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# Asymptotic Behaviour of a Non-classical Heat Conduction Problem for a Semi-infinite Material

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The asymptotic behaviour of a heat conduction problem involving a non-linear heat source depending on the heat-flux occurring in the extremum of a semi-infinite slab is discussed. Conditions are given on the non-linearity so as to accelerate the convergence of the solution to zero. Copyright © 2000 John Wiley & Sons, Ltd.

## 1. Introduction and preliminaries

The following non-classical heat conduction problem for a semi-infinite material was studied in [21]:

$$
\begin{aligned}
&u_t(x,t) - u_{xx}(x,t) = \Phi(x)\ \left(\mathscr{F}(u_x(0,\cdot),\cdot)\right)(t), \quad x>0, \quad t>0\\
&u(0,t) = g(t), \quad t>0\\
&u(x,0) = h(x), \quad x>0
\end{aligned}
\tag{1}
$$

where $\Phi$, $g$, $h$ are real functions defined on $\mathbb{R}^+$ and $\mathscr{F}$ is a function depending on the heat flux at the extremum $x=0$. Non-classical problems like (1) are motivated by the modelling of a system of temperature regulation in isotropic media and the source term $\Phi(x)\ (\mathscr{F}(u_x(0,\cdot))(t)$ describes a cooling or heating effect depending on the properties of $\mathscr{F}$ which are related to the evolution of the heat flux $u_x(0,t)$. It is called the thermostat problem. Related problems are considered in [3–9, 12, 13, 20]. Under suitable assumptions on data, existence, uniqueness and monotone-continuous dependence on the data are established in [21] for problem (1).

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In this paper we shall consider the simple instance of problem (1) given by

$$\begin{aligned} &u_t - u_{xx} = -F(u_x(0,t)), \quad x>0, \quad t>0\\ &u(0,t)=0, \quad t>0\\ &u(x,0)=h(x), \quad x>0 \end{aligned} \tag{2}$$

where $h(x)$, $x>0$, and $F(v)$, $v \in \mathbb{R}$, are continuous functions. The function $F$, henceforth referred as *control function*, is assumed to fulfill the following conditions:

(A) $v\,F(v) \geqslant 0$,
(B) $F(0)=0$,

which intuitively means that the control attempts to stabilize the process at every time. As it is shown in [23] (see also [21, 22]), the solution to problem (2) can be represented by

$$u(x,t) = u_0(x,t) - \int_0^t \operatorname{erf}\left(\frac{x}{2\sqrt{(t-\tau)}}\right) F(V(\tau))\,\mathrm{d}\tau \tag{3}$$

where $u_0 = u_0(x,t)$, defined by

$$u_0(x,t) = \int_0^{+\infty} G(x,t;\,\xi,0)\,h(\xi)\,\mathrm{d}\xi \tag{4}$$

is the solution to problem (2) with null source term. Function $V=V(t)$ in (3) represents the heat flux at the extremum of the slab, i.e.

$$V(t) = u_x(0,t), \quad t>0, \tag{5}$$

and is satisfied by the following Volterra integral equation:

$$V(t) = V_0(t) - \int_0^t \frac{F(V(\tau))}{\sqrt{[\pi(t-\tau)]}}\,\mathrm{d}\tau \tag{6}$$

where the forcing function $V_0(t)$ is given by

$$V_0(t) = \frac{1}{2\sqrt{\pi}\,t^{3/2}} \int_0^{+\infty} \xi \exp\left(-\frac{\xi^2}{4t}\right) h(\xi)\,\mathrm{d}\xi, \quad t>0 \tag{7}$$

Function $G$ in (4) denotes the Green's function of the heat equation in the quarter plane and, as it is well-known, it can be written as $G(x,t;\,\xi,\tau) = K(x,t;\,\xi,\tau) - K(-x,t;\xi,\tau)$, $x, \xi > 0$, $t>\tau>0$, where

$$K(x,t;\xi,\tau) = \frac{1}{\sqrt{[4\pi(t-\tau)]}} \exp\left(-\frac{(x-\xi)^2}{4(t-\tau)}\right)$$

is the one-dimensional heat kernel.

Control function $F$ has been supposed Lipschitzian in [23, 22] in order to prove local existence and uniqueness of the solution to the integral equation (6). However, it is well known that this equation admits a solution under less stringent conditions. In fact, conditions (A) and (B) of $F$ are usually sufficient to guarantee global existence and uniqueness of solution (see, for example, chapter 4 [15]).

From now on, we suppose that $h$ is a non-negative and non-identically null function which, in view of (7), implies $V_0(t) > 0$, $t > 0$. When the control function $F$ satisfies conditions (A) and (B) and, moreover, the initial temperature $h$ is non-negative, then the solution $u(x, t)$ to problem (2) tends to zero when $t \to +\infty$ (see [21, 23]). The present paper is devoted to the study of 'controlling' problem (2) through $F$ so that, by the stabilizing effect of the control, its solution should converge to zero (when the time goes to infinity) faster than that corresponding to problem (2) in the absence of control; i.e.:

$$\lim_{t \to +\infty} \frac{u(x, t)}{u_0(x, t)} = 0$$

As it was observed in [23], from the continuity up to the boundary of the solution $u$ to (2) follows that the heat flux $w(x, t) = u_x(x, t)$ satisfies a classical heat conduction problem with a nonlinear convective condition at $x = 0$, which can be written in the form

$$\begin{aligned} &w_t - w_{xx} = 0, \quad x > 0, \ t > 0 \\ &w_x(0, t) = F(w(0, t)), \quad t > 0 \\ &w(x, 0) = h'(x), \quad x > 0 \end{aligned} \tag{8}$$

The literature concerning problem (8) has constantly increased from the apparition of the papers [14, 18]. Other related problems in this direction are considered in [2, 19, 11]. In [16, 10, 17] the asymptotic behaviour of the solution to (8) is investigated by formal expansion schemes. However, the techniques we shall employ in this paper to manage the same problem are substantially different. Section 2 of this paper is devoted to clarify the connection between the initial temperature $h(x)$ with the forcing function $V_0(t)$ of the Volterra integral equation (6). In section 3, a general study of the above-stated control problem for (2) is undertaken for the case of bounded initial temperatures. Specifically, we find spatially uniform bounds for the quotient $u(x, t)/u_0(x, t)$ which depend on the solution $V(t)$ to integral equation (6), from which becomes apparent that conditions (A) and (B) are not sufficient to attain the objective of the control; i.e., to obtain $\lim_{t \to +\infty} u(x, t)/u_0(x, t) = 0$. In section 4, for linear control functions $F(v) = \lambda v$, we give an example to illustrate that there exists an exact solution to problem (6) providing $u(x, t)/u_0(x, t) \cong 1/(2\lambda^2 t)$, $t \to +\infty$. Generalizing this fact, we also give sufficient conditions on $F$ so that the control goal may be reached.

## 2. Relationships between $V_0(t)$ and $h(x)$

In this section we will make several observations concerning the forcing functions $V_0(t)$ of Volterra integral equation (6). Indeed, the behaviour of the solution $V(t)$ of integral equation (6) depends on the function $V_0(t)$ and, in turn, this depends strongly on $h$ taking different values. Let us begin with the following result, which explains the behaviour of $V_0(t)$ for the small times.

**Lemma 1.** (a) *If h verifies the following assumptions*:

(i) $h \in \mathscr{C}^0[0, +\infty)$,
(ii) *there exist positive constants*, $A$, $B$ *and* $\alpha \in [0, 1)$ *such that* $|h(x)| \leqslant A \exp(Bx^{1+\alpha})$, $x \geqslant x_0$,

*then*

$$\lim_{t \to 0^+} \sqrt{(\pi t)}\, V_0(t) = h(0)$$

(b) *Moreover, if* $h \in \mathscr{C}^{1,\beta}[0, +\infty)$ *then there exists a positive constant C such that*

$$|\sqrt{(\pi t)}\, V_0(t) - h(0)| \leqslant \sqrt{(\pi t)}\, [|h'(0)| + Ct^{\beta/2}]$$

*Proof.* By making the change of variable $\xi = 2\sqrt{(tx)}$ in (7) we obtain

$$\sqrt{(\pi t)}\, V_0(t) = \int_0^{+\infty} h(2\sqrt{(tx)}) \exp(-x)\, \mathrm{d}x \tag{9}$$

Now, from hypothesis (ii) we derive, for $0 \leqslant t < 1$ and certain positive constants $A_0$, $B_0$,

$$|h(2\sqrt{(tx)}) \exp(-x)| \leqslant A_0 \exp[B_0\, x^{(1+\alpha)/2} - x], \quad x \geqslant 0$$

and therefore, the dominated convergence theorem can be applied to the right-hand side of (9) to show the assertion of (a). To prove (b) we make an integration by parts of the right-hand side of (9), getting

$$\sqrt{(\pi t)}\, V_0(t) - h(0) = \sqrt{t} \int_0^{+\infty} h'(2\sqrt{(tx)})\, \frac{\exp(-x)}{\sqrt{x}}\, \mathrm{d}x$$

and so

$$|\sqrt{(\pi t)}\, V_0(t) - h(0)| \leqslant \sqrt{t} \int_0^{+\infty} |h'(2\sqrt{(tx)})|\, \frac{\exp(-x)}{\sqrt{x}}\, \mathrm{d}x \tag{10}$$

Since $h'$ is supposed to be a Hölder-continuous function, we can write

$$|h'(2\sqrt{(tx)})| \leqslant |h'(0)| + K t^{\beta/2} x^{\beta/2}, \quad x, t \geqslant 0 \tag{11}$$

where $K$ is a positive constant. Inequality of (b) easily follows from (10) and (11).

□

We remark that, as a consequence of the previous lemma, the function $V_0(t)$ cannot have a singularity of order greater than $\frac{1}{2}$ at the origin. Our following result provides a useful inequality involving $V_0(t)$.

 

**Lemma 2.** *If h verifies assumptions* (i) *and* (ii) *of Lemma* 1 *and h is a non-decreasing function*, *then* $V_0$ *satisfies the condition*

$$\frac{V_0(\tau)}{V_0(t)} \leqslant \sqrt{\left(\frac{t-s}{\tau-s}\right)}, \quad 0 \leqslant s < \tau < t \tag{12}$$

*Proof.* From (9) and (iii) we obtain, for every $0 \leqslant s < \tau < t$,

$$\frac{V_0(\tau)}{V_0(t)} = \sqrt{\left(\frac{t}{\tau}\right)}\left[\frac{\int_0^{+\infty} h(2\sqrt{(\tau\xi)})\exp(-\xi)\,\mathrm{d}\xi}{\int_0^{+\infty} h(2\sqrt{(t\xi)})\exp(-\xi)\,\mathrm{d}\xi}\right] \leqslant \sqrt{\left(\frac{t}{\tau}\right)} \leqslant \sqrt{\left(\frac{t-s}{\tau-s}\right)} \qquad \square$$

**Corollary 3.** *Under the same assumptions made on h in Lemma* 2, *the solution of the integral equation* (6) *satisfies the following inequality*:

$$0 \leqslant V(t) \leqslant V_0(t), \quad t > 0 \tag{13}$$

*Proof.* Since $V_0(t)$ verifies the inequality (12) and $F$ verifies condition (A) from the introduction, we can apply Theorem 6.1 from [15, chapter 2] to obtain the bounds for $V(t)$ given by (13). $\square$

In the case of constant initial temperature $h(x) = h_0$, $x \geqslant 0$, inequalities (13) become

$$0 \leqslant V(t) \leqslant \frac{h_0}{\sqrt{(\pi t)}}, \quad t > 0$$

that is

$$0 \leqslant \frac{\sqrt{(\pi t)}}{h_0} V(t) \leqslant 1, \quad t > 0$$

It should be noted that the function

$$t \mapsto \int_0^t \frac{V(\tau)}{\sqrt{[\pi(t-\tau)]}}\,\mathrm{d}\tau$$

is decreasing. In fact, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^t \frac{V(\tau)}{\sqrt{[\pi(t-\tau)]}}\,\mathrm{d}\tau\right) = -F(V(t)) \leqslant 0$$

The qualitative analysis of the solution to equation (6) often requires the additional hypothesis of monotonicity on its forcing function $V_0(t)$ (cf. [15]). So, to establish connections between the monotonicity of $V_0(t)$ and properties of the initial temperature $h(x)$ is an interesting matter. With this purpose in mind, let us define a function $H$ by

$$H(x) = xh'(x) - h(x), \quad x > 0$$

 

In the following result, a relationship between the monotonicity of $V_0$ and the sign of $H$ is established.

**Lemma 4.** *If $h \in \mathscr{C}^1\,[0, +\infty)$, then $V_0$ is a non-decreasing [non-increasing] function provided that $H(x) \geqslant 0$, $x > 0$, [$H(x) \leqslant 0$, $x > 0$].*

*Proof.* By taking derivatives in (9) we obtain

$$V_0'(t) = \frac{1}{2\sqrt{\pi}\, t^{3/2}} \int_0^{+\infty} H(2\sqrt{(\tau\xi)}) \exp(-\xi)\, \mathrm{d}\xi, \quad t > 0$$

hence, $V_0'(t) \geqslant 0$, $t > 0$, when $H(x) \geqslant 0$, $x > 0$. □

Consider, for instance, an initial temperature given by $h(x) = Ax^{\alpha}$, $x \geqslant 0$, ($A > 0$, $\alpha \geqslant 0$), then

$$H(x) = A(\alpha - 1)x^{\alpha}, \quad x \geqslant 0$$

$$V_0(t) = A\,\frac{2^{\alpha}}{\sqrt{\pi}}\,\Gamma\left(1 + \frac{\alpha}{2}\right) t^{(\alpha-1)/2}, \quad t > 0$$

that is, the function $V_0$ is increasing [decreasing] if and only if $\alpha > 1$ [$\alpha < 1$] and it is constant ($V_0(t) = A$) when $\alpha = 1$.

## 3. Bounded initial temperatures

In this section we shall consider in some detail the instance of problem (2) corresponding to a bounded initial temperature $h$. The main result could be stated by saying that conditions (A) and (B) on the control function $F$ do not guarantee for themselves that $\lim_{t\to+\infty} u(x, t)/u_0(x, t) = 0$. Examples of this unexpected behaviour are given and auxiliary conditions are imposed on $F$ in order to attain that objective. We reduce the study of the relatively general situation in which $h$ is bounded to the more simpler instance $h(x) = h_0 > 0$, $x \geqslant 0$. For constant temperatures, the solution to problem (2) is represented by (3) with

$$u_0(x, t) = h_0 \operatorname{erf}\left(\frac{x}{2\sqrt{t}}\right), \quad x > 0,\ t > 0 \tag{14}$$

while $V = V(t)$ becomes the solution to the Volterra integral equation

$$V(t) = \frac{h_0}{\sqrt{(\pi t)}} - \int_0^t \frac{F(V(\tau))}{\sqrt{[\pi(t-\tau)]}}\, \mathrm{d}\tau, \quad t > 0 \tag{15}$$

Equations (14) and (15) have been, respectively, obtained from (4) and (6) by simply substituting $h(x) = h_0$. First, we prove two simple lemmas.

 

**Lemma 5.** *If* $V(\tau)$ *denotes the solution to equation* (15), *then the identity*

$$\frac{1}{h_0}\int_0^t \frac{V(\tau)}{\sqrt{[\pi(t-\tau)]}}\,\mathrm{d}\tau = 1 - \frac{1}{h_0}\int_0^t F(V(\tau))\,\mathrm{d}\tau$$

*holds for* $t > 0$.

*Proof.* The identity immediately follows from an application of the Abel transformation to both members of (15). □

**Lemma 6.** *Let* $A$ *and* $B$ *be two positive real constant such that* $B \leqslant A$. *Then, we have*

$$1 \leqslant \frac{\operatorname{erf}(Ax)}{\operatorname{erf}(Bx)} \leqslant \frac{A}{B}, \quad x \in \mathbb{R}$$

Observe that the above inequalities are strict when $B < A$.

*Proof.* It is sufficient to realize that the error function is an odd increasing one and the auxiliary real function $\varphi(x) = A\operatorname{erf}(Bx) - B\operatorname{erf}(Ax)$, $x \in \mathbb{R}$, satisfies

$$\varphi(0) = 0, \quad \varphi(+\infty) = A - B \geqslant 0$$

$$\varphi'(x) = \frac{2AB}{\sqrt{\pi}}\exp(-A^2x^2)\,[\exp((A^2-B^2)x^2) - 1] \geqslant 0, \quad x \in \mathbb{R}$$ □

Our next results will be the cornerstone of the ulterior analysis.

**Theorem 7.** *The following inequalities*:

$$\frac{\sqrt{(\pi t)}}{h_0}V(t) \leqslant \frac{u(x,t)}{u_0(x,t)} \leqslant \frac{1}{h_0}\int_0^t \frac{V(\tau)}{\sqrt{[\pi(t-\tau)]}}\,\mathrm{d}\tau \tag{16}$$

*hold for* $x > 0$, $t > 0$.

*Proof.* From (3) and (14) we deduce

$$\frac{u(x,t)}{u_0(x,t)} = 1 - \frac{1}{h_0}\int_0^t \left[\frac{\operatorname{erf}(x/2\sqrt{(t-\tau)})}{\operatorname{erf}(x/2\sqrt{t})}\right] F(V(\tau))\,\mathrm{d}\tau \tag{17}$$

hence, an application of Lemma 6 gives, for $x > 0$, $t > 0$

$$1 - \frac{1}{h_0}\int_0^t \sqrt{\left(\frac{t}{(t-\tau}\right)}F(V(\tau))\,\mathrm{d}\tau \leqslant \frac{u(x,t)}{u_0(x,t)} \leqslant 1 - \frac{1}{h_0}\int_0^t F(V(\tau))\,\mathrm{d}\tau \tag{18}$$

In view of (15), for the last member of (18) we have

$$1 - \frac{1}{h_0}\int_0^t \sqrt{\left(\frac{t}{t-\tau}\right)}F(V(\tau))\,\mathrm{d}\tau = \frac{\sqrt{(\pi t)}}{h_0}V(t)$$

 

thus proving the first inequality of the statement. In respect to the second one, it easily follows from (18) and Lemma 5. □

As it can be easily seen, the data corresponding to the Volterra integral equation (15) verify the hypothesis of Corollary 3, so that its solution $V(t)$ satisfies the inequalities

$$0 \leqslant \frac{\sqrt{(\pi t)}}{h_0} V(t) \leqslant 1, \quad t > 0 \tag{19}$$

and therefore

$$0 \leqslant \liminf_{t \to +\infty} \left( \frac{\sqrt{(\pi t)}}{h_0} V(t) \right) \leqslant \limsup_{t \to +\infty} \left( \frac{\sqrt{(\pi t)}}{h_0} V(t) \right) \leqslant 1$$

Next result relates the asymptotic behaviour of the quotient $u(x, t)/u_0(x, t)$ with the oscillation limits of $\sqrt{(\pi t)}\, V(t)/h_0$.

**Corollary 8.** (i) *If* $\limsup_{t \to +\infty} (\sqrt{(\pi t)}\, V(t)/h_0) = \delta$ *(with* $0 \leqslant \delta \leqslant 1$*) then*

$$\limsup_{t \to +\infty} \frac{u(x, t)}{u_0(x, t)} = \delta, \quad \textit{uniformly in } x > 0$$

(ii) *If, for some* $x > 0$, $\limsup_{t \to +\infty} (u(x, t)/u_0(x, t)) = \delta$ *(with* $0 \leqslant \delta \leqslant 1$*), then*

$$\limsup_{t \to +\infty} \frac{1}{h_0} \int_0^t \frac{V(\tau)}{\sqrt{[\pi(t - \tau)]}} \, d\tau = \delta$$

(iii) *If* $\lim_{t \to +\infty} \int_0^t (V(\tau)/\sqrt{[\pi(t - \tau)]}) \, d\tau = 0$, *then*

$$\lim_{t \to +\infty} \frac{u(x, t)}{u_0(x, t)} = 0, \quad \textit{uniformly in } x > 0$$

(iv) *If for certain* $x > 0$, $\limsup_{t \to +\infty} (u(x, t)/u_0(x, t)) = 0$, *then*

$$\lim_{t \to +\infty} \left( \frac{\sqrt{(\pi t)}}{h_0} V(t) \right) = \lim_{t \to +\infty} \frac{1}{h_0} \int_0^t \frac{V(\tau)}{\sqrt{[\pi(t - \tau)]}} \, d\tau = 0$$

*Proof.* In order to prove (i) we apply Theorem 7 and Fatou's Lemma obtaining

$$\begin{aligned} \delta \leqslant \limsup_{t \to +\infty} \frac{u(x, t)}{u_0(x, t)} &\leqslant \limsup_{t \to +\infty} \frac{1}{h_0} \int_0^t \frac{V(\tau)}{\sqrt{[\pi(t - \tau)]}} \, d\tau \\ &= \limsup_{t \to +\infty} \frac{1}{h_0} \int_0^1 \frac{\sqrt{t}\, V(\lambda t)}{\sqrt{[\pi(1 - \lambda)]}} \, d\lambda \leqslant \int_0^1 \limsup_{t \to +\infty} \frac{1}{h_0} \frac{\sqrt{(\pi \lambda t)}\, V(\lambda t)}{\sqrt{(\pi \lambda)} \sqrt{[\pi(1 - \lambda)]}} \, d\lambda \\ &= \delta \int_0^1 \frac{d\lambda}{\sqrt{(\pi \lambda)} \sqrt{[\pi(1 - \lambda)]}} = \delta \end{aligned}$$

(ii) It follows from another application of Theorem 7 along the same lines:

$$\limsup_{t\to+\infty} \frac{\sqrt{(\pi t)}}{h_0} V(t) \leqslant \delta \leqslant \limsup_{t\to+\infty} \frac{1}{h_0} \int_0^t \frac{V(\tau)}{\sqrt{[\pi(t-\tau)]}}\, \mathrm{d}\tau$$

$$\leqslant \delta \int_0^1 \limsup_{t\to+\infty} \frac{1}{h_0} \frac{\sqrt{(\pi\lambda t)}\, V(\lambda t)}{\sqrt{(\pi\lambda)}\sqrt{[\pi(1-\lambda)]}}\, \mathrm{d}\lambda$$

$$= \delta \int_0^1 \frac{\mathrm{d}\lambda}{\sqrt{(\pi\lambda)}\sqrt{[\pi(1-\lambda)]}} = \delta$$

(iii) It easily follows from Theorem 7 if we realize that, due to the monotonicity of $t \mapsto \int_0^t (V(\tau)/\sqrt{[\pi(t-\tau)]})\, \mathrm{d}\tau$, the oscillation limit $\limsup_{t\to+\infty} \int_0^t (V(\tau)/\sqrt{[\pi(t-\tau)]})\, \mathrm{d}\tau$ can be replaced by an ordinary one. In fact, we have

$$0 \leqslant \liminf_{t\to+\infty} \frac{u(x,t)}{u_0(x,t)} \leqslant \limsup_{t\to+\infty} \frac{u(x,t)}{u_0(x,t)}$$

$$\leqslant \lim_{t\to+\infty} \frac{1}{h_0} \int_0^t \frac{V(\tau)}{\sqrt{[\pi(t-\tau)]}}\, \mathrm{d}\tau = 0, \quad x > 0$$

In a similar way, we deduce

$$0 \leqslant \liminf_{t\to+\infty} \frac{\sqrt{(\pi t)}}{h_0} V(t) \leqslant \limsup_{t\to+\infty} \frac{\sqrt{(\pi t)}}{h_0} V(t) \leqslant \limsup_{t\to+\infty} \frac{u(x,t)}{u_0(x,t)} = 0$$

so providing (iv). □

It should be observed that a necessary and sufficient condition that implies $\lim_{t\to+\infty}(u(x,t)/u_0(x,t)) = 0$, uniformly in $x \in \mathbb{R}^+$, is that

$$\lim_{t\to+\infty} \int_0^t \frac{V(\tau)}{\sqrt{[\pi(t-\tau)]}}\, \mathrm{d}\tau = 0$$

thanks to Corollary 8. In the following lemma, we shall prove that the left and right bounds provided by Theorem 7 for the quotient $u(x,t)/u_0(x,t)$, actually are its limit values when $x \to 0^+$ and $x \to +\infty$ respectively.

**Lemma 9.** *Let $t > 0$ be given; then we have*

(i)

$$\lim_{x\to 0^+} \frac{u(x,t)}{u_0(x,t)} = \frac{\sqrt{(\pi t)}}{h_0} V(t), \quad t > 0$$

(ii)

$$\lim_{x\to+\infty} \frac{u(x,t)}{u_0(x,t)} = \frac{1}{h_0} \int_0^t \frac{V(\tau)}{\sqrt{(\pi(t-\tau)]}}\, \mathrm{d}\tau, \quad t > 0$$

*Proof.* Taking into account Lemma 6, expression (17) and the fact that the real function

$$\tau \mapsto \sqrt{\left(\frac{t}{t-\tau}\right)}\, F(V(\tau))$$

belongs to $L^1(0, t)$ because

$$\int_0^t \sqrt{\left(\frac{t}{t-\tau}\right)}\, F(V(\tau))\, \mathrm{d}\tau = h_0 - \sqrt{(\pi t)}\, V(t), \quad t > 0$$

an application of the dominated convergence theorem provides

$$\lim_{x\to+\infty} \frac{u(x,t)}{u_0(x,t)} = 1 - \frac{1}{h_0}\int_0^t \lim_{x\to+\infty}\left(\frac{\operatorname{erf}(x/2\sqrt{(t-\tau)})}{\operatorname{erf}(x/2\sqrt{t})}\right) F(V(\tau))\, \mathrm{d}\tau$$

$$= 1 - \frac{1}{h_0}\int_0^t F(V(\tau))\, \mathrm{d}\tau, \quad t > 0$$

from which (ii) is immediate. We proceed analogously to derive (i). Indeed we have

$$\lim_{x\to+0} \frac{u(x,t)}{u_0(x,t)} = 1 - \frac{1}{h_0}\int_0^t \lim_{x\to 0^+}\left(\frac{\operatorname{erf}(x/2\sqrt{(t-\tau)})}{\operatorname{erf}(x/2\sqrt{t})}\right) F(V(\tau))\, \mathrm{d}\tau$$

$$= 1 - \frac{1}{h_0}\int_0^t \sqrt{\left(\frac{t}{t-\tau}\right)}\, F(V(\tau))\, \mathrm{d}\tau = \frac{\sqrt{(\pi t)}}{h_0}\, V(t), \quad t > 0 \qquad \square$$

The proof of Lemma 9 can also be carried out by using the L'Hospital rule. For instance, to derive (i) we set

$$\lim_{x\to 0^+} \frac{u(x,t)}{u_0(x,t)} = \lim_{x\to 0^+} \frac{u_x(x,t)}{u_{0x}(x,t)} = \lim_{x\to 0^+} \frac{\sqrt{(\pi t)}\, u_x(x,t)}{h_0 \exp(-x^2/4t)} = \frac{\sqrt{(\pi t)}}{h_0}\, V(t)$$

The inequalities in Theorem 7 can be used, in certain cases, to estimate the order of the convergence of the quotient $u(x,t)/u_0(x,t)$ to zero. To show this use, now we prove the following lemma.

**Lemma 10.** *Let $C_1$, $C_2$, $\alpha$ be three positive constants such that*

$$\frac{C_1}{t^\alpha} \leqslant \frac{\sqrt{(\pi t)}}{h_0}\, V(t) \leqslant \frac{C_2}{t^\alpha}$$

*then, we have*

$$\frac{u(x,t)}{u_0(x,t)} = O(t^{-\alpha}), \quad \text{as } t \to +\infty$$

*provided that $\alpha < 1/2$.*

*Proof.* From Theorem 7 and the hypothesis, we realize that it is enough to prove that

$$\frac{1}{h_0}\int_0^t \frac{V(\tau)}{\sqrt{[\pi(t-\tau)]}}\,\mathrm{d}\tau = O(t^{-\alpha}), \quad \text{as } t\to+\infty$$

Now, for $t>0$ we have

$$\frac{1}{h_0}\int_0^t \frac{V(\tau)}{\sqrt{[\pi(t-\tau)]}}\,\mathrm{d}\tau = \frac{1}{h_0}\int_0^1 \frac{V(\lambda t)\sqrt{t}}{\sqrt{[\pi(1-\lambda)]}}\,\mathrm{d}\lambda$$

$$\leqslant \frac{C_2}{\pi t^\alpha}\int_0^1 \frac{\mathrm{d}\lambda}{\lambda^{\alpha+1/2}(1-\lambda)^{1/2}} = \frac{C_2 B(\frac{1}{2}-\alpha,\ \frac{1}{2})}{\pi t^\alpha}$$

where $B$ denotes the beta function. Hence, we obtain

$$C_1 t^{-\alpha} \leqslant \frac{u(x,t)}{u_0(x,t)} \leqslant \frac{C_2}{\pi} B\left(\frac{1}{2}-\alpha, \frac{1}{2}\right) t^{-\alpha}, \quad t>0$$

which completes the proof. □

Generally speaking, the quotient $u(x,t)/u_0(x,t)$ does not need to converge to zero for a given control function $F$ that only satisfies conditions (A) and (B) from the introduction. This can be easily realized for the general problem (1) when the initial temperature $h$ is non-constant on $\mathbb{R}^+$. Take, for instance, the following data $h(x)=x$, $F(\xi)=\xi(\xi-1)^2$ in problem (1); thus, we obtain

$$u(x,t)=u_0(x,t)=x, \quad x>0,\ t>0$$

$$V(t)=1, \quad t>0$$

whence

$$\frac{u(x,t)}{u_0(x,t)}=1, \quad x>0,\ t>0$$

The same fact is less obvious for the simplified problem under consideration. Indeed, the following example shows that the quotient $u(x,t)/u_0(x,t)$ may not tend to zero if a general control function $F$ is prescribed in problem (2). Let us consider a real function $F$ which verifies conditions (A) and (B), and the following ones:

- $F$ is non-decreasing;
- $\int_0^{+\infty} \xi^{-3}F(\xi)\,\mathrm{d}\xi=\gamma$ for certain $0<\gamma<\pi/h_0$.

An example of a function satisfying all these conditions with $\gamma=1$ is given by

$$F(\xi)=\begin{cases}\xi|\xi|^{3/2}/3, & |\xi|\leqslant 1\\ \xi/3, & |\xi|\geqslant 1\end{cases}$$

Taking into account that the solution of the integral equation (15) verifies inequality (19) we can write

$$\int_0^{+\infty} F(V(\tau))\,\mathrm{d}\tau \leqslant \int_0^{+\infty} F\left(\frac{h_0}{\sqrt{(\pi\tau)}}\right)\mathrm{d}\tau = \frac{h_0^2}{\pi}\int_0^{+\infty}\frac{F(\eta)}{\eta^3}\,\mathrm{d}\eta = \frac{\gamma h_0^2}{\pi} < h_0$$

where $\gamma < \pi/h_0$. In this case, Lemma 5 shows that the quotient $u(x,t)/u_0(x,t)$ cannot converge to zero when $t$ goes to infinity. Therefore, the necessity of imposing additional restrictions on the control $F$ arise in order that the quotient $u(x,t)/u_0(x,t)$ should tend to zero when $t$ goes to infinity. As a clarifying example where this situation occurs, we shall discuss in the next section the case of linear controls: i.e.,

$$F(v) = \lambda v, \quad (\lambda > 0) \tag{20}$$

## 4. An example and a general result

For the case of a linear control explicited by (20) and in order to obtain the explicit solutions $u$ and $V$ of problems (2) and (15), respectively, we define the real function

$$Q(x) = \sqrt{\pi}\, x\exp(x^2)\operatorname{erfc}(x), \quad x > 0$$

which satisfies the following properties:

$$Q(0) = 0, \quad Q(+\infty) = 1, \quad Q'(x) > 0, \quad x > 0$$

The most important facts on the behaviour of the solution $V(t)$ to equation (15) corresponding to a linear control (20) are collected in the following result.

**Lemma 11.** *If $F$ is given by* (20), *we have*

$$0 < V(t) = \frac{h_0}{\sqrt{(\pi t)}}\,[1 - Q(\lambda\sqrt{t})] < \frac{h_0}{\sqrt{(\pi t)}} \tag{21}$$

$$\int_0^t V(\tau)\,\mathrm{d}\tau = \frac{h_0}{\lambda}\,[1 - \exp(\lambda^2 t)\operatorname{erfc}(\lambda\sqrt{t})] \tag{22}$$

$$1 - \frac{1}{h_0}\int_0^t F(V(\tau))\,\mathrm{d}\tau = \exp(\lambda^2 t)\operatorname{erfc}(\lambda\sqrt{t}) \tag{23}$$

*for all $t > 0$ and*

$$\lim_{t\to+\infty}\frac{u(x,t)}{u_0(x,t)} = 0, \quad \textit{uniformly in } x > 0$$

*Furthermore, we have the estimates*

$$\frac{1}{\pi\lambda^2 t} \leqslant \frac{u(x,t)}{u_0(x,t)} \leqslant \frac{1}{\lambda\sqrt{(\pi t)}} \tag{24}$$

*as $t \to +\infty$.*

*Proof.* The integral equation for $V(t)$ can be transformed into the following initial value problem for a first-order ordinary differential equation

$$\dot{U}(t) = \frac{h_0}{\sqrt{(\pi t)}} - \lambda h_0 + \lambda^2 U(t), \quad t > 0$$
$$U(0) = 0$$

where

$$U(t) = \int_0^t V(\tau)\,\mathrm{d}\tau, \quad t > 0 \tag{25}$$

The solution $U(t)$ of (25) is given by (22) and so, simple calculations show that $V(t)$ is given by (21). Moreover, we have

$$1 - \frac{1}{h_0}\int_0^t F(V(\tau))\,\mathrm{d}\tau = 1 - \frac{\lambda}{h_0}U(t) = \exp(\lambda^2 t)\,\mathrm{erfc}(\lambda\sqrt{t})$$

that is (23). To derive inequalities (24) we need the following (see [1]):

$$\frac{2}{3z} < \frac{2}{z + \sqrt{(z^2 + z)}} \leqslant \sqrt{\pi}\exp(z^2)\,\mathrm{erfc}(z) \leqslant \frac{2}{z + \sqrt{(z^2 + 4/\pi)}} < \frac{1}{z}, \quad z > \sqrt{\frac{2}{3}}$$

from which we derive

$$\frac{1}{\pi z^2} \leqslant 1 - Q(z) \leqslant \frac{1}{2z^2}, \quad z > \sqrt{\frac{2}{3}}$$

Therefore, the following inequalities:

$$\frac{1}{\pi\lambda^2 t} \leqslant \frac{\sqrt{(\pi t)}}{h_0}\,V(t) \leqslant \frac{1}{2\lambda^2 t}$$

$$\frac{2}{3\lambda\sqrt{(\pi t)}} \leqslant 1 - \frac{1}{h_0}\int_0^t F(V(\tau))\,\mathrm{d}\tau \leqslant \frac{1}{\lambda\sqrt{(\pi t)}}$$

hold for $t \to +\infty$. Inequality (24) follows from inequalities (16) of Theorem 7. This completes the proof. □

Now, by means of the explicit expression of the solution $u(x,t)$ to the problem with linear control given by (20), the asymptotic behaviour of the quotient $u(x,t)/u_0(x,t)$ is established in our next result.

**Theorem 12.** *If* $F(v) \equiv \lambda v$ (*with* $\lambda > 0$) *and* $h(x) \equiv h_0 > 0$, *the solution to problem* (2) *can be written in the form*

$$u(x,t) = h_0 \exp(\lambda^2 t)\left[\mathrm{erfc}\,(\lambda\sqrt{t}) - \exp(\lambda x)\,\mathrm{erfc}\left(\lambda\sqrt{t} + \frac{x}{2\sqrt{t}}\right)\right] \tag{26}$$

 

*Furthermore,*

$$\frac{u(x,t)}{u_0(x,t)} \sim 1/(2\lambda t^2) \tag{27}$$

*when* $t \to +\infty$, *uniformly in* $x > 0$.

*Proof.* Let $I = I(x,t)$ be the function defined by

$$I(x,t) = \int_0^t \operatorname{erf}\left(\frac{x}{2\sqrt{(t-\tau)}}\right) V(\tau)\,\mathrm{d}\tau$$

By using the Laplace transformation in the variable $t$, we deduce

$$\hat{I}(x,s) = h_0\left[\frac{1}{s(\lambda+\sqrt{s})} - \frac{\exp(-x\sqrt{s})}{s(\lambda+\sqrt{s})}\right]$$

whence, using tables of Laplace transformation (see, for example, [1]), an exact expression for $I$ is obtained as follows:

$$I(x,t) = \frac{h_0}{\lambda}\left(\operatorname{erf}\left(\frac{x}{2\sqrt{t}}\right) + \exp(\lambda^2 t + \lambda x)\operatorname{erfc}\left(\lambda\sqrt{t} + \frac{x}{2\sqrt{t}}\right)\right.$$
$$\left. - \exp(\lambda^2 t)\operatorname{erfc}(\sqrt{(\lambda t)})\right) \tag{28}$$

Now, taking into account that

$$u(x,t) = h_0 \operatorname{erf}\left(\frac{x}{2\sqrt{t}}\right) - \lambda I(x,t)$$

we deduce the expression (26) for $u$. On the other hand, for the quotient $u(x,t)/u_0(x,t)$ we get

$$\frac{u(x,t)}{u_0(x,t)} = \frac{\exp(\lambda^2 t)\operatorname{erfc}(\lambda\sqrt{t}) - \exp(\lambda^2 t + \lambda x)\operatorname{erfc}((\lambda\sqrt{t}) + (x/2\sqrt{t}))}{\operatorname{erf}(x/2\sqrt{t})}$$

whence, in view of the estimation $\operatorname{erf}(z) \sim 2z/\sqrt{\pi}$ when $z \to 0$, we deduce the estimate

$$\frac{u(x,t)}{u_0(x,t)} \sim \frac{1}{2t\lambda(\lambda + x/(2t))} \sim \frac{1}{2\lambda^2 t}, \quad t \to +\infty$$

as expression (27) states. □

In the next theorem we extend the above study of linear control functions to a large class of functions $F$.

**Theorem 13.** *Let F be a continuous real function verifying conditions* (A) *and* (B) *of the introduction. Furthermore, we assume that*

(C) *F is convex on* $(0, +\infty)$,
(D) $F'(0^+) = \lambda > 0$.

*Then the solution to problem* (2) *with* $h(x) = h_0$, $x > 0$, *satisfies the following asymptotic behaviour*:

$$\lim_{t\to+\infty} \frac{u(x,t)}{u_0(x,t)} = 0$$

*Proof.* From conditions (A), (C) and (D) we can write $F(\xi) = \lambda\xi + \varphi(\xi)$, $\xi > 0$, where $\varphi$ is a non-negative function defined on $(0, +\infty)$ such that $\varphi(0) = 0$ and $\lim_{\xi\to+\infty} \varphi(\xi)/\xi = 0$. Moreover, since

$$0 \leqslant \varphi(\varepsilon\xi) = \varphi((1-\varepsilon)0 + \varepsilon\xi) \leqslant (1-\varepsilon)\varphi(0) + \varepsilon\varphi(\xi) = \varepsilon\varphi(\xi) < \varphi(\xi), \quad \varepsilon \in [0, 1]$$

$\varphi$ is an increasing convex function. If $V$ denotes the solution to integral equation (15), then we can write

$$\lim_{t\to+\infty} \sqrt{t}\varphi\,(V(t)) = 0, \qquad \lim_{t\to+\infty} \int_0^t \frac{\varphi(V(t))}{\sqrt{(t-\tau)}}\,\mathrm{d}\tau = 0$$

Since that $0 \leqslant V(t) \leqslant h_0/\sqrt{(\pi t)}$, $t > 0$, from the monotonicity of $\varphi$ we obtain

$$0 \leqslant \lim_{t+\to\infty} \sqrt{t}\varphi(V(t)) \leqslant \lim_{t\to+\infty} \sqrt{t}\varphi\left(\frac{h_0}{\sqrt{(\pi t)}}\right) = \frac{h_0}{\sqrt{(\pi t)}} \lim_{\xi\to 0^+} \left(\frac{\varphi(\xi)}{\xi}\right) = 0$$

i.e., $\lim_{t\to+\infty} \sqrt{t}\varphi\,(V(t)) = 0$ and $\lim_{t\to+\infty} V(t) = 0$. From this and from the Fatou's lemma we get

$$0 \leqslant \liminf_{t\to+\infty} \int_0^t \frac{\varphi\,(V(\tau))}{\sqrt{(t-\tau)}}\,\mathrm{d}\tau \leqslant \limsup_{t\to+\infty} \int_0^t \frac{\varphi\,(V(\tau))}{\sqrt{(t-\tau)}}\,\mathrm{d}\tau$$

$$= \limsup_{t\to+\infty} \int_0^1 \frac{\sqrt{(\xi t)}\,\varphi(V(\xi t))}{\sqrt{\xi(1-\xi)}}\,\mathrm{d}\xi \leqslant \int_0^1 \frac{\limsup_{t\to+\infty} \sqrt{(\xi t)}\,\varphi\,(V\,(\xi t))}{\sqrt{[\xi(1-\xi)]}}\,\mathrm{d}\xi = 0$$

Now, from integral equation (15) we obtain

$$\int_0^t \frac{V(\tau)}{\sqrt{[\pi(t-\tau)]}}\,\mathrm{d}\tau = \frac{1}{\lambda}\left[\frac{h_0}{\sqrt{(\pi t)}} - V(t) - \int_0^t \frac{\varphi\,(V\,(\tau))}{\sqrt{[\pi(t-\tau)]}}\,\mathrm{d}\tau\right]$$

whence

$$\lim_{t\to+\infty} \int_0^t \frac{V(\tau)}{\sqrt{(t-\tau)}}\,\mathrm{d}\tau = 0$$

Finally, the results follows from the last equation and from the observation immediately below Corollary 8. □

 

We finish this section, and the paper, with a generalization of the previous result to the case of bounded initial temperatures.

**Theorem 14.** *Let u be the solution to problem* (2) *with* $h(x)$, $x > 0$, *being a non-negative, continuous and upper and below bounded function in the sense that* $0 < C_1 \leqslant h(x) \leqslant C_2$, $x > 0$, *for certain positive constants* $C_1$ *and* $C_2$. *If the control function F verifies conditions* (A), (B) *from the introduction and* (C), (D) *from Theorem* 13, *then we have*

$$\lim_{t \to +\infty} \frac{u(x,t)}{u_0(x,t)} = 0$$

*Proof.* A simple application of the maximum principle of problem (2) shows that

$$0 \leqslant u_{C_1}(x,t) \leqslant u(x,t) \leqslant u_{C_2}(x,t), \quad x > 0,\ t > 0$$

where $u_C$ is the solution of (2) for the same control function $F$ and initial constant temperature $C$. From this inequality we deduce

$$\frac{C_1}{C_2} \frac{u_{C_1}(x,t)}{u_{0C_1}(x,t)} \leqslant \frac{u(x,t)}{u_0(x,t)} \leqslant \frac{u_{C_2}(x,t)}{u_{0C_2}(x,\ t)}, \quad x > 0,\ t > 0$$

where $u_{0C}$ is the solution to (2) for null control function $F$ and initial constant temperature $C$. Since $\lim_{t \to +\infty} (u_C(x,t)/u_{0C}(x,t)) = 0$, for any positive constant $C$, by Theorem 13, the result is immediately derived from the last inequalities. □

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