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ON A FREE-MOVING BOUNDARY DIFFUSION PROBLEM IN A CATALYTIC GAS-SOLID SYSTEM WITH CATALYST DECAY*

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Abstract. This paper deals with a free-moving boundary diffusion model which describes a catalytic diffusion-reaction process in a gas-solid system with catalyst decay. We consider a one-phase free boundary diffusion problem for the gaseous poison reactant and a two-phase moving boundary diffusion problem for the main gas reactant.

We prove a local result in time for the existence and uniqueness of the solution of the corresponding free-moving boundary diffusion problem.

Key words. free boundary problems, moving boundary problems, diffusion-reaction gas-solid system, catalyst decay, shrinking core model, integral equations

AMS subject classifications. 35R35, 35C15, 45D05, 45G15

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1. Introduction. It is well known that in many industrial areas, the gas-solid catalytic process plays a significant role. Regarding this type of process, the phenomenon of deactivation of the catalytic sites distributed in a given solid pellet is particularly important from a technological and economical point of view. For some years now, several approaches, devices, mechanisms, and models have been proposed in order to analyze and understand this diffusion-reaction phenomenon. A review can be found in [Ar, FrBi, Ol]. It must be pointed out that there is a lot of experimental evidence that in many gas-solid catalytic industrial processes, the phenomenom of catalyst decay which concerns us seems to occur according to a moving boundary model. This paper deals with a theoretical mathematical analysis of an isothermal free-moving boundary model postulated to describe a gas-solid catalytic system which involves a main reactant gas A, a reactant B absorbed in the solid phase, catalytic active sites S distributed in a solid phase, and a gaseous reactant P at a very low concentration that is only able to interact chemically with active sites S. The species P constitutes a poison for the catalyst in the sense that active catalytic sites are rendered inactive by the absorption of P during the evolution of the process. In this way, it is postulated that once the catalytic pellet is immersed in the medium containing the gaseous mixture $A + P$, the diffusion-reaction process begins from the outside surface of the pellet with a quick and irreversible isothermal chemical reaction. We suppose that the chemical reaction takes place under operative conditions such that the equilibrium constant is very large; this fact accounts for the irreversibility in our model. As a consequence of the decay of the catalyst, at any instant there is a dead region or inert layer, because the catalyst was totally deactivated or completely poi-

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soned, separated by a free boundary or reaction front from the active region (not yet deactivated core). In the formulation of the chemical reaction mechanism we shall suppose that the poisoning reaction is related to the process not by a reaction parallel or consecutive to the main one (that is, for gas A) but through an independent chemical reaction. On the other hand, we suppose that the reactant B absorbed in the solid and the active sites S are uniformly distributed. At any time, the concentration of B is supposed to be in excess with respect to the concentration of A. In section 2 we give the model equations as a free-moving boundary diffusion problem [CaJa, Cr1, Cr2, ElOc, OlPrRa, Pr, Ru, Ta], that is, a one-phase free boundary diffusion problem for the poison P and a two-phase moving boundary diffusion problem for the main gas reactant. The first free boundary problem was solved in [TaVi]. Another formulation is given in [Co, FaPr]. For the difference between a free boundary problem and a moving boundary problem, see [Ta]. We give a new equivalent integral formulation for the second two-phase moving boundary problem. In section 3 we prove a local result in time for the existence and uniqueness of the corresponding second kind Volterra integral equations system.

2. Model equations and integral formulation. Let us consider a solid slab catalytic particle of semithickness L along the gas diffusion direction and with a very low permeability for the gaseous species A and P.

Let us denote by $C_A = C_A(Y,\tau), \widetilde{C}_A = \widetilde{C}_A(Y,\tau), P = P(y,\tau), \widetilde{P} = \widetilde{P}(y,\tau)$ the concentration of gas A and poison P in the inert layer and in the core or active region, respectively. $W = W(Y,\tau)$ denotes the concentration of active sites S in the core or reaction zone; $\sigma = \sigma(\tau)$ denotes the position of the free boundary at time τ (it separates the inert layer from the reaction zone). Then, taking into account the considerations explained in the previous section regarding the mechanism of the system, the corresponding mathematical scheme (Wen's model [TaVi, Vi, We]) can be formulated as follows: Find the functions $C_A = C_A(Y,\tau), \widetilde{C}_A = \widetilde{C}_A(Y,\tau), P = P(y,\tau), \widetilde{P} = \widetilde{P}(y,\tau), W = W(Y,\tau)$ of the spatial variable Y and time variable τ, and the free boundary $\sigma = \sigma(\tau)$ so that they satisfy the following equations and conditions:

$$(1)\quad \left|\begin{array}{lll} \text{(a)}\, \epsilon\dfrac{\partial P}{\partial \tau} = D_P\dfrac{\partial^2 P}{\partial Y^2}, & \sigma(\tau) < Y < L, & \tau_0 < \tau < \tau_1, \\ \text{(b)}\, P(L,\tau) = P_0 = \text{Const.} > 0, & \tau_0 < \tau < \tau_1, & \\ \text{(c)}\, P(Y,\tau_0) = P_1(Y), & L_0 \le Y \le L, & \\ \text{(d)}\, \sigma(\tau_0) = L_0, & & \end{array}\right.$$

$$(2)\quad \left|\begin{array}{lll} \text{(a)}\, \dfrac{\partial \widetilde{P}}{\partial \tau} = \widetilde{D}_P\dfrac{\partial^2 \widetilde{P}}{\partial Y^2} - R(\widetilde{P},W), & 0 < Y < \sigma(\tau), & \tau_0 < \tau < \tau_1, \\ \text{(b)}\, \widetilde{P}(Y,0) = 0, & 0 \le Y \le L_0, & \\ \text{(c)}\, \dfrac{\partial \widetilde{P}}{\partial Y}(0,\tau) = 0, & \tau_0 < \tau < \tau_1, & \end{array}\right.$$

$$(\mathrm{B}(1-2))\quad \left|\begin{array}{l} \text{(a)}\, P(\sigma^+(\tau),\tau) = \widetilde{P}(\sigma^-(\tau),\tau)(\equiv P(\sigma(\tau),\tau)), \\ \text{(b)}\, D_P\dfrac{\partial P}{\partial Y}(\sigma^+(\tau),\tau) - \widetilde{D}_P\dfrac{\partial \widetilde{P}}{\partial Y}(\sigma^-(\tau),\tau) = -\lambda a \overset{\bullet}{\sigma}(\tau), \\ \text{(c)}\, D_P\dfrac{\partial P}{\partial Y}(\sigma^+(\tau),\tau) - \widetilde{D}_P\dfrac{\partial \widetilde{P}}{\partial Y}(\sigma^-(\tau),\tau) = g\left(P(\sigma(\tau),\tau), W(\sigma(\tau),\tau)\right) \\ \text{for}\quad \tau_0 < \tau < \tau_1, \end{array}\right.$$

$$(3)\quad \left|\begin{array}{lll} \text{(a)}\, \dfrac{\partial W}{\partial \tau} = -R\left(\widetilde{P}, W\right), & 0 < Y < \sigma(\tau), & \tau_0 < \tau < \tau_1, \\ \text{(b)}\, W(Y, \tau_0) = W_0, & 0 < Y \le L_0, & \end{array}\right.$$

$$(4)\quad \left|\begin{array}{lll} \text{(a)}\, \epsilon \dfrac{\partial C_A}{\partial \tau} = D_A \dfrac{\partial^2 C_A}{\partial Y^2}, & \sigma(\tau) < Y < L, & \tau_0 < \tau < \tau_1, \\ \text{(b)}\, C_A(L, \tau) = V_0 = \text{Const.} > 0, & \tau_0 < \tau < \tau_1, & \\ \text{(c)}\, C_A(Y, \tau_0) = \Phi(Y), & L_0 \le Y \le L, & \end{array}\right.$$

$$(5)\quad \left|\begin{array}{lll} \text{(a)}\, \dfrac{\partial \widetilde{C}_A}{\partial \tau} = \widetilde{D}_A \dfrac{\partial^2 \widetilde{C}_A}{\partial Y^2} - \gamma \widetilde{C}_A, & 0 < Y < \sigma(\tau), & \tau_0 < \tau < \tau_1, \\ \text{(b)}\, \dfrac{\partial \widetilde{C}_A}{\partial Y}(0, \tau) = 0, & \tau_0 < \tau < \tau_1, & \\ \text{(c)}\, \widetilde{C}_A(Y, 0) = 0, & 0 \le Y \le L_0, & \end{array}\right.$$

$$(\text{B}(4-5))\quad \left|\begin{array}{ll} \text{(a)}\, C_A(\sigma^+(\tau), \tau) = \widetilde{C_A}(\sigma^-(\tau), \tau) (\equiv C_A(\sigma(\tau), \tau)), & \tau_0 < \tau < \tau_1, \\ \text{(b)}\, D_A \dfrac{\partial C_A}{\partial Y}(\sigma^+(\tau), \tau) = \widetilde{D_A} \dfrac{\partial \widetilde{C_A}}{\partial Y}(\sigma^-(\tau), \tau), & \tau_0 < \tau < \tau_1, \end{array}\right.$$

where $\epsilon, D_A, \widetilde{D}_A, D_P, \widetilde{D}_P, a,$ and γ are positive constants denoting the porosity of the inert layer, the effective diffusion coefficients in the porous layer and in the reaction zone, the stoichiometric coefficient, and the constant of chemical reaction velocity, respectively. The constant λ contains the kinetic constant and W_0. In (1)–(5) we are assuming that at the time τ_0 a porous inert layer of thickness $L - L_0$ is already formed and this explains the initial conditions (1c), (2b), (1d), (4c), and (5c). On the other hand, we have considered a first order homogeneous chemical reaction with respect to the gaseous reactant A. Equations (1a), (2a), (4a), and (5a) express the mass balance in the respective domains for gaseous species A and P. On the fixed boundary, conditions are prescribed by (1b), (2c), (4b), and (5c). On the free boundary $Y = \sigma(\tau)$, (B$(1-2)$c) expresses the equality of the rate of mass consumption of the species P in the surface reaction and the incoming mass flux of the same component where g represents the kinetics on the reaction front. We remark that the term proportional to $P(\sigma(\tau), \tau) . \dot{\sigma}(\tau)$ has been considered to be negligible with respect to $\lambda . \dot{\sigma}(\tau)$ in (B$(1-2)$b) (an acceptable assumption in gas-solid systems).

Equation (B$(1-2)$b) states the same preceding balance in terms of the free boundary velocity, taking into account that $-\lambda . \dot{\sigma}(\tau)$ is also the rate of mass consumption of P. Equation (3) gives the mass balance for the active sites in the reaction zone, where R denotes the kinetic expression for the poisoning reaction. It is a known function. It can be seen that (1)–(5) constitute a system of free-moving boundary problems for the diffusion equation. In fact, it is clear that (1), (2), (B$(1-2)$), (3) represent a free boundary problem related to a moving boundary problem given by (4), (5), (B$(4-5)$). As a first stage, in this paper we shall analyze a moving boundary problem which comes from (1)–(5) as a limiting case. This limiting case appears if we assume that the diffusion of the poison P in the reaction zone is negligible, being the amount of homogeneous chemical reaction $R(\widetilde{P}, W)$, which is also negligible in comparison to the surface chemical reaction $g(P, W)$, which is very fast. Under such considerations, at the place of (2), (B$(1-2)$), and (3) we can write

$$
(6)\quad \left|\begin{array}{lll}
\text{(a) } \widetilde{P} \equiv 0, & 0 < Y < \sigma(\tau), & \tau_0 < \tau < \tau_1, \\
\text{(b) } - D_P \dfrac{\partial P}{\partial Y}(\sigma(\tau), \tau) = a\lambda \overset{\bullet}{\sigma}(\tau), & \tau_0 < \tau < \tau_1, & \\
\text{(c) } D_P \dfrac{\partial P}{\partial Y}(\sigma(\tau), \tau) = \widehat{g}(P(\sigma(\tau), \tau)), & \tau_0 < \tau < \tau_1, & \\
\text{(d) } W(\tau) \equiv W_0 = \text{Const.} > 0, & \tau_0 < \tau < \tau_1. &
\end{array}\right.
$$

Hence, taking into account (6), the system (1)–(3) leads to the following one: Find the functions $P = P(Y, \tau)$ and $\sigma = \sigma(\tau)$ such that they satisfy the following conditions:

$$
(7)\quad \left|\begin{array}{lll}
\text{(a) } \epsilon \dfrac{\partial P}{\partial \tau} = D_P \dfrac{\partial^2 P}{\partial Y^2}, & \sigma(\tau) < Y < L, & \tau_0 < \tau < \tau_1, \\
\text{(b) } P(L, \tau) = P_0 = \text{Const.} > 0, & \tau_0 < \tau < \tau_1, & \\
\text{(c) } P(Y, \tau_0) = P_1(Y), & L_0 \le Y \le L, & \\
\text{(d) } - D_P \dfrac{\partial P}{\partial Y}(\sigma(\tau), \tau) = a\lambda \overset{\bullet}{\sigma}(\tau), & \tau_0 < \tau < \tau_1, & \\
\text{(e) } D_P \dfrac{\partial P}{\partial Y}(\sigma(\tau), \tau) = \widehat{g}(P(\sigma(\tau), \tau)), & \tau_0 < \tau < \tau_1, & \\
\text{(f) } \sigma(\tau_0) = L_0. & &
\end{array}\right.
$$

The problems (4), (5), (B(4−5)) remain unchanged in form, but we must take into account that from now on, in these problems the moving boundary $\sigma = \sigma(\tau)$ comes from the solution of the free boundary problem (7). We remark that in the expression of the function $\widehat{g} = \widehat{g}(P(\sigma(\tau), \tau))$ in the right-hand side of (7e), the concentration of the sites appears as the constant W_0. In general, in (7) $\widehat{g}$ may be any real function such that

$$\widehat{g} = 0 \quad \text{if} \quad P(\sigma(\tau), \tau) = 0 \quad \text{or} \quad W_0 = 0.$$

We pointed out that in [TaVi] a local result in time for the existence and uniqueness of the solution of the free boundary problem (7) was obtained under suitable assumptions on the data P_0, P_1, and $\widehat{g}$. In what follows of this paper, we shall study the remaining moving boundary problem (4), (5), (B(4 − 5)) taking into account that the moving boundary $\sigma = \sigma(\tau)$ belongs to a given class of functions (see assumptions below). In order to write the model (4), (5), (B(4 − 5)) in a more suitable form, we introduce the variables x, t; functions $u = u(Y, \tau)$, $U = U(x, t)$, $V = V(x, t)$, $h = h(x)$, $s = s(t)$, and parameters $\widehat{D}_A$, b, T:

$$
(8)\quad \left|\begin{array}{ll}
x = \dfrac{L - Y}{L}, & t = \dfrac{1}{L^2}(\tau - \tau_0), \\
u(Y, \tau) = \exp(\gamma\tau)\widetilde{C}_A(Y, \tau), & U(x, t) = u(1 - x, L^2 t + \tau_0), \\
V(x, t) = C_A(1 - x, L^2 t + \tau_0), & h(x) = \Phi(L(1 - x)), \\
s(t) = \dfrac{L - \sigma(\tau)}{L}, & b = 1 - \dfrac{L_0}{L} \in (0, 1), \\
\widehat{D}_A = \dfrac{D_A}{\epsilon}, & D = \dfrac{\widetilde{D}_A}{\widehat{D}_A},\ T = \dfrac{1}{L^2}(\tau_1 - \tau_0).
\end{array}\right.
$$

Taking into account (8), the problem (4), (5), (B(4 − 5)) is transformed into

$$
(9)\quad \left|\begin{array}{lll}
\text{(a) } V_t = \widehat{D}_A V_{xx} & \text{in} & \Omega_T^1 \equiv \{(x, t) / 0 < x < s(t), 0 < t < T\}, \\
\text{(b) } V(0, t) = V_0 = \text{Const.} > 0, & & 0 < t < T, \\
\text{(c) } V(x, 0) = h(x), & & 0 \le x \le b,
\end{array}\right.
$$

$$(10)\qquad \begin{vmatrix} \text{(a)}\, U_t = \widetilde{D}_A U_{xx} & \text{in} \quad \Omega_T^2 \equiv \{(x,t)\,/s(t) < x < 1\,, 0 < t < T\}, \\ \text{(b)} U_x(1,t) = 0, & 0 < t < T, \\ \text{(c)} U(x,0) = 0, & b \le x \le 1, \end{vmatrix}$$

$$(\mathrm{B}(9-10))\qquad \begin{vmatrix} \text{(a)}\, V(s(t),t) = \exp(-\gamma t) U(s(t),t), \\ \text{(b)}\, V_x(s(t),t) = D\exp(-\gamma\tau) U_x(s(t),t). \end{vmatrix}$$

Remark 1. We point out that the diffusivity of the main reactant D_A and $\widetilde{D}_A$ in the inert layer and in the reaction zone, respectively, are different since the inert layer is a porous media and the reaction zone has very low permeability. Then we suppose that

(A_0) $\qquad D \neq 1.$

From [TaVi] (see also [Co, FaPr]), we assume that the moving boundary $s = s(t)$ is a monotone increasing function which satisfies the following assumptions (for a $T > 0$ small enough):

(A_1) $\quad \begin{array}{lll} s \in C^1[0,T], & s\,(0) = b \in (0,1), & 0 < a \le s(t) \le A < 1, \\ 0 < \alpha_0 \le \dot{s}\,(t) \le \beta_0, & t \in [0,T]. & \end{array}$

(A_2) There exists $\ddot{s}$ such that

$$\left|\ddot{s}\,(t)\right| \le \delta_0, \qquad t \in [0,T],$$

where a, A, α_0, β_0, and δ_0 are suitable positive constants which depend upon the data T, P_0, P_1, and $\widehat{g}$.

We also assume, for the initial concentration $h = h(x)$ in (9),

(A_3) $\quad h \in C^1[0,b], \qquad h \ge 0, \qquad h\,(b) = 0, \qquad h\,(0) = V_0 > 0.$

We say that the functions $V = V(x,t)$ and $U = U(x,t)$ are a solution for the problem (9), (10), and (B(9−10)) if $V \in C^{2,1}(\Omega_T^1), U \in C^{2,1}(\Omega_T^2)$, and they satisfy the (9a) and (10a) and conditions (9b), (9c), (10b), (10c), (B(9−10)a), and (B(9−10)b).

Now we shall obtain an equivalent integral formulation for problem (9), (10), (B(9 − 10)). This formulation will be used in the next section, where the main result of the paper, that is, a theorem on local existence and uniqueness of the solution of the moving boundary problem (9), (10), (B(9 − 10)), is proved.

Let $G_1 = G_1(x,t;\xi,\tau)$, $N_1 = N_1(x,t;\xi,\tau)$ and $G_2 = G_2(x,t;\xi,\tau)$, $N_2 = N_2(x,t;\xi,\tau)$ be the Green's and Neumann's functions for the sets Ω_T^1 and Ω_T^2, respectively, defined by

$$(11)\qquad \begin{vmatrix} G_1(x,t;\xi,\tau) = K_1(x,t;\xi,\tau) - K_1(-x,t;\xi,\tau), \\ N_1(x,t;\xi,\tau) = K_1(x,t;\xi,\tau) + K_1(-x,t;\xi,\tau); \end{vmatrix}$$

$$(12)\qquad \begin{vmatrix} G_2(x,t;\xi,\tau) = K_2(x-1,t;\xi-1,\tau) - K_2(1-x,t;\xi-1,\tau), \\ N_2(x,t;\xi,\tau) = K_2(x-1,t;\xi-1,\tau) + K_2(1-x,t;\xi-1,\tau), \end{vmatrix}$$

where

$$(13)\qquad K_1(x,t;\xi,\tau) = \frac{1}{2\sqrt{\pi\widehat{D}_A}\sqrt{t-\tau}} \exp\left(-\frac{(x-\xi)^2}{4\widehat{D}_A\,(t-\tau)}\right), \quad x,\xi > 0, \quad t > \tau,$$

$$(14)\qquad K_2(x,t;\xi,\tau) = \frac{1}{2\sqrt{\pi\widetilde{D}_A}\sqrt{t-\tau}} \exp\left(-\frac{(x-\xi)^2}{4\widetilde{D}_A\,(t-\tau)}\right), \quad x,\xi > 0, \quad t > \tau.$$

If we take into account (9), (10), (B(9 − 10)), it is well known that if we integrate the identities

$$\widehat{D}_A \frac{\partial}{\partial \xi}\left(G_1 V_\xi - G_{1\xi} V\right) = \frac{\partial}{\partial \tau}(G_1 V), \tag{15}$$

$$\widetilde{D}_A \frac{\partial}{\partial \xi}\left(N_2 U_\xi - N_{2\xi} U\right) = \frac{\partial}{\partial \tau}(N_2 U) \tag{16}$$

over the domains $0 < \xi < s(t)$, $0 < \epsilon < \tau < t - \epsilon$, and $s(\tau) < \xi < 1$, $0 < \epsilon < \tau < t - \epsilon$, respectively, and take the limit $\epsilon \to 0^+$, then the following representation for the functions $V = V(x,t)$ y $U = U(x,t)$ can be obtained:

$$\begin{aligned} V(x,t) = {} & \int_0^b G_1(x,t;\xi,0)\ h(\xi)\, d\xi + V_0 \widehat{D}_A \int_0^t G_{1\xi}(x,t;0,\tau)\ d\tau \\ & + \int_0^t G_1(x,t;s(\tau),\tau)\ V(s(\tau),\tau)\ \overset{\bullet}{s}(\tau)\, d\tau \\ & + \widehat{D}_A \int_0^t \left[G_1(x,t;s(\tau),\tau)\ V_\xi(s(\tau),\tau) - G_{1\xi}(x,t;s(\tau),\tau)\ V(s(\tau),\tau)\right]\ d\tau; \end{aligned} \tag{17}$$

$$\begin{aligned} U(x,t) = {} & \int_0^t \left[\widetilde{D}_A N_{2\xi}(x,t;s(\tau),\tau) - \overset{\bullet}{s}(\tau)\ N_2(x,t;s(\tau),\tau)\right] U(s(\tau),\tau)\ d\tau \\ & - \widetilde{D}_A \int_0^t N_2(x,t;s(\tau),\tau)\ U_\xi(s(\tau),\tau)\ d\tau. \end{aligned} \tag{18}$$

Since $-N_{1x} = G_{1\xi}$, we note that the equality given by (17) can also be written as

$$\begin{aligned} V(x,t) = {} & \int_0^b G_1(x,t;\xi,0)\ h(\xi)\, d\xi + V_0 \widehat{D}_A \int_0^t G_{1\xi}(x,t;0,\tau)\ d\tau \\ & + \int_0^t G_1(x,t;s(\tau),\tau)\ V(s(\tau),\tau)\ \overset{\bullet}{s}(\tau)\, d\tau \\ & + \widehat{D}_A \int_0^t \left[G_1(x,t;s(\tau),\tau)\ V_\xi(s(\tau),\tau) + N_{1x}(x,t;s(\tau),\tau)\ V(s(\tau),\tau)\right]\ d\tau. \end{aligned} \tag{17bis}$$

Remark 2. We pointed out that in our case, the following jump relation holds [Fr, Ru]:

$$\begin{aligned} & \lim_{x \to s(t)^-} \int_0^t \rho(\tau)\ G_{1x}(x,t;s(\tau),\tau)\ d\tau \\ & \quad = \frac{1}{2}\frac{\rho(t)}{\widehat{D}_A} + \int_0^t \rho(\tau)\ G_{1x}(s(t),t;s(\tau),\tau)\ d\tau, \end{aligned} \tag{19}$$

$$\begin{aligned} & \lim_{x \to s(t)^+} \int_0^t \rho(\tau)\ N_{2x}(x,t;s(\tau),\tau)\ d\tau \\ & \quad = \frac{1}{2}\frac{\rho(t)}{\widetilde{D}_A} + \int_0^t \rho(\tau)\ N_{2x}(s(t),t;s(\tau),\tau)\ d\tau. \end{aligned} \tag{20}$$

From (B(9 − 10)), (17), (17bis), and (18), and taking into account (A_1), (A_2), (19), and (20), we obtain the following system of integral equations for the unknown functions $V_x = V_x(s(t),t)$, $F_0 = F_0(s(t))$, $V = V(s(t),t)$ (see the appendix):

$$\text{(21)}\quad \left|\begin{array}{l} \text{(a) } V_x(s(\tau),\tau) = \frac{1}{1-D}\left\{P_1(t) - DP_0(t) + \left[\frac{\gamma}{\dot{s}(\tau)} + \left(\frac{\widehat{D}_A - \widetilde{D}_A}{\widehat{D}_A\widetilde{D}_A}\right)\dot{s}(\tau)\right]P_2(t)\,D\right\}, \\ \text{(b) } F_0(s(\tau)) = \frac{1}{1-D}\left\{P_1(t) - P_0(t) + \left[\frac{D\gamma}{\dot{s}(\tau)} + (1-D)\frac{\ddot{s}(\tau)}{\dot{s}^2(\tau)}\right]P_2(t)\right\}, \\ \text{(c) } V(s(t),t) = P_2(t), \end{array}\right.$$

where

$$\text{(22)}\quad \left|\begin{array}{l} \text{(a) } F_0(s(t)) \equiv \dfrac{1}{\dot{s}(\tau)}\dfrac{d}{dt}V(s(t),t), \quad V(s(t),t) \equiv V(x,t)|_{x=s(t)^-}; \\ \text{(b) } V_x(s(t),t) \equiv V_x(x,t)|_{x=s(t)^-} \end{array}\right.$$

and

$$P_0(t) = \overline{f}_0(t) + \int_0^t \overline{H}_0(t,\tau,V_x(s(\tau),\tau),V(s(\tau),\tau),F_0(s(\tau)))\,d\tau, \tag{23}$$

$$P_1(t) = \overline{f}_1(t) + \int_0^t \overline{H}_1(t,\tau,V_x(s(\tau),\tau),V(s(\tau),\tau),F_0(s(\tau)))\,d\tau, \tag{24}$$

$$P_2(t) = f_2(t) + \int_0^t H_2(t,\tau,V_x(s(\tau),\tau),V(s(\tau),\tau),F_0(s(\tau)))\,d\tau, \tag{25}$$

with

$$\overline{f}_0(t) = 2\int_0^b G_{1x}(s(\tau),\tau;\xi,0)\,h(\xi)\,d\xi - 2V_0\widehat{D}_A N_1(s(t),t;0,0), \tag{26}$$

$$\begin{aligned} \overline{H}_0(t,\tau) &= 2\left[1 + \widehat{D}_A\alpha(\tau)\right]G_{1x}(s(t),t;s(\tau),\tau)\,V(s(\tau),\tau)\,\dot{s}(\tau) \\ &+ \widehat{D}_A\left[F_0(s(\tau)) + V_\xi(s(\tau),\tau)\right]G_{1x}(s(t),t;s(\tau),\tau), \end{aligned} \tag{27}$$

$$\alpha(\tau) = -\frac{\ddot{s}(\tau)}{\dot{s}^3(\tau)}, \tag{28}$$

$$\overline{f}_1(t) \equiv 0, \tag{29}$$

$$\begin{aligned} \overline{H}_1(t,\tau) &= -2\Big\{\exp(-\gamma(t-\tau))\,D\widehat{D}_A F_0(s(t))\,N_{2x}(s(t),t;s(\tau),\tau) \\ &+ \exp(-\gamma t)\,DV(s(\tau),\tau).\Big[\exp(\gamma\tau)\,\dot{s}(\tau).\Big(1 + \widehat{D}_A\alpha(\tau)\Big) \\ &+ \widehat{D}_A\,\beta(\tau)\Big]N_{2x}(s(t),t;s(\tau),\tau)\Big\} \\ &- 2\widetilde{D}_A\exp(-\gamma(t-\tau))\,V_\xi(s(\tau),\tau)N_{2x}(s(t),t;s(\tau),\tau), \end{aligned} \tag{30}$$

$$\beta(\tau) = \frac{\gamma}{\dot{s}(\tau)}\exp(\gamma\tau), \tag{31}$$

$$f_2(t) = 2\int_0^b G_1(s(t),t;\xi,0)\,h(\xi)\,d\xi + 2V_0\widehat{D}_A\int_0^b G_{1\xi}(s(t),t;0,\tau)d\tau, \tag{32}$$

$$H_2(t,\tau) = 2G_1(s(t),t;s(\tau),\tau)\overset{\bullet}{s}(\tau)\,V(s(\tau),\tau) + 2\widehat{D}_A N_{1x}(s(t),t;s(\tau),\tau)V(s(\tau),\tau) + 2\widehat{D}_A G_1(s(t),t;s(\tau),\tau)V_\xi(s(\tau),\tau). \tag{33}$$

Remark 3. Let us denote by $\Phi_0 = \Phi_0(t)$, $\Phi_1 = \Phi_1(t)$, $\Phi_2 = \Phi_2(t)$ the functions defined, respectively, by

$$\Phi_0(t) = V_x(s(t),t), \qquad \Phi_1(t) = F_0(s(t)), \qquad \Phi_2(t) = V(s(t),t) \tag{D}$$

and let us introduce the functions $f_0 = f_0(t)$, $f_1 = f_1(t)$, $\mu_0 = \mu_0(t)$, $\mu_1 = \mu_1(t)$, $H_0 = H_0(t,\tau,\Phi_0(\tau),\Phi_1(\tau),\Phi_2(\tau))$, $H_1 = H_1(t,\tau,\Phi_0(\tau),\Phi_1(\tau),\Phi_2(\tau))$ which are defined by

$$\mu_0(t) = \left(\frac{\gamma}{\overset{\bullet}{s}(t)} + \left(\frac{\widehat{D}_A - \widetilde{D}_A}{\widehat{D}_A\widetilde{D}_A}\right)\overset{\bullet}{s}(t)\right)\frac{D}{1-D}, \tag{34}$$

$$\mu_1(t) = \left(\frac{D\gamma}{\overset{\bullet}{s}(t)} + (1-D)\frac{\overset{\bullet\bullet}{s}(t)}{\overset{\bullet}{s}^2(t)}\right)\frac{1}{1-D}, \tag{35}$$

$$f_0(t) = \mu_0(t)\,f_2(t) - \frac{D}{1-D}\,\overline{f}_0(t), \tag{36}$$

$$f_1(t) = \mu_1(t)\,f_2(t) - \frac{\overline{f}_0(t)}{1-D}, \tag{37}$$

(38)

$$H_1(t,\tau,\Phi_0(\tau),\Phi_1(\tau),\Phi_2(\tau)) = \frac{1}{1-D}\big(\overline{H}_1(t,\tau,\Phi_0(\tau),\Phi_1(\tau),\Phi_2(\tau)) - \overline{H}_0(t,\tau,\Phi_0(\tau),\Phi_1(\tau),\Phi_2(\tau))\big) + \mu_1(t)\,H_2(t,\tau,\Phi_0(\tau),\Phi_1(\tau),\Phi_2(\tau)),$$

(39)

$$H_0(t,\tau,\Phi_0(\tau),\Phi_1(\tau),\Phi_2(\tau)) = \frac{1}{1-D}\big(\overline{H}_1(t,\tau,\Phi_0(\tau),\Phi_1(\tau),\Phi_2(\tau)) - D\overline{H}_0(t,\tau,\Phi_0(\tau),\Phi_1(\tau),\Phi_2(\tau))\big) + \mu_0(t)\,H_2(t,\tau,\Phi_0(\tau),\Phi_1(\tau),\Phi_2(\tau)).$$

Then, the system of integral equations (21) can be written in a compact form as

$$\Phi_i(t) = f_i(t) + \int_0^t H_i(t,\tau,\Phi_0(\tau),\Phi_1(\tau),\Phi_2(\tau))\,d\tau, \quad i = 0,1,2. \tag{40}$$

We have the following lemma.

LEMMA 1. *Finding the solution of problem* (9), (10), *and* (B(9−10)) *is equivalent to the problem of solving the system of integral equations* (40) *(that is,* (21)*).*

Proof. From the previous computation and the appendix we can see that the system of integral equations (21) (that is, (40)) is a necessary condition for finding the solution to problem (9), (10), (B(9 − 10)).

Conversely, let us suppose that for some $w > 0$ (e.g., $w \le T$) the functions $\Phi_0 = \Phi_0(\tau)$, $\Phi_1 = \Phi_1(\tau)$, and $\Phi_2 = \Phi_2(\tau)$ satisfy the system of integral equations

(40) for $t \in [0, w]$. Then we can define the functions $V = V(x,t)$ and $U = U(x,t)$ by

$$
\begin{aligned}
V(x,t) = & \int_0^b G_1(x,t;\xi,0)\,h(\xi)\,d\xi + V_0 \widehat{D}_A \int_0^t G_{1\xi}(x,t;0,\tau)\,d\tau \\
& + \int_0^t \left[G_1(x,t;s(\tau),\tau)\,\overset{\bullet}{s}(\tau) + \widehat{D}_A N_{1x}(x,t;s(\tau),\tau) \right] \Phi_2(\tau)\,d\tau \\
& + \widehat{D}_A \int_0^t G_1(x,t;s(\tau),\tau)\,\Phi_0(\tau)\,d\tau,
\end{aligned} \tag{41}
$$

(42)

$$
\begin{aligned}
U(x,t) = & \int_0^t \left[\widetilde{D}_A N_{2\xi}(x,t;s(\tau),\tau) - \overset{\bullet}{s}(\tau)\,N_2(x,t;s(\tau),\tau) \right] \exp(\gamma\tau)\,\Phi_2(\tau)\,d\tau \\
& - \widehat{D}_A \int_0^t N_2(x,t;s(\tau),\tau)\exp(\gamma\tau)\,\Phi_0(\tau)\,d\tau.
\end{aligned}
$$

It is easy to see that they satisfy (9a) and (10a) and conditions (9b), (9c), (10b), and (10c) because of the basic properties of the functions G_i and N_i $(i = 1, 2)$ and the fact that

$$
\int_0^t G_{1\xi}(x,t;0,\tau)\,d\tau = \frac{1}{\widehat{D}_A} \operatorname{erfc}\left(\frac{x}{2\sqrt{t}} \right), \quad x \geq 0, \quad t > 0, \tag{43}
$$

$$
N_{2x}(1,t;s(\tau),\tau) = 0, \quad t > \tau. \tag{44}
$$

Now it remains to prove that both conditions $(\mathrm{B}(9-10)\mathrm{a})$ and $(\mathrm{B}(9-10)\mathrm{b})$ are satisfied. Taking into account the jump relation (19), taking the limit $x \to s(t)^-$ in (41) we obtain that

$$
\begin{aligned}
V(s(t),t) = \; & \int_0^b G_1(s(t),t;\xi,0)\,h(\xi)\,d\xi + V_0 \widehat{D}_A \int_0^t G_{1\xi}(s(t),t;0,\tau)\,d\tau \\
& + \int_0^t G_1(s(t),t;s(\tau),\tau)\,\overset{\bullet}{s}(\tau)\,\Phi_2(\tau)\,d\tau \\
& + \widehat{D}_A \left(\frac{1}{2}\frac{\Phi_2(t)}{\widehat{D}_A} + \int_0^t N_{1x}(s(t),t;s(\tau),\tau)\,\Phi_2(\tau)\,d\tau \right) \\
& + \widehat{D}_A \int_0^t G_1(s(t),t;s(\tau),\tau)\,\Phi_0(\tau)\,d\tau = \frac{f_2(t)}{2} + \frac{\Phi_2(t)}{2} \\
& + \int_0^t G_1(s(t),t;s(\tau),\tau)\,\overset{\bullet}{s}(\tau)\,\Phi_2(\tau) \\
& + \widehat{D}_A \int_0^t N_{1x}(s(t),t;s(\tau),\tau)\,\Phi_2(\tau)\,d\tau \\
& + \widehat{D}_A \int_0^t G_1(s(t),t;s(\tau),\tau)\,\Phi_0(\tau)\,d\tau = \Phi_2(t), 0 < t < w.
\end{aligned} \tag{45}
$$

In a similar way, taking the limit $x \to s(t)^+$ in (42) we obtain

$$
\begin{aligned}
U(s(t),t) = & -\widetilde{D}_A \Bigg[-\frac{1}{2\widetilde{D}_A} \exp(\gamma\tau)\, \Phi_2(t) \\
& \qquad + \int_0^t G_{2x}(s(t), t; s(\tau), \tau) \exp(\gamma\tau)\, \Phi_2(\tau)\, d\tau \Bigg] \\
& - \int_0^t \overset{\bullet}{s}(\tau)\, N_2(s(t), t; s(\tau), \tau) \exp(\gamma\tau)\, \Phi_2(\tau)\, d\tau \\
& - \widehat{D}_A \int_0^t N_2(s(t), t; s(\tau), \tau) \exp(\gamma\tau)\, \Phi_0(\tau)\, d\tau \\
= & \frac{\exp(\gamma\tau)\, \Phi_2(t)}{2} - \widetilde{D}_A \int_0^t G_{2x}(s(t), t; s(\tau), \tau) \exp(\gamma\tau)\, \Phi_2(\tau)\, d\tau \\
& - \int_0^t \overset{\bullet}{s}(\tau)\, N_2(s(t), t; s(\tau), \tau) \exp(\gamma\tau)\, \Phi_2(\tau)\, d\tau \\
& - \widehat{D}_A \int_0^t N_2(s(t), t; s(\tau), \tau) \exp(\gamma\tau)\, \Phi_0(\tau)\, d\tau.
\end{aligned} \tag{46}
$$

Since $\Phi_2(0) = 0$ from (40) and the computation

(47)

$$
\begin{aligned}
& \int_0^t G_{1\xi}(x, t; s(\tau), \tau)\, \Phi_2(\tau)\, d\tau \\
& = \int_b^{s(t)} G_{1\xi}\left(x, t; \xi, s^{-1}(\xi)\right) \Phi_2\left(s^{-1}(\xi)\right) \frac{ds^{-1}}{d\xi}(\xi)\, d\xi \\
& = -\int_b^{s(t)} G_1\left(x, t; \xi, s^{-1}(\xi)\right) \frac{d}{d\xi}\left(\Phi_2\left(s^{-1}(\xi)\right) \frac{ds^{-1}}{d\xi}(\xi) \right) d\xi \\
& = -\int_b^{s(t)} G_1\left(x, t; \xi, s^{-1}(\xi)\right) \left(F_0(\xi) . \frac{ds^{-1}}{d\xi}(\xi) + \Phi_2\left(s^{-1}(\xi)\right) \alpha(\xi) \right) d\xi \\
& = -\int_0^t G_1(x, t; s(\tau), \tau)\, \Phi_1(\tau)\, d\tau - \int_0^t G_1(x, t; s(\tau), \tau)\, \Phi_2(\tau)\, \alpha(\tau)\, \overset{\bullet}{s}(\tau)\, d\tau,
\end{aligned}
$$

we get

$$
\begin{aligned}
V(x,t) = & \int_0^b G_1(x, t; \xi, 0)\, h(\xi)\, d\xi + V_0 \widehat{D}_A \int_0^t G_{1\xi}(x, t; 0, \tau)\, d\tau \\
& + \int_0^t G_1(x, t; s(\tau), \tau)\, \overset{\bullet}{s}(\tau)\, \Phi_2(\tau)\, d\tau + \widehat{D}_A \int_0^t G_1(x, t; s(\tau), \tau)\, \Phi_0(\tau)\, d\tau \\
& + \widehat{D}_A \int_0^t G_1(x, t; s(\tau), \tau)\, \Phi_1(\tau)\, d\tau \\
& + \widehat{D}_A \int_0^t G_1(x, t; s(\tau), \tau)\, \overset{\bullet}{s}(\tau)\, \alpha(\tau)\, \Phi_2(\tau)\, d\tau.
\end{aligned} \tag{48}
$$

Differentiating (48) with respect to x and taking the limit $x \to s(t)^-$ we get

(49)
$$
\begin{aligned}
&V_x(s(t),t) = \frac{\overline{f_0}\,(t)}{2} + \frac{\overset{\bullet}{s}\,(t)\Phi_2\,(t)}{2\widehat{D}_A} + \frac{\Phi_0\,(t)}{2} + \frac{\Phi_1\,(t)}{2} + \frac{\Phi_2\,(t)\,\alpha\,(t)\,\overset{\bullet}{s}\,(t)}{2} \\
&+\int_0^t G_{1x}\,(s(t),t;s\,(\tau)\,,\tau)\,\overset{\bullet}{s}\,(\tau)\,\Phi_2\,(\tau)\,d\tau \\
&+\widehat{D}_A\int_0^t G_{1x}\,(s(t),t;s\,(\tau)\,,\tau)\,\Phi_0\,(\tau)\,d\tau \\
&+\widehat{D}_A\int_0^t G_{1x}\,(s(t),t;s\,(\tau)\,,\tau)\,\Phi_1\,(\tau)\,d\tau \\
&+\widehat{D}_A\int_0^t G_{1x}\,(s(t),t;s\,(\tau)\,,\tau)\,\Phi_2\,(\tau)\,\alpha\,(\tau)\,\overset{\bullet}{s}\,(\tau)\,d\tau,
\end{aligned}
$$

(50)
$$
\begin{aligned}
&\int_0^t N_{2\xi}\,(x,t;s\,(\tau)\,,\tau)\exp\left(\gamma\tau\right)\Phi_2\,(\tau)\,d\tau \\
&= \int_b^{s(t)} N_{2\xi}\left(x,t;\xi,s^{-1}\,(\xi)\right)\exp\left(\gamma s^{-1}\,(\xi)\right)\Phi_2\left(s^{-1}\,(\xi)\right)\frac{ds^{-1}}{d\xi}\,(\xi)\,d\xi \\
&= -\int_b^{s(t)} N_2\left(x,t;\xi,s^{-1}\,(\xi)\right)\frac{d}{d\xi}\left(\exp\left(\gamma s^{-1}\,(\xi)\right)\Phi_2\left(s^{-1}\,(\xi)\right)\frac{ds^{-1}}{d\xi}\,(\xi)\right)d\xi \\
&= -\int_b^{s(t)} N_2\left(x,t;\xi,s^{-1}\,(\xi)\right) \\
&\qquad\times\left(G_0\,(\xi)\,\tfrac{ds^{-1}}{d\xi}\,(\xi) + \exp\left(\gamma s^{-1}\,(\xi)\right)\Phi_2\left(s^{-1}\,(\xi)\right)\tfrac{d}{d\xi}\left(\tfrac{ds^{-1}}{d\xi}\,(\xi)\right)\right)d\xi \\
&= -\int_0^t N_2\,(x,t;s\,(\tau)\,,\tau)\,G_0\,(s(\tau))\,d\tau \\
&\quad -\int_0^t N_2\,(x,t;s\,(\tau)\,,\tau)\exp\left(\gamma\tau\right)\Phi_2\,(\tau)\,\alpha\,(\tau)\,\overset{\bullet}{s}\,(\tau)\,d\tau,
\end{aligned}
$$

where

(51)
$$
\begin{aligned}
&G_0(\xi) = \frac{d}{d\xi}\left(\exp(\gamma s^{-1}(\xi))\ \Phi_2(s^{-1}(\xi))\right) = \exp(\gamma s^{-1}(\xi))F_0(\xi) \\
&+\Phi_2(s^{-1}(\xi))\gamma\exp(\gamma s^{-1}(\xi))\frac{ds^{-1}}{d\xi}(\xi),
\end{aligned}
$$

(52)
$$
G_0(s(t)) = \exp(\gamma t)F_0(s(t)) + \Phi_2(t)\beta(t) = \exp(\gamma t)\Phi_1(t) + \Phi_2(t)\beta(t).
$$

We therefore have

(53)
$$
\begin{aligned}
&U(x,t) = -\widetilde{D}_A\int_0^t N_2\,(x,t;s\,(\tau)\,,\tau)\,G_0\,(s\,(\tau))\,d\tau \\
&-\widetilde{D}_A\int_0^t N_2\,(x,t;s\,(\tau)\,,\tau)\exp\left(\gamma\tau\right)\Phi_2(\tau)\alpha\,(\tau)\,\overset{\bullet}{s}\,(\tau)\,d\tau \\
&-\int_0^t N_2\,(x,t;s\,(\tau)\,,\tau)\exp\left(\gamma\tau\right)\Phi_2\,(\tau)\,\overset{\bullet}{s}\,(\tau)\,d\tau \\
&-\widehat{D}_A\int_0^t N_2\,(x,t;s\,(\tau)\,,\tau)\exp\left(\gamma\tau\right)\Phi_0\,(\tau)\,d\tau.
\end{aligned}
$$

Differentiating (53) with respect to x and taking the limit $x \to s(t)^+$ we get

(54)
$$\begin{aligned}
&U_x\left(s\left(t\right),t\right) = \frac{\exp\left(\gamma t\right)\Phi_1\left(t\right)+\Phi_2\left(t\right)\beta\left(t\right)}{2} + \frac{\exp\left(\gamma t\right)\alpha\left(t\right)\overset{\bullet}{s}\left(t\right)\Phi_2\left(t\right)}{2}\\
&+\frac{\exp\left(\gamma t\right)\Phi_2\left(t\right)\overset{\bullet}{s}\left(t\right)}{2} + \frac{\exp\left(\gamma t\right)\Phi_0\left(t\right)}{2D}\\
&-\widetilde{D}_A\int_0^t N_{2x}\left(x,t;s\left(\tau\right),\tau\right)\left[\exp\left(\gamma\tau\right)\Phi_1\left(\tau\right)+\Phi_2\left(\tau\right)\beta\left(\tau\right)\right]d\tau\\
&-\widetilde{D}_A\int_0^t N_{2x}\left(x,t;s\left(\tau\right),\tau\right)\exp\left(\gamma\tau\right)\Phi_2\left(\tau\right)\alpha\left(\tau\right)\overset{\bullet}{s}\left(\tau\right)d\tau\\
&-\int_0^t N_{2x}\left(s\left(t\right),t;s\left(\tau\right),\tau\right)\exp\left(\gamma\tau\right)\Phi_2\left(\tau\right)\overset{\bullet}{s}\left(\tau\right)d\tau\\
&-\widehat{D}_A\int_0^t N_{2x}\left(s(t),t;s\left(\tau\right),\tau\right)\exp\left(\gamma\tau\right)\Phi_0\left(\tau\right)d\tau.
\end{aligned}$$

Let the functions $Q = Q(t)$ and $R = R(t)$ be defined by

(55)
$$Q(t) = V(s(t),t) - \exp(\gamma t)U(s(t),t),$$

(56)
$$R(t) = V_x(s(t),t) - D\exp(-\gamma t)U_x(s(t),t), \qquad \left(D = \frac{\widetilde{D}_A}{\widehat{D}_A}\right).$$

We shall prove that $Q \equiv 0$ and $R \equiv 0$, that is, (B(9 − 10)a), (B(9 − 10)b). Subtracting the second and the first equations in (40) (or (A8), below) we have that

(57)
$$P_0(t) - P_1(t) + (1-D)\Phi_1(t) + \Phi_2(t)[(1-D)\alpha(t)\overset{\bullet}{s}(t) - D\beta(t)\exp(-\gamma t)] = 0.$$

From (49), (54), (56), and (57) we obtain

(58)
$$\begin{aligned}
&R(t) = \frac{\overline{f_0}(t)}{2} + \frac{1-D}{2}\Phi_1(t) + \frac{\Phi_2(t)\,\alpha(t)\overset{\bullet}{s}(t)}{2}(1-D) - \frac{D}{2}\Phi_2(t)\beta(t)\exp(-\gamma t)\\
&+(1+\widehat{D}_A)\int_0^t G_{1x}(s(t),t;s(\tau),\tau)\overset{\bullet}{s}(\tau)\,\phi_2(\tau)d\tau\\
&+\widehat{D}_A\int_0^t G_{1x}(s(t),t;s(\tau),\tau)\Phi_0(\tau)d\tau + \widehat{D}_A\int_0^t G_{1x}(s(t),t;s(\tau),\tau)\Phi_1(\tau)d\tau\\
&+D\int_0^t N_{2x}(s(t),t;s(\tau),\tau)\Phi_2(\tau)\exp\left(-\gamma\tau\right)\\
&\times\left[\widetilde{D}_A\beta\left(\tau\right) + \widetilde{D}_A\exp\left(\gamma\tau\right)\alpha\left(\tau\right)\overset{\bullet}{s}\left(\tau\right) + \exp\left(\gamma\tau\right)\overset{\bullet}{s}\left(\tau\right)\right]d\tau\\
&+D\widetilde{D}_A\exp(-\gamma t)\int_0^t N_{2x}(s(t),t;s(\tau),\tau)\exp(\gamma\tau)\Phi_1(\tau)d\tau\\
&+\widetilde{D}_A\int_0^t N_2(s(t),t;s(\tau),\tau)\exp(-\gamma(t-\tau))\Phi_0(\tau)d\tau\\
&= \frac{\overline{f_0}(t)}{2} + \frac{1-D}{2}\Phi_1(t) + \frac{1-D}{2}\Phi_2(t)\alpha(t)\overset{\bullet}{s}(t)\\
&-\frac{D}{2}\Phi_2(t)\beta(t)\exp(-\gamma t) + \frac{1}{2}\int_0^t(\overline{H}_0 - \overline{H}_1)d\tau\\
&= \frac{1}{2}[P_0(t) - P_1(t) + (1-D)\Phi_1(t)\\
&\qquad +\Phi_2(t)((1-D)\alpha(t)\overset{\bullet}{s}(t) - D\beta(t)\exp(-\gamma t))] = 0,
\end{aligned}$$

that is,

$$V_x(s(t),t) = D\exp(-\gamma t)U_x(s(t),t). \tag{59}$$

Since the function $V = V(x,t)$ verifies (9) and (45), we can integrate the identity (15) over the domain $0 < \xi < s(\tau), 0 < \epsilon < \tau < t-\epsilon$, take the limit $\epsilon \to 0^+$ and obtain the following expression:

$$\begin{aligned} V(x,t) = &\int_0^b G_1(x,t;\xi,0)h(\xi)d\xi + V_0\widehat{D}_A\int_0^t G_{1\xi}(x,t;0,\tau)dt \\ &+\int_0^t [G_1(x,t;s(\tau),\tau)\overset{\bullet}{s}(\tau) + \widehat{D}_A N_{1x}(x,t;s(\tau),\tau)]\Phi_2(\tau)d\tau \\ &+\widehat{D}_A\int_0^t G_1(x,t;s(\tau),\tau)V_\xi(s(\tau),\tau)d\tau. \end{aligned} \tag{60}$$

If we compare the two expressions (41) and (60) we deduce that

$$\int_0^t G_1(x,t;s(\tau),\tau)[\Phi_0(\tau) - V_\xi(s(\tau),\tau)]d\tau = 0. \tag{61}$$

If we differentiate (60) with respect to variable x, take the limit $x \to s(t)^-$ and the jump relation (19), for the function $\Psi(t) = \Phi_0(t) - V_x(s(t),t)$ we get the following integral equation:

$$\Psi(t) = -2\widehat{D}_A\int_0^t G_{1x}(s(t),t;s(\tau),\tau)\Psi(\tau)d\tau. \tag{62}$$

Taking into account the inequality $|G_{1x}(s(t),t;s(\tau),\tau)| \le \frac{C_{10}}{\sqrt{t-\tau}}$, from (62) we obtain the following inequality:

$$|\Psi(t)| \le C'_{10}\int_0^t \left|\frac{\Psi(\tau)}{\sqrt{t-\tau}}\right| d\tau, \tag{63}$$

where

$$C_{10} = \frac{1}{2\widehat{D}_A^{\frac{3}{2}}\sqrt{\pi}}\left(\frac{\beta_0}{2} + \frac{A\widehat{D}_A}{e\ a^2}\right), \qquad C'_{10} = 2\widehat{D}_A C_{10}, \tag{64}$$

then $\Psi(t) = 0$, that is,

$$V_x(s(t),t) = \Phi_0(t). \tag{65}$$

Since the function $U = U(x,t)$ verifies (10) and $U_x(s(t),t) = \frac{1}{D}\exp(\gamma t)\Phi_0(t)$, we can integrate the identity (16) over the domain $s(\tau) < \xi < 1,\ 0 < \epsilon < \tau < t-\epsilon$. We take the limit $\epsilon \to 0^+$; then we obtain the following expression

$$\begin{aligned} U(x,t) = &\int_0^t [\widetilde{D}_A N_{2\xi}(x,t;s(\tau),\tau) - \overset{\bullet}{s}(\tau)N_2(x,t;s(\tau),\tau)]U(s(\tau),\tau)d\tau \\ &-\widehat{D}_A\int_0^t N_2(x,t;s(\tau),\tau)\exp(\gamma\tau)\,\Phi_0(\tau)\,d\tau. \end{aligned} \tag{66}$$

If we compare the two expressions (42) and (66) we deduce that ($N_{2\xi} = -G_{2x}$)

$$\int_0^t [\widetilde{D}_A G_{2x}(x,t;s(\tau),\tau) + \overset{\bullet}{s}(\tau) N_2(x,t;s(\tau),\tau)] \exp(\gamma\tau) Q(\tau) d\tau = 0. \tag{67}$$

If we differentiate (67) with respect to variable x, take the limit $x \to s(t)^+$ and the jump relation (20), for the function $\eta(t) = \exp(\gamma t)Q(t)$ we get the following integral equation:

$$\eta(t) = 2\int_0^t [\widetilde{D}_A G_{2x}(s(t),t;s(\tau),\tau) + \overset{\bullet}{s}(\tau) N_2(s(t),t;s(\tau),\tau)] \eta(\tau) d\tau. \tag{68}$$

Taking into account the inequality

$$\left| \widetilde{D}_A G_{2x}(s(t),t;s(\tau),\tau) + \overset{\bullet}{s}(\tau) N_2(s(t),t;s(\tau),\tau) \right| \le \frac{C_{11}}{\sqrt{t-\tau}}, \tag{69}$$

where

$$C_{11} = \frac{1}{2\widetilde{D}_A^{\frac{1}{2}}\sqrt{\pi}} \left(\frac{(1-a)\widetilde{D}_A}{e\,(1-A)^2} + \frac{5}{2}\beta_0 \right),$$

we obtain

$$|\eta(t)| \le C'_{11} \int_0^t \frac{|\eta(\tau)|}{\sqrt{t-\tau}} d\tau, \qquad C'_{11} = 2C_{11}; \tag{70}$$

then $\eta(t) = 0$, that is,

$$\exp(-\gamma t) U(s(t),t) = V(s(t),t) = \Phi_2(t). \tag{71}$$

3. Main results.

Let $X_{w,M}$ be the closed set in $C^0[0,w]$, defined by

$$X_{w,M} = \left\{ f \in C^0([0,w]) / \, \|f\| = \max_{0\le t\le w} |f(t)| \le M \right\}$$

for any positive constants w and M.

LEMMA 2. *Let us assume that the functions $h = h(x)$ and $s = s(t)$ satisfy the assumptions* (A$_1$)–(A$_3$). *Then*

(i) *$\overline{f}_0 = \overline{f}_0(t)$ and $f_2 = f_2(t)$ are bounded continuous functions for any $t \in [0,T]$;*

(ii) *$\overline{H}_0 = \overline{H}_0(t,\tau,\Phi_0(\tau),\Phi_1(\tau),\Phi_2(\tau)), \overline{H}_1 = \overline{H}_1(t,\tau,\Phi_0(\tau),\Phi_1(\tau),\Phi_2(\tau))$, and $H_2 = H_2(t,\tau,\Phi_0(\tau),\Phi_1(\tau),\Phi_2(\tau))$ are continuous functions for all $\Phi_i \in X_{T,M}$ $(i = 0,1,2)$ and $t > \tau$. Moreover, the following estimates hold:*

$$|f_0(t)| \le \gamma_1, \tag{72}$$

$$|f_2(t)| \le \gamma_2, \tag{73}$$

$$\left|\overline{H}_0(t,\tau,\Phi_0(\tau),\Phi_1(\tau),\Phi_2(\tau))\right| \le \frac{\gamma_3}{\sqrt{t-\tau}}, \quad t > \tau, \tag{74}$$

$$\left|\overline{H}_1(t,\tau,\Phi_0(\tau),\Phi_1(\tau),\Phi_2(\tau))\right| \le \frac{\gamma_4}{\sqrt{t-\tau}}, \quad t > \tau, \tag{75}$$

$$|H_2(t,\tau,\Phi_0(\tau),\Phi_1(\tau),\Phi_2(\tau))| \le \frac{\gamma_5}{\sqrt{t-\tau}}, \quad t > \tau, \tag{76}$$

where γ_i are adequate positive constants.

Proof. (i) The proof follows from (26) and (32) taking into account (A_1)–(A_3) and the following inequality:

$$\frac{\exp\left(\frac{-x^2}{\alpha t}\right)}{t^{n/2}} \leq \left(\frac{n\alpha}{2ex^2}\right)^{\frac{n}{2}}, \quad \alpha, x, t > 0, n \in \mathbb{N}. \tag{77}$$

(ii) The proof follows from (26), (27), (30), (32), (33), (A_1)–(A_3) and the fact that

$$N_2(s(t), t, 0, 0) \leq \frac{1}{\sqrt{2\pi e}}\left(\frac{1}{a} + \frac{1}{2-A}\right), \quad t > 0, \tag{78}$$

$$\int_0^b N_1(s(t), t, \xi, 0) d\xi \leq 1, \ t > 0, \tag{79}$$

$$|G_{1x}(s(t), t; s(\tau), \tau)| \leq \frac{C_{10}}{\sqrt{t-\tau}}, \quad t > \tau, \tag{80}$$

$$|N_{2x}(s(t), t; s(\tau), \tau)| \leq \frac{C_{20}}{\sqrt{t-\tau}}, \quad t > \tau, \tag{81}$$

where

$$C_{20} = \frac{1}{2\widetilde{D}_A^{\frac{3}{2}}\sqrt{\pi}}\left(\frac{(1-a)\widetilde{D}_A}{e\,(1-A)^2} + \frac{\beta_0}{2}\right), \tag{82}$$

and C_{10} is defined by (64).

Remark 4. From (27), (30), (33), (38), and (39) it follows that

$$H_i(t, \tau, 0, 0, 0) = 0, \quad t > \tau, \quad i = 0, 1, 2. \tag{83}$$

THEOREM 3. *Let us suppose the assumption* (A_0) *and that the functions* $h = h(x)$ *and* $s = s(t)$ *satisfy the assumptions* (A_1)–(A_3)*. Then, there exists one and only one solution in* $X_{T,M}$ *of the system of integral equations* (40)*, for suitable positive constants* T *and* M*.*

Proof. We note from (A_1), (A_2), and Lemma 2 that $f_0 = f_0(t)$, $f_1 = f_1(t)$, $f_2 = f_2(t)$ defined by (36), (37), and (32), respectively, are continuous functions for any $t \in (0, T]$.

On the other hand, we shall show that the functions

$$H_0 = H_0(t, \tau, \Phi_0(\tau), \Phi_1(\tau), \Phi_2(\tau)), \ H_1 = H_1(t, \tau, \Phi_0(\tau), \Phi_1(\tau), \Phi_2(\tau))$$

verify the condition (C1), that is,

(C1) H_0 and H_1 are continuous functions for all $i \in X_{T,M}, i = 0, 1, 2,$ $t > \tau$.

Furthermore, the condition (C2),

(C2) $\left|H_i(t, \tau, \Phi_0^1, \Phi_1^1, \Phi_2^1) - H_i(t, \tau, \Phi_0^2, \Phi_1^2, \Phi_2^2)\right| \leq L(t, \tau)\left(\sum_{i=0}^{2}\left|\Phi_i^1 - \Phi_i^2\right|\right),$ $i = 0, 1, 2,$

where

$$\int_{t_1}^{t_2} L\,(t_2, \tau)\, d\tau \leq \Theta\,(t_2 - t_1), \qquad t_2 > t_1, \tag{84}$$

for some monotone increasing function Θ with

$$\lim_{z\to 0}\Theta(z)=0 \tag{85}$$

and

$$\int_{t_1}^{t_2}|H_i(t_2,\tau,0,0,0)|\,d\tau\le\Phi(t_2-t_1),\qquad t_2>t_1, \tag{86}$$

for some nonnegative function Φ with

$$\lim_{z\to 0}\Phi(z)=0. \tag{87}$$

Condition (C1) follows from (38) and (39) as a direct consequence of Lemma 2. To prove (C2), first we note that

$$\begin{aligned}&\left|H_0(t,\tau,\Phi_0^1,\Phi_1^1,\Phi_2^1)-H_0(t,\tau,\Phi_0^2,\Phi_1^2,\Phi_2^2)\right|\\ &\le\frac{1}{|1-D|}\left(\left|\overline{H}_1(t,\tau,\Phi_0^1,\Phi_1^1,\Phi_2^1)-\overline{H}_1(t,\tau,\Phi_0^2,\Phi_1^2,\Phi_2^2)\right|\right)\\ &+\frac{D}{|1-D|}\left(\left|\overline{H}_0(t,\tau,\Phi_0^1,\Phi_1^1,\Phi_2^1)-\overline{H}_0(t,\tau,\Phi_0^2,\Phi_1^2,\Phi_2^2)\right|\right)\\ &+\|\mu_0\,(t)\|\left(\left|H_2(t,\tau,\Phi_0^1,\Phi_1^1,\Phi_2^1)-H_2(t,\tau,\Phi_0^2,\Phi_1^2,\Phi_2^2)\right|\right),\end{aligned} \tag{88}$$

where, for each term on the right-hand side of (88), the following estimates are obtained (we denote $\overline{H}_2=H_2$):

$$\left|\overline{H}_i(t,\tau,\Phi_0^1,\Phi_1^1,\Phi_2^1)-\overline{H}_i(t,\tau,\Phi_0^2,\Phi_1^2,\Phi_2^2)\right|\le\frac{J_i}{\sqrt{t-\tau}}\sum_{i=0}^{2}\left|\Phi_i^1-\Phi_i^2\right|,\quad i=0,1,2, \tag{89}$$

where J_i are adequate positive constants.

The inequality (89) follows from (27), (30), (33), (78), (80), and (81). Therefore, we have proved (C_2) where the function $L=L(t,\tau)$ is given by

$$L(t,\tau)=\frac{\gamma}{\sqrt{t-\tau}},\quad t>\tau, \tag{90}$$

where γ is an adequate positive constant.

Because (90), it is evident that conditions (84) and (85) are satisfied, e.g., $\Theta=\Theta(z)$ can be taken as $\Theta(z)=\text{Const.}\sqrt{z}$. Conditions (86) and (87) follow immediately from (33), (38), (39), and Remark 4. At this point, the conclusion of the theorem follows from Corollary 8.2.1 of Theorem 8.2.1 of [Ca].

Appendix. We shall explain some calculations leading to the computation of the system of integral equations (21). First, taking into account the jump relation (19), taking $x\to s(t)^-$ in (17bis), we obtain

(A1)

$$\begin{aligned}V(s(t),t)&=2\int_0^b G_1(s(t),t;\xi,0)h\,(\xi)\,d\xi+2V_0\widehat{D}_A\int_0^t G_{1\xi}(s(t),t;0,\tau)d\tau\\ &+2\int_0^t G_1(s(t),t;s(\tau),\tau)V(s(\tau),\tau)\,\dot{s}\,(\tau)\,d\tau+2\widehat{D}_A\int_0^t N_{1x}(s(t),t;s(\tau),\tau)V(s(\tau),\tau)d\tau\\ &+2\widehat{D}_A\int_0^t G_1(s(t),t;s(\tau),\tau)V_\xi(s(\tau),\tau)d\tau.\end{aligned}$$

Next, the fifth term on the right-hand side of (17) can be written as [BoTw]

(A2)
$$\int_0^t G_{1\xi}(x,t;s(\tau),\tau)V(s(\tau),\tau)d\tau = -G_1(x,t;b,0)V(b,0)\frac{ds^{-1}}{d\xi}\,|_{\xi=b}$$
$$-\int_0^t G_1(x,t;s(\tau),\tau)F(s(\tau))\,\dot{s}\,(\tau)\,d\tau,$$

because the transformation $\tau = s^{-1}(\xi)$ (or $\xi = s(\tau)$) and the expression

(A3)
$$F(\xi) = \frac{d}{d\xi}\left(V(\xi, s^{-1}(\xi))\frac{ds^{-1}}{d\xi}(\xi)\right).$$

Inserting expression (A2) in (17), we have

(A4)
$$V(x,t) = \int_0^b G_1(x,t;\xi,0)h\,(\xi)\,d\xi + V_0\widehat{D}_A\int_0^t G_{1\xi}(x,t;0,\tau)d\tau$$
$$+ \widehat{D}_A\; G_1(x,t;b,0)V(b,0)\frac{ds^{-1}}{d\xi}\,|_{\xi=b} + \int_0^t G_1(x,t;s(\tau),\tau)V(s(\tau),\tau)\,\dot{s}\,(\tau)d\tau$$
$$+ \widehat{D}_A\int_0^t G_1(x,t;s(\tau),\tau)V(s(\tau),\tau)\alpha(\tau)\,\dot{s}\,(\tau)d\tau$$
$$+ \widehat{D}_A\int_0^t G_1(x,t;s(\tau),\tau)F_0(s(\tau))d\tau + \widehat{D}_A\int_0^t G_1(x,t;s(\tau),\tau)V_\xi(s(\tau),\tau)d\tau,$$

where

(A5)
$$\left|\begin{array}{l}\alpha(\tau) = -\dfrac{\ddot{s}\,(\tau)}{\left[\dot{s}\,(\tau)\right]^3} = \dfrac{d}{d\xi}\left(\dfrac{d}{d\xi}s^{-1}(\xi)\right)|_{\xi=s(\tau)},\\[2ex] F_0(\xi) = \dfrac{d}{d\xi}\big(V\left(\xi, s^{-1}(\xi)\right)\big).\end{array}\right.$$

We differentiate (A4) with respect to x, we take into account that $G_{1_{\xi x}} = N_{1_\tau}$ and the jump relation (19), and taking $x \to s(t)$, we deduce

(A6)
$$V_x(s(t),t) = 2\int_0^b G_{1x}(s(t),t;\xi,0)h(\xi)d\xi - 2V_0\widehat{D}_A N_1(s(t),t;0,0)d\tau$$
$$+2\widehat{D}_A G_x(s(t),t;b,0)h(b)\left.\frac{ds^{-1}}{d\xi}\right|_{\xi=b} + \left(\frac{1}{\widehat{D}_A} + \alpha\,(t)\right)V(s(t),t)\,\dot{s}\,(t)$$
$$+2\int_0^t\left[1+\widehat{D}_A\alpha\,(\tau)\right]G_{1x}\,(s\,(t)\,,t;s\,(\tau)\,,\tau)\,V\,(s\,(\tau)\,,\tau)\,\dot{s}\,(\tau)\,d\tau + F_0\,(s\,(t))$$
$$+\widehat{D}_A\int_0^t\left[F_0\,(s\,(\tau)) + V_\xi\,(s\,(\tau)\,,\tau)\right]G_{1x}\,(s\,(t)\,,t;s\,(\tau)\,,\tau)\,d\tau.$$

Following a procedure similar to the preceding one, we have

(A7)

$$
\begin{aligned}
&U_x(s(t),t) = \exp(\gamma t)F_0(s(t)) - 2\widetilde{D}_A \int_0^t \exp(\gamma\tau) N_{2x}(s(t),t;s(\tau),\tau)d\tau \\
&+V(s(t),t)\left(\exp(\gamma t)\overset{\bullet}{s}(t)\left(\frac{1}{\widetilde{D}_A}+\alpha(t)\right)+\beta(t)\right) \\
&-2\int_0^t V(s(t),t)\left[\exp(\gamma t)\overset{\bullet}{s}(t)\left(1+\widetilde{D}_A\alpha(t)\right)+\widetilde{D}_A\beta(t)\right] N_{2x}(s(t),t;s(\tau),\tau)d\tau \\
&-2\widetilde{D}_A\int_0^t N_{2x}(s(t),t;s(\tau),\tau)U_\xi(s(t),t)\,d\tau.
\end{aligned}
$$

By using the relation (B(9 − 10)b), (A1), (A6), and (A7) we get

$$
\text{(A8)} \qquad \left|\begin{array}{l}
V_x(s(t),t) - F_0(s(t)) - \left(\dfrac{1}{\widehat{D}_A}+\alpha(t)\right)\overset{\bullet}{s}(t)V(s(t),t) = P_0(t), \\
V_x(s(t),t) - DF_0(s(t)) - D\delta(t)\,V(s(t),t) = P_1(t), \\
V(s(t),t) = P_2(t),
\end{array}\right.
$$

where P_0, P_1, and P_2 are defined by (23), (24), and (25), respectively, and

$$
\text{(A9)} \qquad \delta(t) = \overset{\bullet}{s}(t)\left(\frac{1}{\widehat{D}_A}+\alpha(t)\right)+\exp(-\gamma t)\beta(t).
$$

Owing to $D \neq 1$, from (A8) we deduce (21).

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