Chebyshev polynomials. If the unknown functions $\widetilde{y}_1=\widetilde{y}_1(\eta)$ and $\widetilde{y}_2=\widetilde{y}_2(\eta)$ are expressed in terms of the shifted Chebyshev polynomials as in (4.8):

$$\widetilde{y}_1(\eta) \approx \widetilde{y}_{1_N}(\eta) = \sum_{j=0}^{N} c_j T_j^{[0,\widetilde{\lambda}_N]}(\eta) = C\,T^{[0,\widetilde{\lambda}_N]}(\eta), \qquad \eta\in[0,\widetilde{\lambda}_N], \tag{4.39}$$

$$\widetilde{y}_2(\eta) \approx \widetilde{y}_{2_N}(\eta) = \sum_{j=0}^{N} d_j T_j^{[\widetilde{\lambda}_N,\widetilde{\mu}_N]}(\eta) = D\,T^{[\widetilde{\lambda}_N,\widetilde{\mu}_N]}(\eta), \qquad \eta\in[\widetilde{\lambda}_N,\widetilde{\mu}_N], \tag{4.40}$$

where $C=\left[c_0\ c_1\ \dots\ c_N\right]$ and $D=\left[d_0\ d_1\ \dots\ d_N\right]$ are row $(N+1)$-vectors to be determined, according to (4.9), we have that

$$\widetilde{y}_{1_N}(\eta) = C\,T\,W_{0,\widetilde{\lambda}_N}X(\eta) \qquad \eta\in[0,\widetilde{\lambda}_N], \tag{4.41}$$

$$\widetilde{y}_{2_N}(\eta) = D\,T\,W_{\widetilde{\lambda}_N,\widetilde{\mu}_N}X(\eta), \qquad \eta\in[\widetilde{\lambda}_N,\widetilde{\mu}_N], \tag{4.42}$$

where T is given by (4.11), $W_{a,b}$ is defined by (4.12) and X is given by (4.10).

From (4.41) and (4.42), and taking into account that the derivatives are approximated as in (4.15), the residual $R_{N,\widetilde{y}_{i_N}}(\eta)$ for $i=1,2$ is defined as:

$$\begin{aligned} R_{N,\widetilde{y}_{1_N}}(\eta) = {} & 2\alpha\eta\left(1+\delta(C\,T\,W_{0,\widetilde{\lambda}_N}X)^p\right)CTW_{0,\widetilde{\lambda}_N}BX \\ & + \delta p(CTW_{0,\widetilde{\lambda}_N}X)^{p-1}CTW_{0,\widetilde{\lambda}_N}BXCTW_{0,\widetilde{\lambda}_N}BX \end{aligned}$$

$$+\left(1+\delta(C\,T\,W_{0,\widetilde{\lambda}_N}X)^p\right)CTW_{0,\widetilde{\lambda}_N}B^2X, \tag{4.43}$$

$$\begin{aligned}R_{N,\widetilde{y}_{2_N}}(\eta) = {} & 2\eta\left(1+\delta(D\,T\,W_{\widetilde{\lambda}_N,\widetilde{\mu}_N}X)^p\right)DTW_{\widetilde{\lambda}_N,\widetilde{\mu}_N}BX\\ & +\delta p(DTW_{\widetilde{\lambda}_N,\widetilde{\mu}_N}X)^{p-1}DTW_{\widetilde{\lambda}_N,\widetilde{\mu}_N}BX\,DTW_{\widetilde{\lambda}_N,\widetilde{\mu}_N}BX\\ & +\left(1+\delta(D\,T\,W_{\widetilde{\lambda}_N,\widetilde{\mu}_N}X)^p\right)DTW_{\widetilde{\lambda}_N,\widetilde{\mu}_N}B^2X.\end{aligned} \tag{4.44}$$

By virtue of the two equations above, from now on, we will assume $p \geq 1$. According to Tau method, to minimize the residuals in the sense that the first $(N+1)$ terms of its spectral series are 0, we generate the following $2N-2$ non-linear equations:

$$\left\langle R_{N,\widetilde{y}_{1_N}}(\eta),\ T_j^{[0,\widetilde{\lambda}_N]}(\eta)\right\rangle = \int_0^{\widetilde{\lambda}_N} R_{N,\widetilde{y}_{1_N}}(\eta)T_j^{[0,\widetilde{\lambda}_N]}(\eta)d\eta = 0, \tag{4.45}$$

$$\left\langle R_{N,\widetilde{y}_{2_N}}(\eta),\ T_j^{[\widetilde{\lambda}_N,\widetilde{\mu}_N]}(\eta)\right\rangle = \int_{\widetilde{\lambda}_N}^{\widetilde{\mu}_N} R_{N,\widetilde{y}_{2_N}}(\eta)T_j^{[\widetilde{\lambda}_N,\widetilde{\mu}_N]}(\eta)d\eta = 0, \tag{4.46}$$

for $j = 0, 1, \dots, N-2$, where $\langle\cdot,\cdot\rangle$ is the usual inner product in $L^2(a,b)$.

Also, by using (4.41) and taking into account conditions (4.32) and (4.33), we obtain:

$$CTW_{0,\widetilde{\lambda}_N}X(0) = \beta_1, \qquad CTW_{0,\widetilde{\lambda}_N}X(\widetilde{\lambda}_N) = 1, \tag{4.47}$$

respectively. In the same manner, using (4.42) and from conditions (4.35), (4.36) and (4.37), we get

$$DTW_{\widetilde{\lambda}_N,\widetilde{\mu}_N}X(\widetilde{\lambda}_N) = 1, \quad DTW_{\widetilde{\lambda}_N,\widetilde{\mu}_N}X(\widetilde{\mu}_N) = 0, \quad DTW_{\widetilde{\lambda}_N,\widetilde{\mu}_N}BX(\widetilde{\mu}_N) = 0, \tag{4.48}$$

respectively. And finally, using (4.41) and (4.42) and taking into account the condition (4.38), we obtain that

$$\left(CTW_{0,\widetilde{\lambda}_N} - \frac{k_2^0}{k_1^0}DTW_{\widetilde{\lambda}_N,\widetilde{\mu}_N}\right)BX(\widetilde{\lambda}_N) = -\frac{2\alpha}{(1+\delta)\mathrm{Ste}}\widetilde{\lambda}_N. \tag{4.49}$$

Therefore, in order to obtain the $(N+1)$-vectors C and D, and the positive parameters $\widetilde{\lambda}_N$ and $\widetilde{\mu}_N$, we have to solve the non-linear system of $2N+4$ equations defined by (4.45)–(4.49).

In order to numerically solve the nonlinear system of Eqs. (4.45)–(4.49), we proceed as follows:

Algorithm 1 Numerical solution of the nonlinear system (4.45)–(4.49)

1: **Input:** Integer N; initial guess $x_0 = (C_0, D_0, \lambda_0, \mu_0) \in \mathbb{R}^{2N+4}$
2: Define function $F : \mathbb{R}^{2N+4} \to \mathbb{R}^{2N+4}$ as follows:
3: Let $x = (C, D, \lambda, \mu)$, where $C, D \in \mathbb{R}^{N+1}$ and $\lambda, \mu \in \mathbb{R}$
4: Return vector $F(x)$ corresponding to the system (4.45)–(4.49) set to zero (approximate the integrals in equations (4.45) and (4.46) using the composite trapezoidal rule)
5: Use `fsolve` from Scilab with inputs:
6: Initial guess x_0
7: Function F
8: Obtain numerical solution $x_{sol} = (C, D, \lambda, \mu)$ such that $F(x_{sol}) \approx 0$

The solver `fsolve` typically employs variations of the Newton–Raphson method or quasi-Newton methods. A future improvement to the code could involve gaining more control over the nonlinear solver, or even developing specific algorithms to solve this system.

5. Numerical results

In virtue of the equivalences given in the previous Theorems, in this section, we will analyze the accuracy of the approximate solutions obtained for problem (ODPηN) by applying the Tau method comparing them with the exact solution to problem (ODPη) presented in Section 3.

Firstly, we will address the specific scenario involving constant thermal coefficients. Secondly, we will shift our attention to the broader context involving power-type temperature-dependent thermal coefficients.

5.1. Constant thermal coefficients

As presented in Section 3, the exact solution to problem (ODPη) for the positive constant thermal coefficients $k_i = k_i^0$, $c_i = c_i^0$, (i.e. $\delta = 0$) is given by Eqs. (2.14)–(2.16) where $\lambda \in \mathbb{R}^+$ is the unique solution to the following equation

$$(\beta_1 - 1)\frac{\exp(-\alpha z^2)}{\operatorname{erf}(\sqrt{\alpha}z)} - \frac{k_2^0}{k_1^0}\frac{1}{\sqrt{\alpha}}\frac{\exp(-z^2)}{\operatorname{erfc}(z)} = \frac{\sqrt{\alpha\pi}}{\mathrm{Ste}}z, \quad z \in \mathbb{R}^+, \tag{5.1}$$

Table 1
Solutions to the non-linear system (4.45)-(4.49) for $\delta = 0$.

N	$\widetilde{\lambda}_N$	$\widetilde{\mu}_N$	Vectors C and D for coefficients of $\widetilde{y}_{i_N}$, $i = 1,2$ respectively
2	0.4068404	1.6330352	[2.9636606 − 2.0000000 0.0363394] [0.3750000 − 0.5000000 0.1250000]
3	0.4002221	2.1869234	[2.9648161 − 2.0055373 0.0351947 0.0055324] [0.3108273 − 0.4679717 0.1888369 − 0.032445]
4	0.4007662	2.2194037	[2.9642863 − 2.0055161 0.0359064 0.0055716 − 0.0002002] [0.2981045 − 0.4684118 0.2057383 − 0.0328739 − 0.0043209]
5	0.4005504	2.4391824	[2.9642907 − 2.0055606 0.0358661 0.0055745 − 0.0001568 − 0.0000139] [0.2871744 − 0.448296 0.2122791 − 0.0551898 0.0005464 0.0034858]

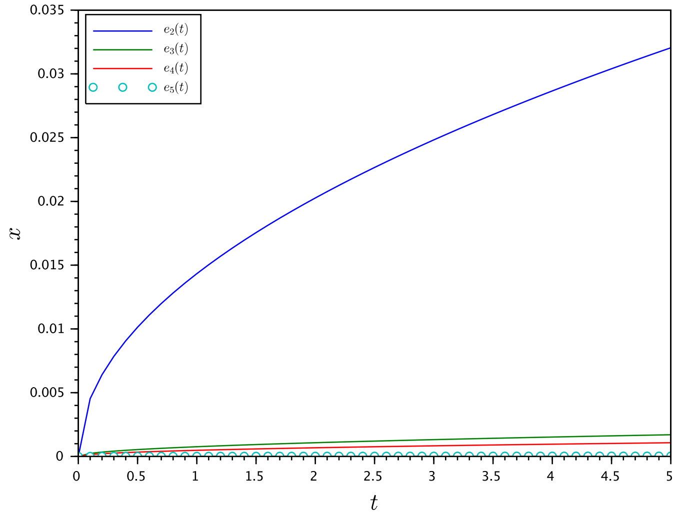

Fig. 1. Absolute error $e_N(t)$, $N = 2,3,4,5$ up to $t = 5$ s for $\alpha_2 = 1.3\ \frac{\mathrm{m}^2}{\mathrm{s}}$.

β_1, α and Ste are given by (2.24) and

$$y_1(\eta) = \beta_1 + \frac{1-\beta_1}{\operatorname{erf}\left(\sqrt{\alpha}\lambda\right)} \operatorname{erf}\left(\sqrt{\alpha}\eta\right), \qquad \eta \in [0,\lambda], \tag{5.2}$$

$$y_2(\eta) = \frac{\operatorname{erfc}(\eta)}{\operatorname{erfc}(\lambda)}, \qquad \eta \in [\lambda, +\infty). \tag{5.3}$$

If we fix the values $\mathrm{Ste} = 0.1$, $\alpha = 0.9$, $u_\infty = 2$ °C, $u_s = 1.8$ °C, $u_0 = 1$ °C, $\alpha_2 = 1.3 \frac{m^2}{s}$ and $\frac{k_2^0}{k_1^0} = 1.1$, then the unique solution to Eq. (5.1) is

$$\lambda = 0.4005556. \tag{5.4}$$

The approximate solution $(\widetilde{y}_{1_N}, \widetilde{y}_{2_N}, \widetilde{\lambda}_N, \widetilde{\mu}_N)$ to the problem (ODPηN) is obtained by solving the non-linear system defined by (4.45)–(4.49) where

$$R_{N,\widetilde{y}_{1_N}}(\eta) = C\ T\ W_{0,\widetilde{\lambda}_N}\left(2\alpha\eta B + B^2\right) X(\eta), \qquad \eta \in [0, \widetilde{\lambda}_N], \tag{5.5}$$

$$R_{N,\widetilde{y}_{2_N}}(\eta) = D\ T\ W_{\widetilde{\lambda}_N,\widetilde{\mu}_N}\left(2\eta B + B^2\right) X(\eta), \qquad \eta \in [\widetilde{\lambda}_N, \widetilde{\mu}_N]. \tag{5.6}$$

For $N = 2$, $N = 3$, $N = 4$ and $N = 5$, the solutions to the non-linear systems (4.45)–(4.49) of $2N + 4$ equations is given in Table 1.

Due to the similarity of the coefficients λ and $\widetilde{\lambda}_N$, in order to appreciate the difference between the approximate and exact free boundaries, Fig. 1 displays the absolute error given by $e_N(t) = |s(t) - \widetilde{s}_N(t)|$ where $\widetilde{s}_N(t) = 2\widetilde{\lambda}_N\sqrt{\alpha_2 t}$, for $N = 2,3,4,5$ on the time interval $t \in (0,5)$ s. As expected, the absolute error grows over time t and decreases as N increases.

Note that the relative error associated with the free boundaries

$$er_N(t) = \left|\frac{s(t) - \widetilde{s}_N(t)}{s(t)}\right| = \left|\frac{\lambda - \widetilde{\lambda}_N}{\lambda}\right|$$

remains constant over time. Table 2 presents the relative errors for $N = 2,3,4$, and 5. As observed, the relative error decreases as N increases.

Table 2
Relative errors for different values of N.

N	Relative Error
2	0.0156902
3	0.0008326
4	0.0005258
5	0.0000130

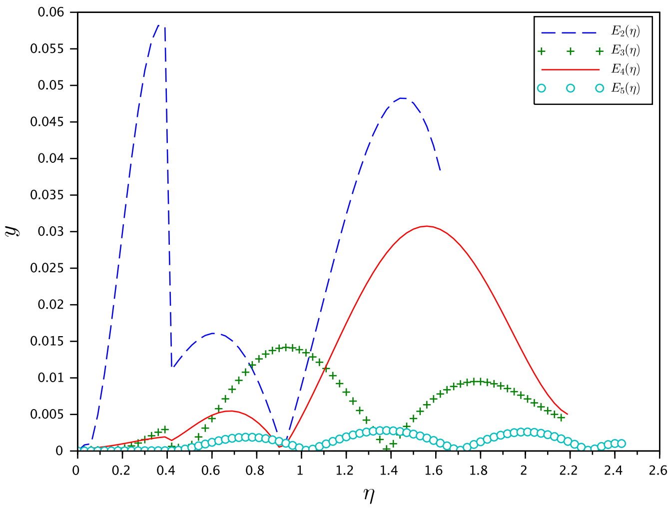

Fig. 2. Absolute error $E_N(\eta)$ for $\eta \in [0, \widetilde{\mu}_N]$, $N = 2, 3, 4, 5$.

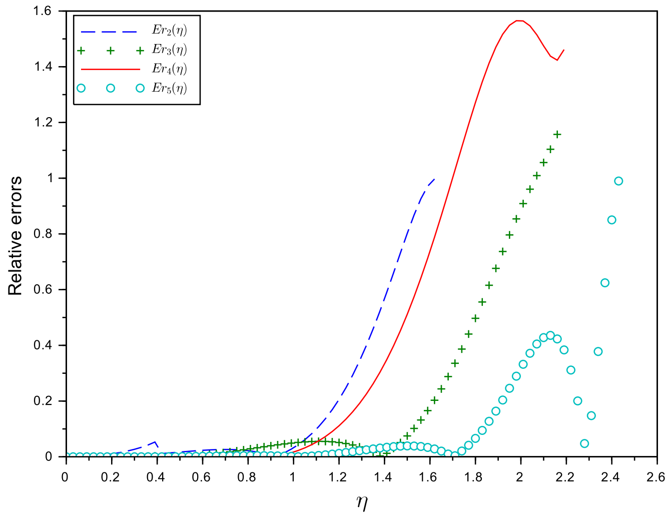

Fig. 3. Relative error $Er_N(\eta)$ for $\eta \in [0, \widetilde{\mu}_N]$, $N = 2, 3, 4, 5$.

Fig. 2 shows the absolute error $E_N(\eta) = |y(\eta) - \widetilde{y}_N(\eta)|$ against $\eta \in [0, \widetilde{\mu}_N]$ for $N = 2, 3, 4, 5$, where

$$y(\eta) = \begin{cases} y_1(\eta) & \text{if} \quad \eta \in [0, \lambda], \\ y_2(\eta) & \text{if} \quad \eta \in [\lambda, +\infty), \end{cases} \quad \text{and} \quad \widetilde{y}_N(\eta) = \begin{cases} \widetilde{y}_{1_N}(\eta) & \text{if} \quad \eta \in [0, \widetilde{\lambda}_N], \\ \widetilde{y}_{2_N}(\eta) & \text{if} \quad \eta \in [\widetilde{\lambda}_N, \widetilde{\mu}_N] \end{cases}$$

and Fig. 3 shows the relative error $E_N(\eta) = \left| \frac{y(\eta) - \widetilde{y}_N(\eta)}{y(\eta)} \right|$.

From Figs. 1 and 2, it is clear that the smallest absolute errors, for both the free boundary s and the function y, occur when $N = 5$. For this case, in Fig. 4 we plot the exact function $y = y(\eta)$ given by (3.1) and (3.2) and the approximate one $y = \widetilde{y}_5(\eta)$ given by (4.41) and (4.42) against $\eta \in [0, \widetilde{\mu}_5]$ with $\widetilde{\mu}_5 = 2.4391824$. In accordance with expectations, the derivatives of the functions y and $\widetilde{y}_5$ have a discontinuity at λ and $\widetilde{\lambda}_5$, respectively. This arises due to the discontinuity of the heat flux at the phase-change interface,

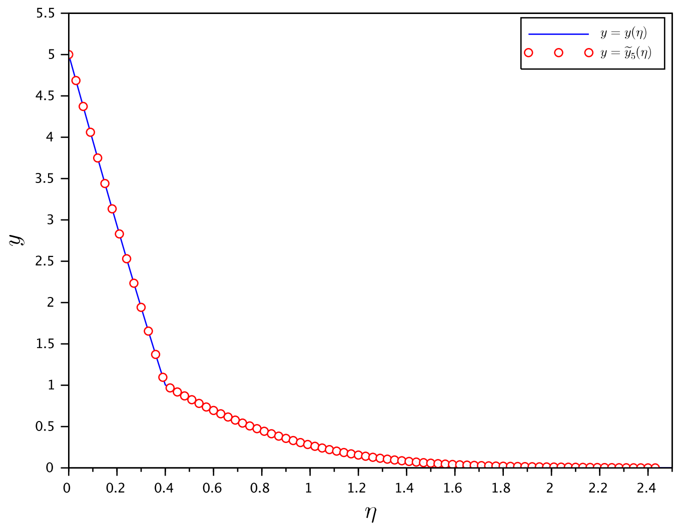

Fig. 4. Exact $y = y(\eta)$ and approximate $y = \widetilde{y}_5(\eta)$ functions for $\eta \in [0, \widetilde{\mu}_5]$ and $\delta = 0$.

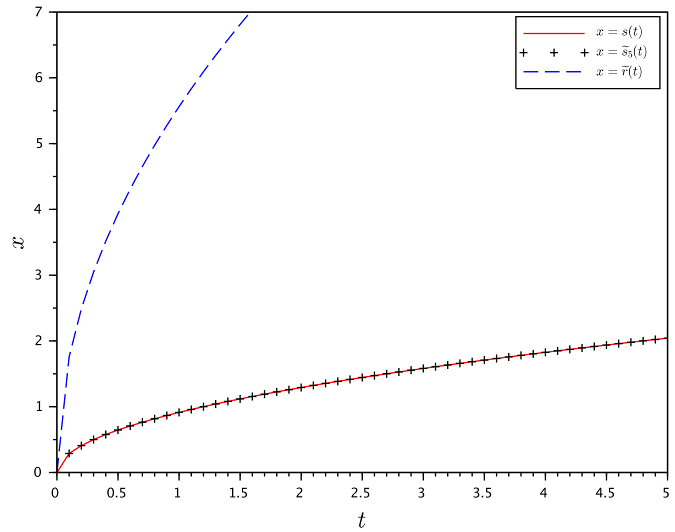

Fig. 5. Exact $x = s(t)$ and approximate $x = \widetilde{s}_5(t)$ free boundaries and thermal layer $x = \widetilde{r}(t)$ against time $t \in (0, 5)s$, for $\alpha_2 = 1.3\ \frac{\mathrm{m}^2}{\mathrm{s}}$ and $\delta = 0$.

$\Gamma(t)$. Finally, Fig. 5 illustrates the agreement between the exact free boundary $x = s(t)$ and its numerical approximation $x = \widetilde{s}_5(t)$ as well as the evolution of the thermal layer $\widetilde{r}(t) = 2\widetilde{\mu}_5\sqrt{\alpha_2 t}$ for $t \in (0, 5)$ s. We can see that $x = \widetilde{r}(t)$ simulates the infinity in the Stefan problem, i.e., condition (2.3).

These results confirm that the numerical method used for obtaining the approximated solutions is efficient and accurate in the case of constant thermal coefficients. Also, as expected, we conclude that the higher the order of the matrix of differentiation is, the more accurate the results are.

5.2. Variable thermal coefficients

Now that the accuracy of the Tau method for the particular case that the thermal coefficients are constants has been established, we are going to approximate the solution to the problem (ODPη) through the solution to the problem (ODPηN), for the variable thermal coefficients given by (2.8) and (2.9).

Let us take the fixed values Ste $= 0.1$, $\alpha = 0.9$, $u_\infty = 2$ °C, $u_s = 1.8$ °C, $u_0 = 1$ °C, $\alpha_2 = 1.3\frac{m^2}{s}$, $\frac{k_2^0}{k_1^0} = 1.1$ and $\delta = 0.1$. Then, we are going to consider two cases: linear ($p = 1$) and quadratic ($p = 2$) temperature-dependent thermal coefficients.

For both cases, the approximate solutions $(\widetilde{y}_{1_N}, \widetilde{y}_{2_N}, \widetilde{\lambda}_N, \widetilde{\mu}_N)$ are obtained by solving the non-linear system defined by (4.45)–(4.49).

Table 3
Solutions to the non-linear system (4.45)-(4.49) for $p = 1$.

N	$\widetilde{\lambda}_N$	$\widetilde{\mu}_N$	Vectors C and D for coefficients of $\widetilde{y}_{i_N}$, $i = 1, 2$ respectively
2	0.4575845	1.6797615	[3.0312449 − 2 − 0.0312449] [0.375 − 0.5 0.125]
3	0.4515848	2.1589221	[3.0310540 − 2.0044779 − 0.0310540 0.0044779] [0.3196039 − 0.4723020 0.1803961 − 0.0276980]
4	0.4519006	2.2690646	[3.0312505 − 2.0044871 − 0.0312823 0.0044763 0.0000996] [0.2988557 − 0.4675062 0.2046120 − 0.0341250 − 0.0040214]
5	0.4518571	2.4586225	[3.0312152 − 2.0045138 − 0.0312902 0.0045240 0.0000750 − 0.0000102] [0.2884689 − 0.4496585 0.2118087 − 0.0540876 − 0.0002775 0.0037461]

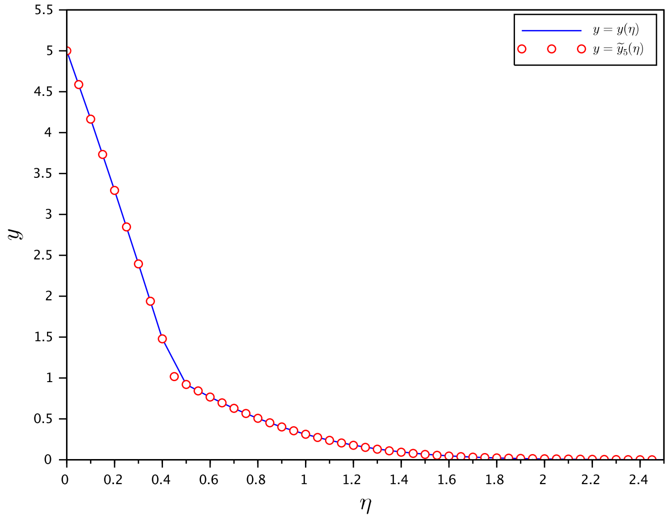

Fig. 6. Exact $y = y(\eta)$ and approximate $y = \widetilde{y}_5(\eta)$ functions for $\eta \in [0, \widetilde{\mu}_5]$ and $p = 1$.

5.2.1. Case 1: Linear temperature-dependent thermal coefficients

This case corresponds to setting $p = 1$ in (2.8) and (2.9). First, we obtain the value of the coefficient that characterizes the exact free boundary, which is the unique solution to Eq. (3.3), and it is given by

$$\lambda = 0.4518618. \tag{5.7}$$

On the other hand, for $N = 2, 3, 4, 5$, the approximate solutions obtained through the solution to the non-linear system (4.45)–(4.49) are presented in Table 3.

Due to the similarity between the exact and approximate solutions for $N = 2, 3, 4, 5$, it becomes visually challenging to observe the difference among them. Therefore, in Fig. 6 we only consider the particular case $N = 5$, and plot the exact function $y = y(\eta)$ given by (3.1) and (3.2) and the approximate one $y = \widetilde{y}_5(\eta)$ given by (4.41) and (4.42) for values of $\eta \in [0, \widetilde{\mu}_5]$ with $\widetilde{\mu}_5 = 2.4586225$. Additionally for this case, in Fig. 7 we show the exact free boundary $x = s(t)$, the approximate free boundary $x = \widetilde{s}_5(t)$ and the thermal layer $\widetilde{r}(t) = 2\widetilde{\mu}_5\sqrt{\alpha_2 t}$ for $t \in (0, 5)$ s.

Moreover, when calculating the global error in $L^2(0, +\infty)$, defined as

$$E_N^G(\eta) = \int_0^{+\infty} \left(y(\eta) - \widetilde{y}_N(\eta)\right)^2 d\eta, \tag{5.8}$$

we observe that for $N = 2$, the error is on the order of 10^{-1}, while for $N = 3$ and $N = 4$, it is of the order of 10^{-3}. For $N = 5$, the global error is on the order of 10^{-4}. These results clearly demonstrate that the approximate values are sufficiently accurate and in good agreement with the exact solution for thermal coefficients that depend linearly on temperature when N increases.

5.2.2. Case 2: Quadratic temperature-dependent thermal coefficients

If we consider the thermal coefficients given by (2.8) and (2.9) with $p = 2$ then the parameter that characterizes the exact free boundary is the unique solution to Eq. (3.3) and is given by

$$\lambda = 0.5544354. \tag{5.9}$$

Moreover, the approximate solutions for $N = 2, 3, 4, 5$, are obtained through the solution to the non-linear system (4.45)–(4.49) and they are given in Table 4.

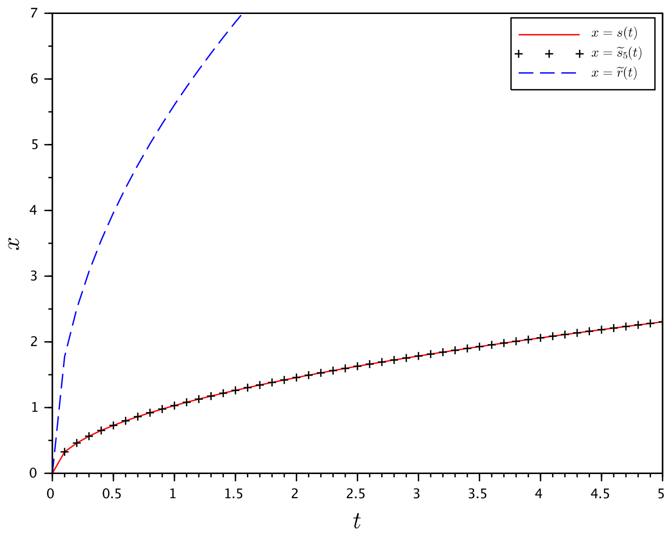

Fig. 7. Exact $x = s(t)$ and approximate $x = \widetilde{s}_5(t)$ free boundaries and thermal layer $x = \widetilde{r}(t)$ against time $t \in (0, 5)$ s, for $\alpha_2 = 1.3\ \frac{\text{m}^2}{\text{s}}$ and $p = 1$.

Table 4
Solutions to the non-linear system (4.45)-(4.49) for $p = 2$.

N	$\widetilde{\lambda_N}$	$\widetilde{\mu}_N$	Vectors C and D for coefficients of $\widetilde{y}_{i_N}$, $i = 1, 2$ respectively
2	0.5331905	1.6976789	[3.2074222 − 2.0000000 − 0.2074222] [0.3750000 − 0.5000000 0.1250000]
3	0.5542882	2.2253863	[3.2245372 − 1.9789331 − 0.2245041 − 0.0210198] [0.3110687 − 0.4680610 0.1888094 − 0.0323157]
4	0.5545971	2.2352235	[3.2255878 − 1.9790813 − 0.2260442 − 0.0209858 0.0004544] [0.2954854 − 0.4689558 0.2091319 − 0.0330290 − 0.0052971]
5	0.5544276	2.4512475	[3.2265636 − 1.9759522 − 0.2273316 − 0.0256361 0.0007680 0.0015883] [0.2878182 − 0.4479734 0.2117631 − 0.0559304 0.0004187 0.0039038]

In this case, the global error in $L^2(0, +\infty)$, as given by Eq. (5.8), is on the order of 10^{-1} for $N = 2$, while for $N = 3$ and $N = 4$, it is on the order of 10^{-3}. Furthermore, for $N = 5$, we observe that the error is on the order of 10^{-4}. For this reason, we consider the particular case $N = 5$ as an example.

In Fig. 8 we plot the exact function $y = y(\eta)$ given by (3.1) and (3.2) and the approximate one $y = \widetilde{y}_5(\eta)$ given by (4.41) and (4.42) against $\eta \in [0, \widetilde{\mu}_5]$ with $\widetilde{\mu}_5 = 2.4512475$. Moreover, for this case, in Fig. 9 we show the exact free boundary $x = s(t)$, the approximate free boundary $x = \widetilde{s}_5(t)$ and the thermal layer $\widetilde{r}(t) = 2\widetilde{\mu}_5\sqrt{\alpha_2 t}$ for $t \in (0, 5)$ s.

From Figs. 8 and 9 we can see that the solution obtained by numerical scheme is nearly equal to the exact solution.

From the analyzed cases, we observed that as N increases, the value of λ_N gets closer to the exact value λ. In particular, for $N = 10$, we obtained $\widetilde{\lambda}_{10} = 0.5544353$ and $\widetilde{\mu}_{10} = 3.7137011$, resulting in an absolute error in λ on the order of 10^{-6}. For this case, a temperature color map of the approximate $\widetilde{u}$ is shown in Fig. 10, and a 3D plot of the same is presented in Fig. 11.

Remark 5.1. The parameter p introduces a nonlinear effect in the model; however, within the tested range and numerical resolution, this influence does not significantly affect the numerical accuracy in the considered examples. As a result, the convergence errors for $p = 1$ and $p = 2$ remain nearly identical.

Conclusions

A one-dimensional two-phase Stefan problem that models the solidification process of a semi-infinite material with temperature-dependent thermal conductivity and specific heat assuming a Dirichlet boundary condition at the fixed face was analyzed. An equivalent ordinary differential problem was obtained through a similarity transformation. Then, a functional problem was deduced and the existence and uniqueness of solution was proved. Furthermore, numerical approximations were obtained using the Tau method, which relies on differentiation using shifted Chebyshev operational matrices. The accuracy of this approach was validated against exact solutions of test problems, highlighting the attractiveness of the Tau method based on shifted Chebyshev polynomials. This validation underscores its ability to achieve excellent agreement between approximate and exact values in numerical examples.

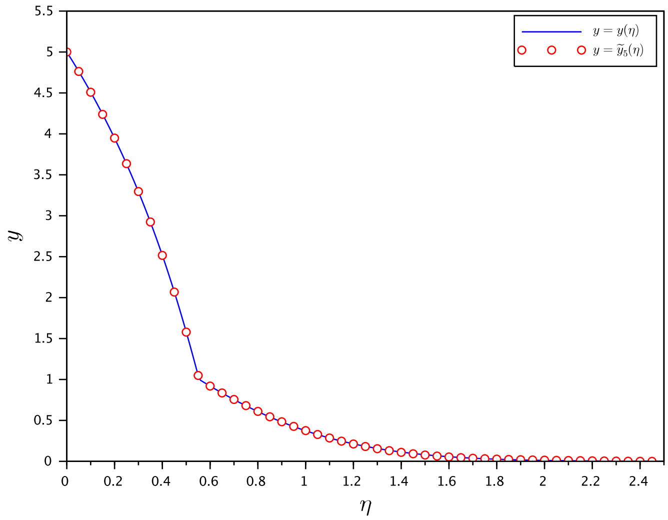

Fig. 8. Exact $y = y(\eta)$ and approximate $y = \widetilde{y}_5(\eta)$ functions for $\eta \in [0, \widetilde{\mu}_5]$ and $p = 2$.

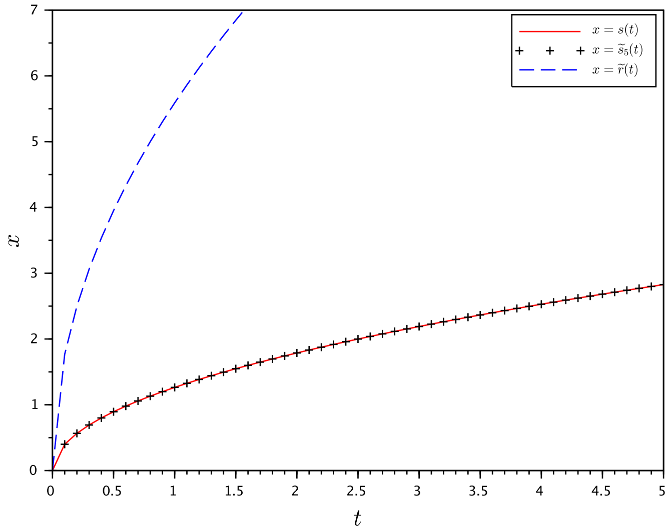

Fig. 9. Exact $x = s(t)$ and approximate $x = \widetilde{s}_5(t)$ free boundaries and thermal layer $x = \widetilde{r}(t)$ against time $t \in (0, 5)$ s, for $\alpha_2 = 1.3\ \frac{\mathrm{m}^2}{\mathrm{s}}$ and $p = 2$.

CRediT authorship contribution statement

Julieta Bollati: Conceptualization, Methodology, Software, Validation, Writing – review & editing. **M. Teresa Cao-Rial:** Conceptualization, Methodology, Software, Validation, Writing – review & editing. **María F. Natale:** Conceptualization, Methodology, Validation, Writing – review & editing. **José A. Semitiel:** Conceptualization, Methodology, Software, Validation, Writing – review & editing. **Domingo A. Tarzia:** Conceptualization, Methodology, Validation, Writing – review & editing, Supervision, Formal analysis, Resources.

Acknowledgment

This work has been partially funded by the European Union's Horizon 2020 Research and Innovation Program under the Marie Sklodowska-Curie grant agreement 823731 CONMECH and the Projects O06-24CI1901 and O06-24CI1903 from Universidad Austral, Rosario, Argentina. Funding for open acces charge was provided by Universidade da Coruña/CISUG.

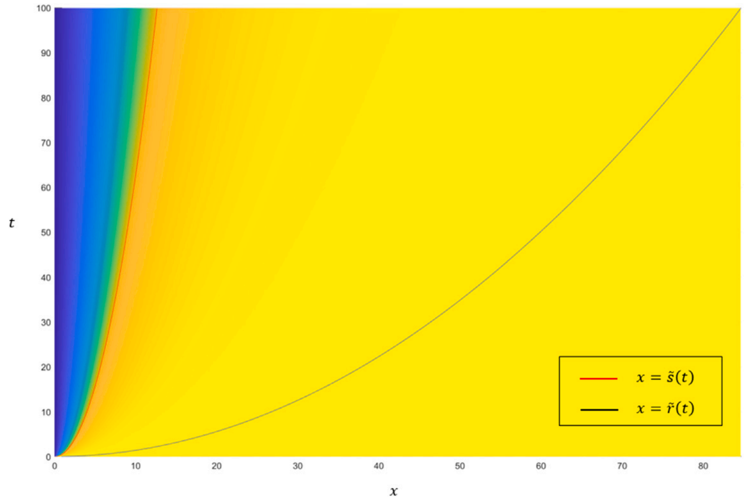

Fig. 10. Approximate temperature color map for $\alpha_2 = 1.3\ \frac{\mathrm{m}^2}{\mathrm{s}}$, $p = 2$ and $N = 10$.

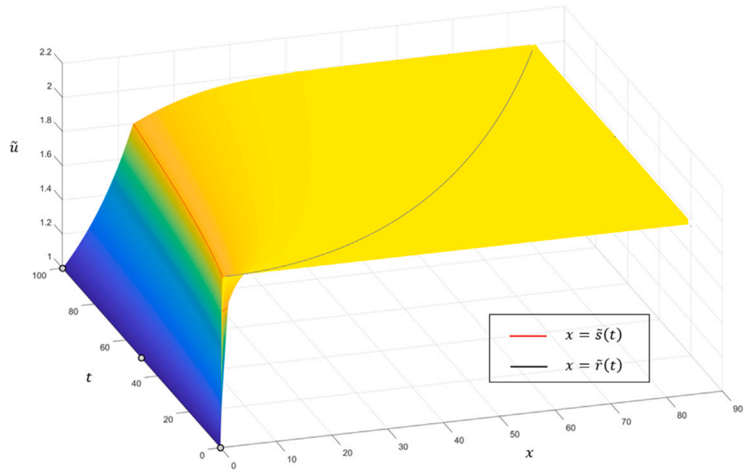

Fig. 11. Approximate temperature $\widetilde{u} = \widetilde{u}(x,t)$ for $\alpha_2 = 1.3\ \frac{\mathrm{m}^2}{\mathrm{s}}$, $p = 2$ and $N = 10$.

Data availability

Data will be made available on request.

References

[1] V. Alexiades, A.D. Solomon, Mathematical Modeling of Melting and Freezing Processes, Hemisphere Publishing Corporation, Washington, 1993.
[2] D.A. Tarzia, A bibliography on moving-free boundary problems for heat diffusion equation. The Stefan problem, in: MAT-Serie A, vol. 2, 2000, pp. 1–297.
[3] F. Brosa Planella, C.P. Please, R.A. Van Gorder, Extended Stefan problem for the solidification of binay alloys in a sphere, European J. Appl. Math. 32 (2) (2021) 242–279.
[4] Y. Belhamadia, G.O. Cassol, S. Dubljevic, Numerical modelling of hyperbolic phase change problems: Application to continuous casting, Int. J. Heat Mass Transfer 209 (2023).
[5] M.P. Dalwadi, S.L. Waters, H.M. Byrne, I.J. Hewitt, A mathematical framework for developing freezing protocols in the cryopreservation of cells, SIAM J. Appl. Math. 80 (2020) 667–689.
[6] V.R. Voller, J.B. Swenson, C. Paola, An analytical solution for a Stefan problem with variable latent heat, Int. J. Heat Mass Transfer 47 (2004) 5387–5390.
[7] S.C. Gupta, The Classical Stefan Problem. Basic Concepts, Modelling and Analysis, Elsevier, Amsterdam, 2018.
[8] J. Bollati, M.F. Natale, J.A. Semitiel, D.A. Tarzia, in: J. Hristov, R. Bennacer (Eds.), Approximate Solutions to the One-Phase Stefan Problem with Temperature-Dependent Thermal Conductivity, Nova Science Publishers, Inc., 2019, pp. 1–20.
[9] J. Bollati, M.F. Natale, J.A. Semitiel, D.A. Tarzia, A two-phase Stefan problem with power-type temperature-dependent thermal conductivity. Existence of a solution by two fixed points and numerical results, in: Special Issue Free or Moving Boundary Problems in Fluid Mechanics, vol. 9, no. 8, AIMS Mathematics, 2024, pp. 21189–21211.

[10] L. Bougoffa, S. Bougouffa, A. Khanfer, An analysis of the one-phase Stefan problem with variable thermal coefficients of order p, Axioms 12 (2023).
[11] T. Nauryz, S. Kharin, Existence and uniqueness for one-phase spherical Stefan problem with nonlinear thermal coefficients and heat flux condition, Int. J. Appl. Math. 35 (2022) 645–659.
[12] D.A. Tarzia, Relationship between Neumann solutions for two-phase Lamé-Clapeyron-Stefan problems with convective and temperature boundary conditions, Therm. Sci. 21–1 (2017) 187–197.
[13] L.D. Venturato, M.B. Cirelli, D.A. Tarzia, Explicit solutions related to the Rubinstein binary-alloy solidification problem with a heat flux or a convective condition at the fixed face, Math. Methods Appl. Sci. 47 (2024) 6770–6788.
[14] J. Bollati, M.F. Natale, J.A. Semitiel, D.A. Tarzia, Exact solution for non-classical one-phase Stefan problem with variable thermal coefficients and two different heat source terms, Comput. Appl. Math. 41 (2022) 1–11.
[15] A.C. Briozzo, C. Rogers, D.A. Tarzia, A class of moving boundary problems with a source term: Application of a reciprocal transformation, Acta Mech. 234 (2023) 1889–1900.
[16] L. Bougoffa, R.C. Rach, A. Mennouni, On the existence, uniqueness, and new analytic approximate solution of the modified error function in two-phase Stefan problems, Math. Methods Appl. Sci. 44 (2021) 10948–10956.
[17] A. Kumar, A.K. Singh, R. Rajeev, A moving boundary problem with variable specific heat and thermal conductivity, J. King Saud Univ. - Sci. 32 (2020) 384–389.
[18] N.N. Salva, D.A. Tarzia, Explicit solution for a Stefan problem with variable latent heat and constant heat flux boundary conditions, J. Math. Anal. Appl. 379 (2011) 240–244.
[19] H. Hekmatjou, Z. Zeng, J. Shen, J.P. Oliveira, H. Naffakh-Moosavy, A comparative study of analytical rosenthal, finite element, and experimental approaches in laser welding of AA5456 alloy, Metals 10 (4) (2020).
[20] A. Kumar, A.K. Singh, R. Rajeev, A Stefan problem with temperature and time dependent thermal conductivity, J. King Saud Univ. - Sci. 32 (2020) 97–101.
[21] S.L. Mitchell, M. Vynnycky, On the numerical solution of two-phase Stefan problems with heat-flux boundary conditions, J. Comput. Appl. Math. 264 (2014) 49–64.
[22] R. Hidki, L. El Moutaouakil, M. Boukendil, Z. Charqui, B. Jamal, Application of an artificial intelligence model for natural convection of nano-encapsulated phase change materials (NEPCMs) confined in a porous square enclosure with an inclined elliptical heated block, Int. Commun. Heat Mass Transfer 155 (107546) (2024) 1–13.
[23] C. Lanczos, Trigonometric interpolation of empirical and analytical functions, Stud. Appl. Math. 17 (1–4) (1938) 123–199.
[24] E.L. Ortiz, The Tau method, SIAM J. Numer. Anal. 6 (3) (1969) 480–492.
[25] I.-R. Horng, J.-H. Chou, Application of shifted Chebyshev series to the optimal control of linear distributed-parameter systems, Internat. J. Control 42 (1) (1985) 233–241.
[26] S. Abbasbandy, A. Taati, Numerical solution of the system of nonlinear Volterra integro-differential equations with nonlinear differential part by the operational Tau method and error estimation, J. Comput. Appl. Math. 231 (1) (2009) 106–113.
[27] E. Hesameddini, M. Riahi, Shifted Chebyshev polynomial method for solving systems of linear and nonlinear Fredhom-Volterra integro-differential equations, J. Math. Ext. 12 (3) (2018) 55–79.
[28] K. Raslan, K. Ali, E. Mohamed, J. Younis, M. Abd El salam, An operational matrix technique based on Chebyshev polynomials for solving mixed Volterra-Fredholm delay integro-differential equations of variable-order, J. Funct. Spaces 2022 (6203440) (2022) 1–15.
[29] M. Sahlan, H. Feyzollahzadeh, Operational matrices of Chebyshev polynomials for solving singular Volterra integral equations, Math. Sci. 11 (2017) 165–171.
[30] H. Samara, Resolución numérica de ecuaciones diferenciales. Formulación variacional del método Tau, Cuadernos Instituto Matemático Beppo Levi, Rosario, 1979.
[31] E. Doha, A. Bhrawy, S. Ezz-Eldien, Efficient Chebyshev spectral methods for solving multi-term fractional orders differential equations, Appl. Math. Model. 35 (12) (2011) 5662–5672.
[32] E. Ortiz, H. Samara, An operational approach to the Tau method for the numerical solution of non-linear differential equations, Computing 27 (1981) 15–25.
[33] D. Abdelhamied, M. Abdelhakem, M. El-Kady, M. Youssri, Adapted shifted ChebyshevU operational matrix of derivates: Two algorithms for solving even-order BVPs, Appl. Math. Inf. 17 (3) (2023) 575–581.
[34] K. Parand, M. Razzaghi, Rational Chebyshev Tau method for solving higher-order ordinary differential equations, Int. J. Comput. Math. 81 (2004) 73–80.
[35] E. Doha, A. Bhrawy, S. Ezz-Eldien, A Chebyshev spectral method based on operational matrix for initial and boundary value problems of fractional order, Comput. Math. Appl. 62 (5) (2011) 2364–2373.
[36] F. Ghoreishi, S. Yazdani, An extension of the spectral Tau method for numerical solution of multi-order fractional differential equations with convergence analysis, Comput. Math. Appl. 61 (2011) 30–43.
[37] M. AliAbadi, E. Ortiz, Numerical treatment of moving and free boundary value problems with the Tau method, Comput. Math. Appl. 35 (8) (1998) 53–61.
[38] A. Kumar, R. Rajeev, A Stefan problem with moving phase change material, variable thermal conductivity and periodic boundary condition, Appl. Math. Comput. 386 (125490) (2020) 1–13.
[39] A. Kumar, A.K. Singh, R. Rajeev, A Stefan problem with variable thermal coefficients and moving phase change material, J. King Saud Univ. Sci. 31 (2019) 1064–1069.
[40] D. Trif, Matrix based operatorial approach to differential and integral problems, in: C.M. Ionescu (Ed.), MATLAB, IntechOpen, Rijeka, 2011.
[41] A.M. Gonzalez, D.A. Tarzia, Determination of unknown coefficients of a semi-infinite material through a simple mushy zone model for the two-phase Stefan problem, Internat. J. Engrg. Sci. 34 (1996) 799–817.