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Stefan problems for the diffusion–convection equation with temperature-dependent thermal coefficients

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ABSTRACT

Different one-phase Stefan problems for a semi-infinite slab are considered, involving a moving phase change material as well as temperature dependent thermal coefficients. Existence of at least one similarity solution is proved imposing a Dirichlet, Neumann, Robin or radiative–convective boundary condition at the fixed face. The velocity that arises in the convective term of the diffusion–convection equation is assumed to depend on temperature and time. In each case, an equivalent ordinary differential problem is obtained giving rise to a system of an integral equation coupled with a condition for the parameter that characterizes the free boundary, which is solved through a double-fixed point analysis. Some solutions for particular thermal coefficients are provided.

1. Introduction

Stefan problems constitute a broad field of study since they arise in different areas of engineering, biology, geoscience and industry [1–4]. The classical one-phase Stefan problem models a phase-change thermal process that aims to describe the temperature of the material as well as the location of the interface that separate both phases. Mathematically it consists on finding a solution to the heat-conduction equation in an unknown region which has also to be determined, imposing an initial condition, boundary conditions, and the Stefan condition at the moving interface.

In many physical processes, the phase change material is allowed to move when the phase change occurs. Recently, in [5] a Stefan problem which models the undergoing phase transition of a moving material where the phase change and heat distribution in the medium are affected from both the conduction and convection of heat was considered.

The diffusion–convection equation has multiple applications, for example, to ground water hydrology, oil reservoir engineering and drug propagation in the arterial tissues. In [6] a one-phase Stefan problem for the diffusion–convection equation with a particular temperature-dependent thermal conductivity and a particular unidirectional speed was studied. More articles where a convective term is involved in the parabolic equation are [7–10].

In particular, in [10] a Stefan problem with variable thermal coefficients and moving phase change material was studied. It was considered a thermal conductivity and a specific heat whose dependence on the temperature was assumed to be linear.

Motivated by [5,10], in this paper we consider a free boundary problem in a semi-infinite domain $x > 0$ for the nonlinear diffusion–convection equation with a convective term that involves temperature-dependent thermal coefficients.

The problem consists in finding the temperature $T = T(x,t)$ in the liquid region and the free boundary $x = s(t)$ such that:

$$\rho(T)c(T)\frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(k(T)\frac{\partial T}{\partial x}\right) - v(T)\frac{\partial T}{\partial x}, \quad 0 < x < s(t), \quad t > 0, \tag{1.1a}$$

$$T(0,t) = T^*, \quad t > 0, \tag{1.1b}$$

$$T(s(t),t) = T_m, \quad t > 0, \tag{1.1c}$$

$$k\left(T(s(t),t)\right)\frac{\partial T}{\partial x}(s(t),t) = -\rho_0 \ell \dot{s}(t), \quad t > 0, \tag{1.1d}$$

$$s(0) = 0, \tag{1.1e}$$

where $\rho(T)$, $c(T)$ and $k(T)$ are the mass density, the specific heat and the thermal conductivity of the body, respectively defined as

$$\rho(T)(x,t) = \rho(T(x,t)), \quad c(T)(x,t) = c(T(x,t)), \quad k(T)(x,t) = k(T(x,t)). \tag{1.2}$$

Some models involving temperature-dependent thermal coefficients can be found in [11–17].

The presence of a convection term in Eq. (1.1a) represents the fact that the phase change material is allowed to move with an unidirectionally speed $v = v(T)$. Throughout this paper the unidirectional speed v

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is given by

$$v(T) = \frac{\mu(T)}{\sqrt{t}} \tag{1.3}$$

with $\mu(T)(x,t) = \mu(T(x,t))$.

It should be noticed that in [6] it was considered that $k(T,x) = \frac{\rho c(1+dx)}{(a+bT)^2}$ and $v(T) = \frac{\rho c d}{2(a+bT)^2}$, where c is the specific heat and a,b,d are positive constants. Due to this particular assumptions a parametric representation of the solution of the similarity type was obtained.

We assume that T_m is the phase change temperature and $T^* > T_m$ is the temperature imposed at the fixed face $x=0$. Condition (1.1d) represents the Stefan condition where $\rho_0 > 0$ is a constant mass density and $\ell > 0$ is the latent heat of fusion per unit mass.

We will consider three more problems that arise replacing the Dirichlet condition at the fixed face by other type of conditions.

On one hand we stablish the problem with a Neumann condition at the fixed face, replacing (1.1b) by

$$k(T(0,t))\frac{\partial T}{\partial x}(0,t) = -\frac{q}{\sqrt{t}}, \qquad t>0, \tag{1.1b$\star$}$$

where $q>0$ is a given constant and $-\frac{q}{\sqrt{t}}$ represents the prescribed flux at $x=0$. Some bibliography imposing this kind of condition can be found in [18–22]

On the other hand, we consider a problem governed by (1.1a) and (1.1c)–(1.1e) where a Robin condition is imposed:

$$k(T(0,t))\frac{\partial T}{\partial x}(0,t) = \frac{h}{\sqrt{t}}\left[T(0,t)-T^*\right], \qquad t>0, \tag{1.1b†}$$

being h the coefficient that characterizes the heat transfer at the fixed face and T^* the bulk temperature applied at a neighbourhood of $x=0$ with $T^* > T(0,t) > T_m$ [23–27].

Finally we define the problem that arises replacing the Dirichlet condition (1.1b) by a radiative and convective condition

$$k(T(0,t))\frac{\partial T}{\partial x}(0,t) = \frac{h}{\sqrt{t}}\left[T(0,t)-T^*\right] + \frac{\sigma\epsilon}{\sqrt{t}}\left[T^4(0,t)-T^{*4}\right], \qquad t>0, \tag{1.1b††}$$

where σ is the Stefan–Boltzmann constant and $\epsilon > 0$ is the coefficient that characterizes the radiation shape factor, assuming $T^* > T(0,t) > T_m$. Notice that the first term of the r.h.s of condition (1.1b††) coincides to the r.h.s of the Robin condition (1.1b†). This kind of boundary condition also appears in [28–31]

The aim of this paper is to provide, following the methodology of [13,19,24], sufficient conditions on data in order to guarantee existence of at least one solution of a similarity type to four different problems that differ from each other in the boundary condition imposed at the fixed face: temperature, flux, convective or radiative–convective condition.

The manuscript is organized as follows. In Section 2 we analyse the existence of at least one similarity solution to the problem governed by (1.1a)–(1.1e) where a constant temperature is imposed at the fixed face. Introducing the similarity variable, an equivalent ordinary differential problem is obtained, giving rise to nonlinear integral equation coupled with a condition for the parameter that characterizes the free boundary. This system is solved by a double fixed point analysis. In a similar way, in Section 3 we stablish the existence of solution to the Stefan problem that arises when we replace the Dirichlet condition (1.1b) by a Neumann one (1.1b$\star$). Section 4 is devoted to the study the problem where a Robin condition (1.1b†) is imposed at $x=0$. Moreover, we analyse the convergence of the solution when $h \to +\infty$, and recover for $v=0$ a result given in [24]. Finally, in Section 5 we generalize Section 4 by showing that there exists at least one solution to the problem that arises when considering a radiative–convective condition (1.1b††) at the fixed face.

In the last section we present different solutions obtained for some particular cases. On one hand, we consider constant thermal coefficients and a velocity given by $v(T) = \frac{\mu(T)}{\sqrt{t}}$ with

$$\mu(T) = \rho_0 c_0 \sqrt{\alpha_0}\,\mathrm{Pe}, \tag{1.4}$$

where Pe denotes the Peclet number. The solutions given by [5] for Dirichlet and Neumann condition at fixed face are recovered. On the other hand, we analyse the particular case when the thermal coefficients involved are linear functions of the temperature as in [10].

2. Dirichlet Condition

The following section is devoted to the analysis of the Stefan problem given by (1.1a)–(1.1e).

Let us define $\mathscr{C} = C^0(\mathbb{R}_0^+ \times \mathbb{R}^+) \cap L^\infty(\mathbb{R}_0^+ \times \mathbb{R}^+)$. We denote the norm of $\widetilde{T} \in C$ by

$$\|\widetilde{T}\| = \max_{(x,t)\in\mathbb{R}_0^+\times\mathbb{R}^+} |\widetilde{T}(x,t)|.$$

We will assume:

$$\begin{cases} \text{(a) There exists } k_m > 0 \text{ and } k_M > 0 \text{ such that} \\ \quad k_m \le k(\widetilde{T}(x,t)) \le k_M, \qquad \forall\, \widetilde{T} \in \mathscr{C}, \qquad \forall (x,t) \in \mathbb{R}_0^+\times\mathbb{R}^+. \\ \text{(b) There exists } \widetilde{k} > 0 \text{ such that} \\ \quad |k(\widetilde{T}_1(x,t)) - k(\widetilde{T}_2(x,t))| \le \widetilde{k}\, \|\widetilde{T}_1 - \widetilde{T}_2\|, \\ \quad \forall\, \widetilde{T}_1, \widetilde{T}_2 \in \mathscr{C}, \qquad \forall (x,t) \in \mathbb{R}_0^+\times\mathbb{R}^+. \end{cases} \tag{2.1}$$

$$\begin{cases} \text{(a) There exists } \gamma_m > 0 \text{ and } \gamma_M > 0 \text{ such that} \\ \quad \gamma_m \le \rho(\widetilde{T}(x,t))\widetilde{c}(\widetilde{T}(x,t)) \le \gamma_M, \qquad \forall\, \widetilde{T} \in \mathscr{C}, \quad \forall (x,t) \in \mathbb{R}_0^+\times\mathbb{R}^+. \\ \text{(b) There exists } \widetilde{\gamma} > 0 \text{ such that} \\ \quad |\rho(\widetilde{T}_1(x,t))c(\widetilde{T}_1(x,t)) - \rho(\widetilde{T}_2(x,t))c(\widetilde{T}_2(x,t))| \le \widetilde{\gamma}\|\widetilde{T}_1 - \widetilde{T}_2\|, \\ \quad \forall\, \widetilde{T}_1, \widetilde{T}_2 \in \mathscr{C}, \qquad \forall (x,t) \in \mathbb{R}_0^+\times\mathbb{R}^+. \end{cases} \tag{2.2}$$

$$\begin{cases} \text{(a) There exists } v_m > 0 \text{ and } v_M > 0 \text{ such that} \\ \quad v_m \le \mu(\widetilde{T}(x,t)) \le v_M, \qquad \forall\, \widetilde{T} \in \mathscr{C}, \qquad \forall (x,t) \in \mathbb{R}_0^+\times\mathbb{R}^+. \\ \text{(b) There exists } \widetilde{v} > 0 \text{ such that} \\ \quad |\mu(\widetilde{T}_1(x,t)) - \mu(\widetilde{T}_2(x,t))| \le \widetilde{v}\, \|\widetilde{T}_1 - \widetilde{T}_2\|, \\ \quad \forall\, \widetilde{T}_1, \widetilde{T}_2 \in \mathscr{C}, \qquad \forall (x,t) \in \mathbb{R}_0^+\times\mathbb{R}^+. \end{cases} \tag{2.3}$$

If we introduce the following change of variables:

$$\theta = \frac{T-T^*}{T_m - T^*} > 0, \tag{2.4}$$

we have

$$\frac{\partial T}{\partial t} = (T_m - T^*)\frac{\partial \theta}{\partial t}, \qquad \frac{\partial T}{\partial x} = (T_m - T^*)\frac{\partial \theta}{\partial x}, \qquad \frac{\partial^2 T}{\partial x^2} = (T_m - T^*)\frac{\partial^2 \theta}{\partial x^2}.$$

Taking into account that $T = T(\theta) = (T_m - T^*)\theta + T^*$ we can define the following functions:

$$\overline{L}(\theta) = L(T(\theta)), \qquad \overline{N}(\theta) = N(T(\theta)), \qquad \overline{v}(\theta) = \frac{v(T(\theta))}{\rho_0 c_0} \tag{2.5}$$

where

$$L(T) = \frac{k(T)}{k_0}, \qquad N(T) = \frac{\rho(T)c(T)}{\rho_0 c_0} \tag{2.6}$$

and k_0, ρ_0, c_0 and $\alpha_0 = \dfrac{k_0}{\rho_0 c_0}$ are the reference thermal conductivity, mass density, specific heat and thermal diffusivity, respectively.

Therefore, the problem (1.1a)–(1.1e) becomes:

$$\overline{N}(\theta)\frac{\partial\theta}{\partial t} = \alpha_0 \frac{\partial}{\partial x}\left(\overline{L}(\theta)\frac{\partial\theta}{\partial x}\right) - \overline{v}(\theta)\frac{\partial\theta}{\partial x}, \qquad 0 < x < s(t), \quad t > 0, \tag{2.7a}$$

$$\theta(0,t) = 0, \qquad t > 0, \tag{2.7b}$$

$$\theta(s(t),t) = 1, \qquad t > 0, \tag{2.7c}$$

$$\overline{L}(\theta(s(t),t))\frac{\partial\theta}{\partial x}(s(t),t) = \frac{\dot{s}(t)}{\alpha_0 \mathrm{Ste}}, \qquad t > 0, \tag{2.7d}$$

$$s(0) = 0, \tag{2.7e}$$

where $\mathrm{Ste} = \frac{(T^* - T_m)c_0}{\ell} > 0$ is the Stefan number.

If we introduce the similarity variable $\xi = \frac{x}{2\sqrt{\alpha_0 t}}$ and assume a similarity type solution

$$\theta(x,t) = f(\xi), \tag{2.8}$$

then conditions (2.7c)–(2.7d) yield a free boundary given by

$$s(t) = 2\lambda\sqrt{\alpha_0 t}. \tag{2.9}$$

where $\lambda > 0$ is a constant to be determined.

Let us define

$$L^*(f) = \overline{L}(\theta), \qquad N^*(f) = \overline{N}(\theta), \qquad v^*(f) = \overline{v}(\theta). \tag{2.10}$$

Then, (1.1a) turns into the following ordinary differential equation

$$\left(L^*(f)f'(\xi)\right)' + 2N^*(f)\,\xi\, f'(\xi) - \frac{2\sqrt{t}}{\sqrt{\alpha_0}}v^*(f)\, f'(\xi) = 0, \qquad 0 < \xi < \lambda. \tag{2.11}$$

Taking into account that the unidirectional speed v is given by (1.3), and from (2.5) and (2.10), we define

$$\mu^*(f) = \frac{v^*(f)\sqrt{t}}{\sqrt{\alpha_0}} = \frac{\mu(T)}{\sqrt{\rho_0 c_0 k_0}}. \tag{2.12}$$

Therefore we reduce the problem (2.7a)–(2.7e) into an ordinary differential problem defined by:

$$\left(L^*(f)f'(\xi)\right)' + 2f'(\xi)\left(N^*(f)\xi - \mu^*(f)\right) = 0, \qquad 0 < \xi < \lambda, \tag{2.13a}$$

$$f(0) = 0, \tag{2.13b}$$

$$f(\lambda) = 1, \tag{2.13c}$$

$$L^*(f(\lambda))f'(\lambda) = \frac{2\lambda}{\mathrm{Ste}}. \tag{2.13d}$$

Let us define

$$g(\xi) = L^*(f(\xi))f'(\xi). \tag{2.14}$$

Then (2.13a) turns equivalent to

$$g'(\xi) + 2g(\xi)\left(\xi\frac{N^*(f)}{L^*(f)} - \frac{\mu^*(f)}{L^*(f)}\right) = 0,$$

whose solution is given by

$$g(\xi) = A_1 E(f)(\xi)$$

where A_1 is a constant and

$$E(f)(z) = \frac{U(f)(z)}{I(f)(z)}, \tag{2.15}$$

with

$$U(f)(z) = \exp\left(2\int_0^z \frac{\mu^*(f)(\sigma)}{L^*(f)(\sigma)}\,\mathrm{d}\sigma\right),$$
$$I(f)(z) = \exp\left(2\int_0^z \frac{\sigma N^*(f)(\sigma)}{L^*(f)(\sigma)}\,\mathrm{d}\sigma\right). \tag{2.16}$$

From (2.14) we obtain that

$$f(\xi) = A_1\Phi(f)(\xi) + A_2$$

where

$$\Phi(f)(\xi) = \int_0^\xi \frac{E(f)(z)}{L^*(f)(z)}\,\mathrm{d}z. \tag{2.17}$$

From (2.13b)–(2.13c) we get that f must satisfy the following integral equation

$$f(\xi) = \frac{\Phi(f)(\xi)}{\Phi(f)(\lambda)}, \qquad 0 \le \xi \le \lambda. \tag{2.18}$$

Condition (2.13d) leads to the following condition for λ:

$$\frac{\mathrm{Ste}}{2}\frac{E(f)(\lambda)}{\Phi(f)(\lambda)} = \lambda. \tag{2.19}$$

In order to analyse the existence of solution to the problem (2.18)–(2.19), let us study in the first instance, the integral equation (2.18) assuming that $\lambda > 0$ is a fixed given constant.

Consider the space $C^0[0,\lambda]$ of continuous real-valued functions defined on $[0,\lambda]$ endowed with the supremum norm

$$\|f\| = \max_{\xi\in[0,\lambda]} |f(\xi)|. \tag{2.20}$$

Let us define the operator $\mathcal{H}$ on $C^0[0,\lambda]$ given by the r.h.s of Eq. (2.18):

$$\mathcal{H}(f)(\xi) = \frac{\Phi(f)(\xi)}{\Phi(f)(\lambda)}. \tag{2.21}$$

Then, as $(C^0[0,\lambda], \|\cdot\|)$ is a Banach space we use the fixed point Banach theorem to prove that for each $\lambda > 0$ there exists a unique f such that

$$\mathcal{H}(f)(\xi) = f(\xi), \qquad 0 \le \xi \le \lambda \tag{2.22}$$

which is the solution to (2.18).

From the assumptions (2.1), (2.2) and (2.3) we obtain that L^*, N^* and μ^* are bounded and Lipschitz continuous. That is to say

$$\left\{\begin{array}{l} L^* \text{ is such that:} \\ \text{(a) There exists } L_m = \frac{k_m}{k_0} > 0 \text{ and } L_M = \frac{k_M}{k_0} > 0 \text{ such that} \\ \qquad L_m \le L^*(\widetilde{f}(\xi)) \le L_M, \qquad \forall \widetilde{f} \in C^0(\mathbb{R}_0^+) \cap L^\infty(\mathbb{R}_0^+),\ \forall\xi \in \mathbb{R}_0^+. \\ \text{(b) There exists } \widetilde{L} = \frac{\widetilde{k}(T^*-T_m)}{k_0} > 0 \text{ such that} \\ \qquad |L^*(\widetilde{f}_1(\xi)) - L^*(\widetilde{f}_2(\xi))| \le \widetilde{L}\|\widetilde{f}_1 - \widetilde{f}_2\|, \\ \qquad \forall \widetilde{f}_1, \widetilde{f}_2 \in C^0(\mathbb{R}_0^+) \cap L^\infty(\mathbb{R}_0^+) \quad \forall\xi \in \mathbb{R}_0^+. \end{array}\right. \tag{2.23}$$

$$\left\{\begin{array}{l} N^* \text{ is such that:} \\ \text{(a) There exists } N_m = \frac{\gamma_m}{\rho_0 c_0} > 0 \text{ and } N_M = \frac{\gamma_M}{\rho_0 c_0} > 0 \text{ such that} \\ \qquad N_m \le N^*(\widetilde{f}(\xi)) \le N_M, \quad \forall \widetilde{f} \in C^0(\mathbb{R}_0^+) \cap L^\infty(\mathbb{R}_0^+),\ \xi \in \mathbb{R}_0^+. \\ \text{(b) There exists } \widetilde{N} = \frac{\widetilde{\gamma}(T^*-T_m)}{\rho_0 c_0} > 0 \text{ such that} \\ \qquad |N^*(\widetilde{f}_1(\xi)) - N^*(\widetilde{f}_2(\xi))| \le \widetilde{N}\|\widetilde{f}_1 - \widetilde{f}_2\|, \\ \qquad \forall \widetilde{f}_1, \widetilde{f}_2 \in C^0(\mathbb{R}_0^+) \cap L^\infty(\mathbb{R}_0^+),\ \xi \in \mathbb{R}_0^+. \end{array}\right. \tag{2.24}$$

$$\left\{\begin{array}{l} \mu^* \text{ is such that:} \\ \text{(a) There exists } \mu_m = \frac{v_m}{\sqrt{\rho_0 c_0 k_0}} > 0 \text{ and } \mu_M = \frac{v_M}{\sqrt{\rho_0 c_0 k_0}} > 0 \text{ such that} \\ \qquad \mu_m \le \mu^*(\widetilde{f}(\xi)) \le \mu_M, \qquad \forall f \in C^0(\mathbb{R}_0^+) \cap L^\infty(\mathbb{R}_0^+),\ \xi \in \mathbb{R}_0^+. \\ \text{(b) There exists } \widetilde{\mu} = \frac{\widetilde{v}(T^*-T_m)}{\sqrt{\rho_0 c_0 k_0}} > 0 \text{ such that} \\ \qquad |\mu^*(\widetilde{f}_1(\xi)) - \mu^*(\widetilde{f}_2(\xi))| \le \widetilde{\mu}\,\|\widetilde{f}_1 - \widetilde{f}_2\|, \\ \qquad \forall \widetilde{f}_1, \widetilde{f}_2 \in C^0(\mathbb{R}_0^+) \cap L^\infty(\mathbb{R}_0^+),\ \xi \in \mathbb{R}_0^+. \end{array}\right. \tag{2.25}$$

Let us present now some preliminary results that will allow us to prove the existence and uniqueness of solution to Eq. (2.22).

Lemma 2.1. *For all $z \in [0, \lambda]$ the following bounds hold*

$$\exp\left(2\frac{\mu_m}{L_M}z\right) \le U(f)(z) \le \exp\left(2\frac{\mu_M}{L_m}z\right), \tag{2.26}$$

$$\exp\left(\frac{N_m}{L_M}z^2\right) \le I(f)(z) \le \exp\left(\frac{N_M}{L_m}z^2\right), \tag{2.27}$$

$$\exp\left(-\frac{N_M}{L_m}z^2\right) \le \frac{\exp\left(2\frac{\mu_m}{L_M}z\right)}{\exp\left(\frac{N_M}{L_m}z^2\right)} \le E(f)(z) \le \frac{\exp\left(2\frac{\mu_M}{L_m}z\right)}{\exp\left(\frac{N_m}{L_M}z^2\right)} \le \exp\left(2\frac{\mu_M}{L_m}z\right), \tag{2.28}$$

$$\frac{z}{L_M}\exp\left(-\frac{N_M}{L_m}z^2\right) \le \frac{\sqrt{\pi}}{2}\frac{\sqrt{L_m}}{L_M\sqrt{N_M}}\operatorname{erf}\left(\sqrt{\frac{N_M}{L_m}}z\right) \le \Phi(f)(z) \le \frac{1}{2\mu_M}\exp\left(2\frac{\mu_M}{L_m}z\right). \tag{2.29}$$

Proof. The proof follows immediately from the definitions of U, I, E, Φ using assumptions (2.23)–(2.25). □

Lemma 2.2. *Given $\lambda > 0$, for all $z \in [0, \lambda]$ and $f_1, f_2 \in C^0[0, \lambda]$ the following inequalities hold*

$$|U(f_1)(z) - U(f_2)(z)| \le D_1(z)\|f_1 - f_2\|, \tag{2.30}$$

$$|I(f_1)(z) - I(f_2)(z)| \le D_2(z)\|f_1 - f_2\|, \tag{2.31}$$

$$|E(f_1)(z) - E(f_2)(z)| \le D_3(z)\|f_1 - f_2\|, \tag{2.32}$$

$$|\Phi(f_1)(z) - \Phi(f_2)(z)| \le \lambda\, D_4(\lambda)\|f_1 - f_2\|, \tag{2.33}$$

where

$$\begin{aligned}
D_1(z) &= \frac{2\exp\left(\frac{2\mu_M}{L_m}\right)}{L_m^2} z\left(\mu_M \widetilde{L} + L_m\widetilde{\mu}\right),\\
D_2(z) &= \frac{\exp\left(\frac{N_M}{L_m}z^2\right)}{L_m^2} z^2\left(N_M\widetilde{L} + L_m\widetilde{N}\right),\\
D_3(z) &= \exp\left(\frac{N_M}{L_m}z^2\right) D_1(z) + \exp\left(2\frac{\mu_M}{L_m}z\right) D_2(z),\\
D_4(\lambda) &= \frac{1}{L_m^2}\left(\widetilde{L}\exp\left(2\lambda\frac{\mu_M}{L_m}\right) + L_m D_3(\lambda)\right).
\end{aligned} \tag{2.34}$$

Proof. Applying the mean value theorem and taking into account assumptions (2.23)–(2.25) we obtain that

$$\begin{aligned}
|U(f_1)(z) - U(f_2)(z)| &\le 2\exp\left(2\frac{\mu_M}{L_m}z\right)\int_0^z \left|\frac{\mu^*(f_1)(\sigma)}{L^*(f_1)(\sigma)} - \frac{\mu^*(f_2)(\sigma)}{L^*(f_2)(\sigma)}\right| d\sigma\\
&\le 2\exp\left(2\frac{\mu_M}{L_m}z\right)\left\{\int_0^z \left|\tfrac{\mu^*(f_1)(\sigma)}{L^*(f_1)(\sigma)L^*(f_2)(\sigma)}\right| \left|L^*(f_2)(\sigma) - L^*(f_1)(\sigma)\right| d\sigma\right.\\
&\quad \left. + \int_0^z \left|\tfrac{1}{L^*(f_2)(\sigma)}\right|\left|\mu^*(f_2)(\sigma) - \mu^*(f_1)(\sigma)\right| d\sigma\right\}\\
&\le 2\exp\left(2\frac{\mu_M}{L_m}z\right) z\left(\frac{\mu_M\widetilde{L}}{L_m^2}\|f_1 - f_2\| + \frac{\widetilde{\mu}}{L_m}\|f_1 - f_2\|\right) \le D_1(z)\|f_1 - f_2\|.
\end{aligned}$$

In a similar way, by using the mean value theorem again we get

$$\begin{aligned}
|I(f_1)(z) - I(f_2)(z)| &\le 2\exp\left(\frac{N_M}{L_m}z^2\right)\int_0^z \sigma\left|\frac{N^*(f_1)(\sigma)}{L^*(f_1)(\sigma)} - \frac{N^*(f_2)(\sigma)}{L^*(f_2)(\sigma)}\right| d\sigma\\
&\le 2\exp\left(\frac{N_M}{L_m}z^2\right)\left\{\int_0^z \sigma\left|\tfrac{N^*(f_1)(\sigma)}{L^*(f_1)(\sigma)L^*(f_2)(\sigma)}\right|\left|L^*(f_2)(\sigma) - L^*(f_1)(\sigma)\right| d\sigma\right.\\
&\quad \left. + \int_0^z \sigma\left|\tfrac{1}{L^*(f_2)(\sigma)}\right|\left|N^*(f_2)(\sigma) - N^*(f_1)(\sigma)\right| d\sigma\right\}\\
&\le 2\exp\left(\frac{N_M}{L_m}z^2\right)\frac{z^2}{2}\left(\frac{N_M\widetilde{L}}{L_m^2}\|f_1 - f_2\| + \frac{\widetilde{N}}{L_m}\|f_1 - f_2\|\right) \le D_2(z)\|f_1 - f_2\|.
\end{aligned}$$

Taking into account that E defined by (2.15) depends on U and I we use (2.26), (2.27) and the properties (2.30) and (2.31) that we have just proved in order to get the inequality (2.32).

$$\begin{aligned}
|E(f_1)(z) - E(f_2)(z)| &\le \tfrac{1}{|I(f_1)(z)||I(f_2)(z)|}\left|U(f_1)(z)I(f_2)(z) - I(f_1)(z)U(f_2)(z)\right|\\
&\le \exp\left(-2\frac{N_m}{L_M}z^2\right)\left\{|U(f_1)(z)||I(f_2)(z) - I(f_1)(z)|\right.\\
&\quad \left. + |I(f_1)(z)||U(f_1)(z) - U(f_2)(z)|\right\}\\
&\le \exp\left(2\tfrac{\mu_M}{L_m}z\right) D_2(z)\|f_1 - f_2\|\\
&\quad + \exp\left(\frac{N_M}{L_m}z^2\right) D_1(z)\|f_1 - f_2\| \le D_3(z)\|f_1 - f_2\|.
\end{aligned}$$

Finally, by virtue of the definition (2.17) of Φ and inequalities (2.28) and (2.32) we obtain

$$\begin{aligned}
|\Phi(f_1)(z) - \Phi(f_2)(z)| &\le \int_0^z \left\{\frac{|E(f_1)(\eta)|}{|L^*(f_1)(\eta)L^*(f_2)(\eta)|}|L^*(f_2)(\eta) - L^*(f_1)(\eta)|\right.\\
&\quad \left. + \frac{1}{|L^*(f_2)(\eta)|}|E(f_1)(\eta) - E(f_2)(\eta)|\right\} d\eta\\
&\le \int_0^z \left\{\frac{\exp\left(2\frac{\mu_M}{L_m}\eta\right)}{L_m^2}\widetilde{L}\|f_1 - f_2\| + \frac{1}{L_m}D_3(\eta)\|f_1 - f_2\|\right\} d\eta\\
&\le \int_0^\lambda \left\{\frac{\exp\left(2\frac{\mu_M}{L_m}\eta\right)}{L_m^2}\widetilde{L}\|f_1 - f_2\| + \frac{1}{L_m}D_3(\eta)\|f_1 - f_2\|\right\} dz\\
&\le \lambda\left(\frac{\exp\left(2\frac{\mu_M}{L_m}\lambda\right)}{L_m^2}\widetilde{L} + \frac{D_3(\lambda)}{L_m}\right)\|f_1 - f_2\| = \lambda\, D_4(\lambda)\|f_1 - f_2\|. \quad \square
\end{aligned}$$

We are able now to state the following theorem

Theorem 2.3. *Suppose that (2.1)–(2.3) hold and*

$$\frac{2L_M\widetilde{L}}{L_m^2} < 1. \tag{2.35}$$

If $0 < \lambda < \overline{\lambda}$ where $\overline{\lambda} > 0$ is defined as the unique solution to $\mathcal{E}(z) = 1$ with

$$\mathcal{E}(z) := 2D_4(z)L_M\exp\left(\frac{N_M}{L_m}z^2\right), \tag{2.36}$$

then there exists a unique solution $f \in C^0[0, \lambda]$ for the integral equation (2.18), i.e. (2.22).

Proof. As Eq. (2.18) is equivalent to (2.22), we will show that $\mathcal{H}$ given by (2.21) is a contracting self-map of $C^0[0, \lambda]$.

On one hand, notice that for each $\lambda > 0$, taking into account the definition of $\mathcal{H}$ and the hypothesis on L^*, N^* and μ^* we can easily check that $\mathcal{H}$ maps $C^0[0, \lambda]$ onto itself. On the other hand, let $f_1, f_2 \in C^0[0, \lambda]$, from (2.29) and (2.33), for each $0 < \xi < \lambda$ we get

$$\begin{aligned}
|\mathcal{H}(f_1)(\xi) - \mathcal{H}(f_2)(\xi)| &\le \frac{|\Phi(f_1)(\xi)|}{|\Phi(f_1)(\lambda)||\Phi(f_2)(\lambda)|}|\Phi(f_2)(\lambda) - \Phi(f_1)(\lambda)|\\
&\quad + \frac{1}{|\Phi(f_2)(\lambda)|}|\Phi(f_1)(\xi) - \Phi(f_2)(\xi)|\\
&\le \frac{1}{|\Phi(f_2)(\lambda)|}\Big(|\Phi(f_2)(\lambda) - \Phi(f_1)(\lambda)| + |\Phi(f_2)(\xi) - \Phi(f_1)(\xi)|\Big)\\
&\le \frac{L_M}{\lambda}\exp\left(\frac{N_M}{L_m}\lambda^2\right)2\lambda D_4(\lambda)\|f_1 - f_2\|.
\end{aligned}$$

Therefore we obtain

$$\|\mathcal{H}(f_1) - \mathcal{H}(f_2)\| \le \mathcal{E}(\lambda)\|f_1 - f_2\|,$$

with $\mathcal{E}$ defined by (2.36). Notice that $\mathcal{E}$ satisfies

$$\mathcal{E}(0) = \frac{2L_M\widetilde{L}}{L_m^2}, \qquad \mathcal{E}(+\infty) = +\infty, \qquad \mathcal{E}'(z) > 0, \quad \forall z > 0.$$

Under the assumption (2.35) we deduce that there exists a unique $\overline{\lambda} > 0$ such that $\mathcal{E}(\overline{\lambda}) = 1$. Moreover but most significantly we get

$$\mathcal{E}(z) < 1, \quad \forall\, 0 < z < \overline{\lambda} \qquad \text{and} \qquad \mathcal{E}(z) > 1, \quad \forall\, z > \overline{\lambda}.$$

In conclusion, if λ is such that $0 < \lambda < \overline{\lambda}$ then $\mathcal{E}(\lambda) < 1$, and so $\mathcal{H}$ becomes a contraction mapping. By the fixed point Banach theorem we can say that there exists a unique solution $f \in C^0[0, \lambda]$ to the integral equation (2.22), i.e. to the integral equation (2.18). □

Remark 2.4. The solution f of (2.18) depends implicitly on the positive number λ. This means that $f(\xi)=f_\lambda(\xi)=f(\xi,\lambda),\ \forall\, 0<\xi<\lambda$.

So far, we have proved, for a fixed $0<\lambda<\overline{\lambda}$, the existence of a unique solution to Eq. (2.18), which will be referred as $f_\lambda(\xi)$ in view of the dependence outlined in the prior remark.

It remains to analyse the existence of a solution $(f_{\widetilde{\lambda}},\widetilde{\lambda})$ to the system (2.18)–(2.19). So we will focus now on condition (2.19).

Let us define the function $\mathcal{V}(\lambda)=\mathcal{V}(f_\lambda,\lambda)$ as

$$\mathcal{V}(\lambda):=\frac{\text{Ste}}{2}\frac{E(f_\lambda)(\lambda)}{\Phi(f_\lambda)(\lambda)},\qquad 0<\lambda<\overline{\lambda}. \tag{2.37}$$

Then equation (2.19) is equivalent to

$$\mathcal{V}(\lambda)=\lambda,\qquad 0<\lambda<\overline{\lambda}. \tag{2.38}$$

The study of Eq. (2.38) requires the following results:

Lemma 2.5. *Assume that* (2.1)–(2.3) and (2.35) *hold. Then for all* $\lambda\in(0,\overline{\lambda})$ *we have that*

$$\mathcal{V}_1(\lambda)\le\mathcal{V}(\lambda)\le\mathcal{V}_2(\lambda) \tag{2.39}$$

where $\mathcal{V}_1$ *and* $\mathcal{V}_2$ *are functions defined by*

$$\begin{aligned}\mathcal{V}_1(\lambda)&=\text{Ste}\ \mu_M\exp\left(-2\lambda\tfrac{\mu_M}{L_m}-2\lambda^2\tfrac{N_M}{L_m}\right), && \lambda>0\\ \mathcal{V}_2(\lambda)&=\frac{\text{Ste}}{\sqrt{\pi}}\frac{\sqrt{N_M}}{\sqrt{L_m}}L_M\frac{\exp\left(2\overline{\lambda}\tfrac{\mu_M}{L_m}\right)}{\operatorname{erf}\left(\sqrt{\tfrac{N_M}{L_m}}\lambda\right)}, && \lambda>0,\end{aligned} \tag{2.40}$$

that satisfy the following properties

$$\begin{aligned}&\mathcal{V}_1(0)=\text{Ste},\ \mu_M>0 && \mathcal{V}_1'(\lambda)<0, && \forall\lambda>0,\\ &\mathcal{V}_2(0)=+\infty, && V_2'(\lambda)<0, && \forall\lambda>0.\end{aligned} \tag{2.41}$$

Proof. Inequality (2.39) arises immediately from (2.28) and (2.29). The properties for $\mathcal{V}_1$ and $\mathcal{V}_2$ can be easily checked by their own definitions. □

Lemma 2.6. *If*

$$\mathcal{V}_2(\overline{\lambda})<\overline{\lambda}, \tag{2.42}$$

then, there exists a unique solution $\lambda_1<\overline{\lambda}$ *to the equation*

$$\mathcal{V}_1(\lambda)=\lambda,\quad \lambda>0, \tag{2.43}$$

and there exists a unique solution $\lambda_1<\lambda_2<\overline{\lambda}$ *to the equation*

$$\mathcal{V}_2(\lambda)=\lambda,\quad \lambda>0. \tag{2.44}$$

Proof. The proof is straightforward taking into account the properties of $\mathcal{V}_1$ and $\mathcal{V}_2$ presented in Lemma 2.5. □

Remark 2.7. Taking into account the definition of $\mathcal{V}_2$ and Ste, assumption (2.42) is equivalent to the following inequality for the latent heat

$$\ell>\frac{c_0(T^*-T_m)}{\sqrt{\pi}}\frac{\sqrt{N_M}}{\sqrt{L_m}}L_M\frac{\exp\left(2\overline{\lambda}\tfrac{\mu_M}{L_m}\right)}{\operatorname{erf}\left(\sqrt{\tfrac{N_M}{L_m}}\overline{\lambda}\right)} \tag{2.45}$$

Theorem 2.8. *Assume that* (2.1)–(2.3), (2.35) and (2.45) *hold. Consider* λ_1 *and* λ_2 *given by* (2.43) and (2.44)*, respectively. Then, there exists at least one solution* $\widetilde{\lambda}\in(\lambda_1,\lambda_2)$ *to Eq.* (2.38).

Proof. Under the hypothesis of Lemma 2.5, if (2.45) holds, we have that for each $\lambda_1\le\lambda\le\lambda_2<\overline{\lambda}$ the inequality (2.39) holds and $\mathcal{E}(\lambda)<1$. As $\mathcal{V}$ is a continuous function we obtain that there exists at least one solution $\widetilde{\lambda}$ to the equation $\mathcal{V}(\lambda)=\lambda$ that belongs to the interval (λ_1,λ_2). □

We have found sufficient conditions that guarantee the existence of at least one solution $(f_{\widetilde{\lambda}},\widetilde{\lambda})$ to the problem (2.18)–(2.19). Let us return the original problem (1.1a)–(1.1e).

From the prior analysis, and taking into account that condition (2.35) can be rewritten as

$$\frac{2k_M\widetilde{k}(T^*-T_m)}{k_m^2}<1, \tag{2.46}$$

we are able to state the main result of this section.

Theorem 2.9. *Assume that* k, ρ, c, ℓ *and* v *are such that* (2.1), (2.2), (2.3), (2.45)*, Hipotesis-Epsilon-Temp hold. Then, there exists at least one solution to the Stefan problem* (1.1a)–(1.1e)*, where the free boundary is given by*

$$s(t)=2\widetilde{\lambda}\sqrt{\alpha_0 t},\qquad t>0, \tag{2.47}$$

with $\widetilde{\lambda}$ *defined by Theorem 2.8, and the temperature is given by*

$$T(x,t)=(T_m-T^*)f_{\widetilde{\lambda}}(\xi)+T^*,\qquad 0\le\xi\le\widetilde{\lambda} \tag{2.48}$$

being $\xi=\frac{x}{2\sqrt{\alpha_0 t}}$ *the similarity variable and* $f_{\widetilde{\lambda}}$ *the unique solution of the integral equation* (2.18) *which was established in Theorem 2.3.*

3. Neumann Condition

In this section we will study the Stefan problem that arises when we consider a Neumann condition at the fixed face.

Let us notice that in case we consider the Neumann condition (1.1b*), instead of the Dirichlet condition (1.1b), after the change of variables

$$\theta=\frac{T-T_m}{T_m}\qquad (T(\theta)=T_m\theta+T_m) \tag{3.1}$$

the problem governed by (1.1a), (1.1b*), (1.1c)–(1.1e) becomes

$$\overline{N}(\theta)\frac{\partial\theta}{\partial t}=\alpha_0\frac{\partial}{\partial x}\left(\overline{L}(\theta)\frac{\partial\theta}{\partial x}\right)-\overline{v}(\theta)\frac{\partial\theta}{\partial x},\qquad 0<x<s(t),\quad t>0, \tag{3.2a}$$

$$\overline{L}(\theta(0,t))\frac{\partial\theta}{\partial x}(0,t)=\frac{-q}{k_0T_m\sqrt{t}},\qquad t>0, \tag{3.2b}$$

$$\theta(s(t),t)=0,\qquad t>0, \tag{3.2c}$$

$$\overline{L}(\theta(s(t),t))\frac{\partial\theta}{\partial x}(s(t),t)=\frac{-\rho_0\ell}{k_0T_m}\dot{s}(t),\qquad t>0, \tag{3.2d}$$

$$s(0)=0, \tag{3.2e}$$

where $\overline{N},\overline{L}$ and $\overline{v}$ are given by (2.5) with $T(\theta)=T_m\theta+T_m$.

Then, if we introduce the similarity transformation (2.8), we obtain the following ordinary differential problem

$$\left(L^*(f)f'(\xi)\right)'+2f'(\xi)\left(N^*(f)\xi-\mu^*(f)\right)=0,\qquad 0<\xi<\lambda \tag{3.3a}$$

$$L^*(f(0))f'(0)=-q^*, \tag{3.3b}$$

$$f(\lambda)=0, \tag{3.3c}$$

$$f'(\lambda)=-M\lambda, \tag{3.3d}$$

where $\xi=\frac{x}{2\sqrt{\alpha_0 t}}$ is the similarity variable and L^*, N^* and μ^* are given by (2.10) and (2.12), respectively. Moreover, q^* and M are defined by $q^*=\frac{2q\sqrt{\alpha_0}}{k_0T_m}$ and $M=\frac{2\ell k_0}{T_mc_0\,k(T_m)}$.

We can deduce that (f,λ) is a solution to the ordinary differential problem (3.3a)–(3.3d) if and only if (f,λ) satisfies the following integral equation

$$f(\xi)=q^*\left(\Phi(f)(\lambda)-\Phi(f)(\xi)\right),\qquad 0\le\xi\le\lambda, \tag{3.4}$$

together with the condition

$$f'(\lambda)=-M\lambda \tag{3.5}$$

where $\Phi(f)$ is defined by (2.17).

In order to analyse the existence of solution to this problem, let us study first, for a fixed $\lambda > 0$ the integral equation (3.4) for f.

In the same manner as we did in the first section, we consider the space $C^0[0,\lambda]$ of continuous real-valued functions defined on $[0,\lambda]$ endowed with the supremum norm and we define the operator $\mathcal{H}^q$ on $C^0[0,\lambda]$ given by the r.h.s of Eq. (3.4):

$$\mathcal{H}^q(f)(\xi) = q^*\Big(\Phi(f)(\lambda) - \Phi(f)(\xi)\Big). \tag{3.6}$$

Then, as $(C^0[0,\lambda], \|\cdot\|)$ is a Banach space we use the fixed point Banach theorem to prove that for each $\lambda > 0$ there exists a unique f such that

$$\mathcal{H}^q(f)(\xi) = f(\xi), \qquad 0 \le \xi \le \lambda, \tag{3.7}$$

which is the solution to (3.4).

If we assume that k, ρ, c and v satisfy (2.1), (2.2) and (2.3), respectively then L^*, N^* and μ^* verify (2.23), (2.24) and (2.25) where in this case

$$\widetilde{L} = \frac{\widetilde{k}|T_m|}{k_0}, \qquad \widetilde{N} = \frac{\widetilde{\gamma}|T_m|}{\rho_0 c_0}, \qquad \widetilde{\mu} = \frac{\widetilde{v}|T_m|}{\sqrt{\rho_0 c_0 k_0}}. \tag{3.8}$$

Therefore we are able to use the bounds in Lemma 2.1 and the Lipschitz continuities obtained in Lemma 2.2. Hence, we can state the following theorem

Theorem 3.1. *Assume that (2.1)–(2.3) hold. If $0 < \lambda < \overline{\lambda_q}$ where $\overline{\lambda_q} > 0$ is defined as the unique solution to $\mathcal{E}_q(z) = 1$ with*

$$\mathcal{E}_q(z) := 2q^* z D_4(z), \tag{3.9}$$

and D_4 is given by (2.34), then there exists a unique solution $f \in C^0[0,\lambda]$ for the integral equation (3.4), i.e. (3.7).

Proof. As solving equation (3.4) is equivalent to find a fixed point to the operator $\mathcal{H}^q$ given by (3.6), we will show that it is a contracting self-map of $C^0[0,\lambda]$.

On one hand, notice that for each $\lambda > 0$, taking into account the definition of $\mathcal{H}^q$ and the hypothesis on L^*, N^* and μ^* we can easily check that $\mathcal{H}^q$ maps $C^0[0,\lambda]$ onto itself. On the other hand, let $f_1, f_2 \in C^0[0,\lambda]$, from (2.33), for each $0 \le \xi \le \lambda$ we get

$$|\mathcal{H}^q(f_1)(\xi) - \mathcal{H}^q(f_2)(\xi)| \le \left| q^*\Big(\Phi(f_1)(\lambda) - \Phi(f_1)(\xi)\Big) - q^*\Big(\Phi(f_2)(\lambda) - \Phi(f_2)(\xi)\Big)\right|$$
$$\le q^*\Big(|\Phi(f_1)(\lambda) - \Phi(f_2)(\lambda)| + |\Phi(f_1)(\xi) - \Phi(f_2)(\xi)|\Big) \le 2q^*\lambda D_4(\lambda)\|f_1 - f_2\|.$$

Therefore it follows that

$$\|\mathcal{H}^q(f_1) - \mathcal{H}^q(f_2)\| \le \mathcal{E}_q(\lambda)\|f_1 - f_2\|,$$

where $\mathcal{E}_q$ defined by (3.9), is an increasing function that goes from 0 to $+\infty$ when z goes from 0 to $+\infty$. Thus there exists a unique $\overline{\lambda_q} > 0$ such that $\mathcal{E}_q(\overline{\lambda_q}) = 1$. Moreover but most significantly we get

$$\mathcal{E}_q(z) < 1, \quad \forall\, 0 < z < \overline{\lambda_q} \qquad \text{and} \qquad \mathcal{E}_q(z) > 1, \quad \forall\, z > \overline{\lambda_q}.$$

Then, if λ is such that $0 < \lambda < \overline{\lambda_q}$ we get that $\mathcal{E}_q(\lambda) < 1$ and so the operator $\mathcal{H}^q$ becomes a contraction mapping. By the fixed point Banach theorem it must exists a unique solution $f \in C^0[0,\lambda]$ to the integral equation (3.7), i.e. to the integral equation (3.4). □

For each $0 < \lambda < \overline{\lambda_q}$ fixed, we have a unique solution to Eq. (3.4), which will be referred as $f(\xi) = f_\lambda(\xi)$ to make visible its dependence on λ. Notice that we have

$$f'_\lambda(\xi) = -q^*\frac{E(f_\lambda)(\xi)}{L^*(f_\lambda)(\xi)}. \tag{3.10}$$

Then the condition (3.5), which remains to be analysed, becomes equivalent to

$$\mathcal{V}^q(\lambda) = \lambda, \tag{3.11}$$

where

$$\mathcal{V}^q(\lambda) = \mathcal{V}^q(f_\lambda, \lambda) := \frac{q^*}{M L^*(f_\lambda)(\lambda)} E_{f_\lambda}(\lambda). \tag{3.12}$$

Lemma 3.2. *Assume that (2.1)–(2.3) hold. Then for all $\lambda \in (0, \overline{\lambda}_q)$ we have that*

$$\mathcal{V}_1^q(\lambda) \le \mathcal{V}^q(\lambda) \le \mathcal{V}_2^q(\lambda) \tag{3.13}$$

where $\mathcal{V}_1^q$ and $\mathcal{V}_2^q$ are functions defined by

$$\begin{aligned} \mathcal{V}_1^q(\lambda) &= \frac{q^*}{M L_M}\exp\left(-\lambda^2 \tfrac{N_M}{L_M}\right), && \lambda > 0 \\ \mathcal{V}_2^q(\lambda) &= \frac{q^*}{M L_m}\exp\left(2\overline{\lambda}_q \tfrac{\mu_M}{L_m} - \lambda^2 \tfrac{N_M}{L_m}\right), && \lambda > 0, \end{aligned} \tag{3.14}$$

that satisfy the following properties:

$$\begin{aligned} &\mathcal{V}_1^q(0) = \frac{q^*}{M L_M} > 0, \quad \mathcal{V}_1^q(+\infty) = 0, \quad \mathcal{V}_1^{q\prime}(\lambda) < 0, \quad \forall \lambda > 0, \\ &\mathcal{V}_2^q(0) = \frac{q^*}{M L_m} > 0, \quad \mathcal{V}_2^q(+\infty) = 0, \quad \mathcal{V}_2^{q\prime}(\lambda) < 0, \quad \forall \lambda > 0. \end{aligned} \tag{3.15}$$

Proof. Inequality (3.13) follows directly from the bounds obtained in (2.28). The properties for $\mathcal{V}_1^q$ and $\mathcal{V}_2^q$ arise straightforwardly from their definitions. □

Lemma 3.3. *If*

$$\mathcal{V}_2^q(\overline{\lambda}_q) < \overline{\lambda}_q, \tag{3.16}$$

then, there exists a unique solution $0 < \lambda_{1q} < \overline{\lambda}_q$ to the equation

$$\mathcal{V}_1^q(\lambda) = \lambda, \quad \lambda > 0, \tag{3.17}$$

and there exists a unique solution $\lambda_{1q} < \lambda_{2q} < \overline{\lambda}_q$ to the equation

$$\mathcal{V}_2^q(\lambda) = \lambda, \quad \lambda > 0. \tag{3.18}$$

Proof. It is immediate taking into account the properties of $\mathcal{V}_1^q$ and $\mathcal{V}_2^q$ shown in Lemma 3.2. □

Remark 3.4. Taking into account the definition of $\mathcal{V}_2^q$ and M, assumption (3.16) is equivalent to the following inequality for the latent heat

$$\ell > \frac{c_0\, k(T_m) q\sqrt{\alpha_0}}{L_m k_0^2}\,\frac{\exp\left(2\overline{\lambda}_q\frac{\mu_M}{L_m} - \overline{\lambda}_q^2\frac{N_M}{L_m}\right)}{\overline{\lambda}_q} \tag{3.19}$$

Theorem 3.5. *Assume that (2.1)–(2.3) and (3.19) hold. Then there exists at least one solution $\widetilde{\lambda}_q \in (\lambda_{1q}, \lambda_{2q})$ to Eq. (3.11).*

Proof. It is similar to the proof given in Theorem 2.8. □

We have found sufficient conditions that guarantee the existence of at least one solution $(f_{\widetilde{\lambda}}, \widetilde{\lambda})$ to the problem (3.4)–(3.5). Let us return the original problem (1.1a), (1.1b⋆), (1.1d)–(1.1e). After the prior analysis we are able to state the following result.

Theorem 3.6. *Assume that k, ρ, c, ℓ and v are such that (2.1)–(2.3) and (3.19) hold. Then, there exists at least one solution to the Stefan problem (1.1a), (1.1b⋆), (1.1d)–(1.1e), where the free boundary is given by*

$$s(t) = 2\widetilde{\lambda}_q\sqrt{\alpha_0 t}, \qquad t > 0, \tag{3.20}$$

with $\widetilde{\lambda}_q$ defined by Theorem 3.5, and the temperature is given by

$$T(x,t) = T_m f_{\widetilde{\lambda}_q}(\xi) + T_m, \qquad 0 \le \xi \le \widetilde{\lambda}_q \tag{3.21}$$

being $\xi = \frac{x}{2\sqrt{\alpha_0 t}}$ the similarity variable and $f_{\widetilde{\lambda}_q}$ the unique solution of the integral equation (3.4) which was established in Theorem 3.1.

4. Robin condition

The following section is devoted to the analysis of the Stefan problem that arises when we consider a Robin condition at the fixed face $x = 0$.

Let us consider the problem (1.1a), (1.1b†), (1.1c)–(1.1e). After introducing the change of variables (2.4) we get

$$\overline{N}(\theta)\frac{\partial\theta}{\partial t} = \alpha_0\frac{\partial}{\partial x}\left(\overline{L}(\theta)\frac{\partial\theta}{\partial x}\right) - \overline{v}(\theta)\frac{\partial\theta}{\partial x}, \qquad 0 < x < s(t), \quad t > 0, \tag{4.1a}$$

$$\overline{L}(\theta(0,t))\frac{\partial\theta}{\partial x}(0,t) = \frac{h}{k_0\sqrt{t}}\theta(0,t), \qquad t > 0, \tag{4.1b}$$

$$\theta(s(t),t) = 0, \qquad t > 0, \tag{4.1c}$$

$$\overline{L}(\theta(s(t),t))\frac{\partial\theta}{\partial x}(s(t),t) = \frac{\dot{s}(t)}{\alpha_0 \text{Ste}}, \qquad t > 0, \tag{4.1d}$$

$$s(0) = 0, \tag{4.1e}$$

where $\overline{N}, \overline{L}$ and $\overline{v}$ are given by (2.5) with $T(\theta) = (T_m - T^*)\theta + T^*$.

The similarity transformation (2.8) yields to an ordinary differential problem defined by

$$\Big(L^*(f)f'(\xi)\Big)' + 2f'(\xi)\Big(N^*(f)\xi - \mu^*(f)\Big) = 0, \qquad 0 < \xi < \lambda \tag{4.2a}$$

$$L^*(f(0))f'(0) = 2\text{Bi}\, f(0), \tag{4.2b}$$

$$f(\lambda) = 1, \tag{4.2c}$$

$$L^*(f(\lambda))f'(\lambda) = \frac{2\lambda}{\text{Ste}}, \tag{4.2d}$$

where $\xi = \frac{x}{2\sqrt{\alpha_0 t}}$ is the similarity variable, L^*, N^* and μ^* are given by (2.10) and (2.12), respectively and $\text{Bi} = \frac{h\sqrt{\alpha_0}}{k_0}$ is the Biot number.

In addition, this ordinary differential problem is equivalent to find (f, λ) such that the integral equation:

$$f(\xi) = \frac{1 + 2\text{Bi}\,\Phi(f)(\xi)}{1 + 2\text{Bi}\,\Phi(f)(\lambda)}, \qquad 0 \le \xi \le \lambda, \tag{4.3}$$

together with the condition

$$f'(\lambda) = \frac{2}{L^*(f)(\lambda)\text{Ste}}\lambda \tag{4.4}$$

hold, where $\Phi(f)$ is given by (2.17).

Let us consider a fixed $\lambda > 0$, then we can define the operator $\mathcal{H}^h$ on $C^0[0,\lambda]$ as

$$\mathcal{H}^h(f)(\xi) = \frac{1 + 2\text{Bi}\,\Phi(f)(\xi)}{1 + 2\text{Bi}\,\Phi(f)(\lambda)}. \tag{4.5}$$

Therefore, the integral equation (4.3) can be rewritten as

$$\mathcal{H}^h(f)(\xi) = f(\xi), \qquad 0 \le \xi \le \lambda. \tag{4.6}$$

Let us assume that k, ρ, c and v satisfy (2.1), (2.2) and (2.3), respectively then L^*, N^* and μ^* verify (2.23), (2.24) and (2.25). As a consequence, we can state the following results.

Theorem 4.1. *Suppose that (2.1)–(2.3) and (2.35) hold. If $0 < \lambda < \overline{\lambda}_h$ where $\overline{\lambda}_h > 0$ is defined as the unique solution to $\mathcal{E}_h(z) = 1$ with*

$$\mathcal{E}_h(z) := 2L_M D_4(z)\exp\left(z^2\frac{N_M}{L_m}\right), \tag{4.7}$$

and D_4 is given by (2.34), then there exists a unique solution $f \in C^0[0,\lambda]$ for the integral equation (4.3), i.e. (4.6).

Remark 4.2. Notice that $\mathcal{E}_h = \mathcal{E}$ where $\mathcal{E}$ was defined in Theorem 2.3, when we studied the problem with a Dirichlet condition at the fixed face. Hence, we have that $\overline{\lambda}_h = \overline{\lambda}$.

We have obtained, for each $0 < \lambda < \overline{\lambda}_h$ fixed, a unique solution to Eq. (3.4) , $f(\xi) = f_\lambda(\xi)$. If we compute its derivative, we get

$$f'_\lambda(\xi) = \frac{2\text{Bi}}{\big(1 + 2\text{Bi}\,\Phi(f_\lambda)(\lambda)\big)}\frac{E(f_\lambda)(\xi)}{L^*(f_\lambda)(\xi)}. \tag{4.8}$$

Then the condition (4.19), which remains to be analysed, becomes equivalent to

$$\mathcal{V}^h(\lambda) = \lambda, \tag{4.9}$$

where

$$\mathcal{V}^h(\lambda) = \mathcal{V}^h(f_\lambda,\lambda) := \frac{\text{Ste Bi}\; E(f_\lambda)(\lambda)}{1 + 2\text{Bi}\,\Phi(f_\lambda)(\lambda)}. \tag{4.10}$$

We have the following results

Lemma 4.3. *Assume that (2.1)–(2.3),Hipotesis-ExtraContraccion hold. Then for all $\lambda \in (0,\overline{\lambda}_h)$ we have that*

$$0 \le \mathcal{V}^h(\lambda) \le \mathcal{V}_2(\lambda), \tag{4.11}$$

where $\mathcal{V}_2$ is given by (2.39).

Proof. The proof follows straightforwardly by taking into account the bounds given in Lemma 2.1. □

Let us notice that, due to the properties of $\mathcal{V}_2$ studied in Lemma 2.6, we know that if (2.42) holds there exists a unique solution $0 < \lambda_2 < \overline{\lambda}_h$ to the equation

$$\mathcal{V}_2(\lambda) = \lambda, \quad \lambda > 0. \tag{4.12}$$

Theorem 4.4. *Assume that (2.1)–(2.3), (2.35) and (2.42) hold. Then, there exists at least one solution $\widetilde{\lambda}_h \in (0,\lambda_2)$ to Eq. (4.9).*

Proof. It is similar to the proof given in Theorem 2.8. □

Let us return the original problem (1.1a), (1.1b†), (1.1d)–(1.1e). Notice that condition (2.35) can be rewritten as

$$\frac{2k_M\widetilde{k}(T^* - T_m)}{k_m^2} < 1, \tag{4.13}$$

and condition (2.42) is equivalent to (2.45). Then, we state the following main theorem.

Theorem 4.5. *Assume that k, ρ, c, ℓ and v are such that (2.1)–(2.3), (2.45),Hipotesis-Extra-krhoc-Convectivo hold. Then, there exists at least one solution to the Stefan problem (1.1a), (1.1b†), (1.1d)–(1.1e), where the free boundary is given by*

$$s(t) = 2\widetilde{\lambda}_h\sqrt{\alpha_0 t}, \qquad t > 0, \tag{4.14}$$

with $\widetilde{\lambda}_h$ defined by Theorem 4.4, and the temperature is given by

$$T(x,t) = (T_m - T^*)f_{\widetilde{\lambda}_h}(\xi) + T^*, \qquad 0 \le \xi \le \widetilde{\lambda}_h \tag{4.15}$$

being $\xi = \frac{x}{2\sqrt{\alpha_0 t}}$ the similarity variable and $f_{\widetilde{\lambda}_h}$ the unique solution of the integral equation (4.3) which was established in Theorem 4.1.

Remark 4.6. When the coefficient $h \to +\infty$, i.e. $Bi \to +\infty$ the integral equation (4.3) becomes (2.18) and Eq. (4.9) becomes (2.19). Then, the solution given by Theorem 4.1 for the problem with a Robin condition at the fixed face converges to the solution to the problem with a Dirichlet condition at $x = 0$ given by Theorem 2.3 when $h \to \infty$.

Remark 4.7. If we consider that the speed of the convective term in the heat equation (1.1a) is $v(T) \equiv 0$, we can recover the solution obtained in [24] for a null control function which is obtained as a solution to the following integral equation

$$f(\xi) = \frac{L^*(f(0)) + 2\text{Bi}\,\Phi_0(f)(\xi)}{L^*(f(0)) + 2\text{Bi}\,\Phi_0(f)(\lambda)} = \frac{1 + 2\text{Bi}\,\Phi(f)(\xi)}{1 + 2\text{Bi}\,\Phi(f)(\lambda)}, \qquad 0 \le \xi \le \lambda, \tag{4.16}$$

where

$$\Phi_0(f)(\xi) = \int_0^\xi \frac{L^*(f(0))}{L^*(f)(z)I(f)(z)}\,\mathrm{d}z = L^*(f(0))\int_0^\xi \frac{E(f)(z)}{L^*(f(z))}\,\mathrm{d}z = L^*(f(0))\Phi(f)(\xi), \tag{4.17}$$

and

$$E(f)(z) = \frac{1}{I(f)(z)} \tag{4.18}$$

together with the condition for λ given by

$$f'(\lambda) = \frac{2}{L^*(f)(\lambda)\mathrm{Ste}}\lambda. \tag{4.19}$$

5. Radiative–convective condition

We will proceed to the analysis of the Stefan problem that arises when we assume a radiative–convective condition at the fixed face $x = 0$.

If we consider the problem (1.1a), (1.1b††), (1.1c)–(1.1e). After the change of variables (2.4) and introducing the similarity transformation (2.8) we obtain the following ordinary differential problem defined by

$$\left(L^*(f)f'(\xi)\right)' + 2f'(\xi)\left(N^*(f)\xi - \mu^*(f)\right) = 0, \qquad 0 < \xi < \lambda \tag{5.1a}$$

$$L^*(f(0))f'(0) = 2\mathrm{Bi}\, f(0) + rF(f)(0), \tag{5.1b}$$

$$f(\lambda) = 1, \tag{5.1c}$$

$$L^*(f(\lambda))f'(\lambda) = \frac{2\lambda}{\mathrm{Ste}}, \tag{5.1d}$$

where $\xi = \frac{x}{2\sqrt{\alpha_0 t}}$ is the similarity variable, L^*, N^* and μ^* are given by (2.10) and (2.12), respectively and where $r = \dfrac{2\sigma\epsilon\sqrt{\alpha_0}}{k_0(T^* - T_m)} > 0$ and $F(f)(0) = T^{*4} - \left((T_m - T^*)f(0) + T^*\right)^4$.

It is easy to see that this ordinary differential problem is equivalent to find (f, λ) such that the integral equation holds:

$$\mathcal{H}^r(f)(\xi) = f(\xi), \qquad 0 \le \xi \le \lambda, \tag{5.2}$$

together with the condition

$$f'(\lambda) = \frac{2}{L^*(f(\lambda))\mathrm{Ste}}\lambda, \tag{5.3}$$

where the operator $\mathcal{H}^r$ on $C^0[0,\lambda]$ is defined by

$$\mathcal{H}^r(f)(\xi) := 1 - G(f)(0)\left(\Phi(f)(\lambda) - \Phi(f)(\xi)\right). \tag{5.4}$$

with $G(f)(0) = 2\mathrm{Bi} f(0) + rF(f)(0)$ and $\Phi(f)$ is given by (2.17).

In order to solve the fixed point equation (5.2) for a fixed $\lambda > 0$, let us consider the set X given by all non-negative functions bounded by 1, i.e

$$X = \{f \in C^0[0,\lambda] : f \ge 0,\ \|f\| \le 1\}. \tag{5.5}$$

Notice that X is a non-empty closed subset of the Banach space $\left(C^0[0,\lambda], \|\cdot\|\right)$.

Theorem 5.1. *Suppose that (2.1)–(2.3) hold as well as*

$$\frac{2\mathrm{Bi} + rT^{*4}}{L_m\sqrt{\frac{N_m}{L_M}}}\sqrt{\pi}\exp\left(\frac{\mu_M^2 L_M}{L_m^2 N_m}\right) \le 1, \tag{5.6}$$

and

$$\frac{2\mathrm{Bi} + rD_5}{\mu_M} < 1. \tag{5.7}$$

If $0 < \lambda < \overline{\lambda}_r$ where $\overline{\lambda}_r > 0$ is defined as the unique solution to $\mathcal{E}_r(z) = 1$ with

$$\mathcal{E}_r(z) := 2\left(2\mathrm{Bi} + rT^{*4}\right)zD_4(z) + \exp\left(2\tfrac{\mu_M}{L_m}z\right)\frac{(2\mathrm{Bi} + rD_5)}{\mu_M}, \tag{5.8}$$

where D_4 is given by (2.34) and $D_5 = 4(T^* - T_m)|T^*|^3$, then there exists a unique solution $f \in X$ for the integral equation (5.2).

Proof. Let us split the proof into two steps. In the first one, we will see that $\mathcal{H}^r$ is a self-map of X while in the second step we will see that it is a contracting mapping.

Let us show that $\mathcal{H}^r(X) \subset X$. Consider $f \in X$, then have that $0 < F(f)(0) < T^{*4}$ and so,

$$0 < G(f)(0) \le 2\mathrm{Bi} + rT^{*4}$$

From (2.28) and assumption (5.6), for every $\xi \in [0,\lambda]$ we can check that

$$\begin{aligned}
0 &\le G(f)(0)\left(\Phi(f)(\lambda) - \Phi(f)(\xi)\right) < \frac{\left(2\mathrm{Bi} + rT^{*4}\right)}{L_m}\int_\xi^\lambda E(f)(z)\mathrm{d}z \\
&\le \frac{\left(2\mathrm{Bi} + rT^{*4}\right)}{L_m}\int_\xi^\lambda \exp\left(2z\tfrac{\mu_M}{L_m} - z^2\tfrac{N_m}{L_M}\right)\mathrm{d}z \\
&\le \frac{\left(2\mathrm{Bi} + rT^{*4}\right)}{L_m}\,\frac{\sqrt{\pi}\exp\left(\frac{\mu_M^2 L_M}{L_m^2 N_m}\right)\left(\mathrm{erf}\left(\frac{\frac{\lambda N_m}{L_M} - \frac{\mu_M}{L_m}}{\sqrt{\frac{N_m}{L_M}}}\right) + \mathrm{erf}\left(\frac{\frac{\mu_M}{L_m} - \frac{\xi N_m}{L_M}}{\sqrt{\frac{N_m}{L_M}}}\right)\right)}{2\sqrt{\frac{N_m}{L_M}}} \\
&\le \frac{\left(2\mathrm{Bi} + rT^{*4}\right)}{L_m}\,\frac{\sqrt{\pi}\exp\left(\frac{\mu_M^2 L_M}{L_m^2 N_m}\right)}{\sqrt{\frac{N_m}{L_M}}} < 1
\end{aligned}$$

Therefore, $0 \le \mathcal{H}^r(f)(\xi) \le 1$. It is clear that $\mathcal{H}^r(f)$ belongs to $C^0[0,\lambda]$, hence we get that $\mathcal{H}^r(f) \in X$ for every $f \in X$.

Now let us proceed to show that $\mathcal{H}^r$ is a contracting mapping. Consider f_1 and f_2 in X. From the mean value theorem applied to the function $g(x) = \left((T_m - T^*)x + T^*\right)^4$, for $x_1 = f_1(0)$ and $x_2 = f_2(0)$ we have that

$$\begin{aligned}
|g(x_1) - g(x_2)| = |g'(x^*)||x_1 - x_2| &= 4\left((T_m - T^*)x^* + T^*\right)^3|T_m - T^*| \\
&\times |x_1 - x_2|,
\end{aligned}$$

where x^* is between x_1 and x_2, i.e. $0 \le x^* \le 1$. As a consequence it follows that

$$\begin{aligned}
|F(f_1)(0) - F(f_2)(0)| &\le |g(x_1) - g(x_2)| \le 4\left((T_m - T^*)x^* + T^*\right)^3 \\
&\times |T_m - T^*||f_1(0) - f_2(0)| \\
&\le 4|T^* - T_m|\ |T^*|^3\ \|f_1 - f_2\| = D_5\|f_1 - f_2\|.
\end{aligned}$$

and so

$$|G(f_1)(0) - G(f_2)(0)| \le \left(2\ \mathrm{Bi} + rD_5\right)\ \|f_1 - f_2\|.$$

Then, for each $0 \le \xi \le \lambda$ we have that

$$\begin{aligned}
|\mathcal{H}^r(f_1)(\xi) - \mathcal{H}^r(f_2)(\xi)| &\le |G(f_1)(0)|\ |\Phi(f_1)(\lambda) - \Phi(f_2)(\lambda)| \\
&+ |\Phi(f_2)(\lambda)|\ |G(f_1)(0) - G(f_2)(0)| \\
&+ |G(f_1)(0)|\ |\Phi(f_1)(\xi) - \Phi(f_2)(\xi)| + |\Phi(f_2)(\xi)|\ |G(f_1)(0) - G(f_2)(0)| \\
&\le \left(2(2\mathrm{Bi} + rT^{*4})\lambda D_4(\lambda) + \tfrac{(2\mathrm{Bi}+rD_5)}{\mu_M}\exp\left(2\tfrac{\mu_M}{L_m}\lambda\right)\right)\|f_1 - f_2\| \\
&= \mathcal{E}_r(\lambda)\|f_1 - f_2\|.
\end{aligned}$$

Under the assumption (5.7) we have that $\mathcal{E}_r$ satisfies the following properties

$$0 < \mathcal{E}_r(0) = \frac{2\mathrm{Bi} + rD_5}{\mu_M} < 1, \qquad \mathcal{E}_r(+\infty) = +\infty,$$

$$\mathcal{E}_r'(z) > 0, \qquad z \ge 0.$$

Therefore there exists a unique $\overline{\lambda}_r$ such that $\mathcal{E}_r(\overline{\lambda}_r) = 1$. In addition, we obtain

$$\mathcal{E}_r(z) < 1, \quad \forall\, 0 < z < \overline{\lambda}_r \qquad \text{and} \qquad \mathcal{E}_r(z) > 1, \quad \forall z > \overline{\lambda}_r.$$

From the fixed Banach theorem we can state that for a fixed $\lambda \in (0, \overline{\lambda}_r)$ there exists a unique solution $f \in X$ to the integral equation (5.2). □

For each given constant $0 < \lambda < \overline{\lambda}_r$, the unique solution to Eq. (5.2), $f(\xi) = f_\lambda(\xi)$ satisfies

$$f'_\lambda(\xi) = G(f)(0)\frac{E(f_\lambda)(\xi)}{L^*(f_\lambda)(\xi)}. \tag{5.9}$$

Then the condition (5.3) becomes equivalent to solve

$$\mathcal{V}^r(\lambda) = \lambda, \tag{5.10}$$

where

$$\mathcal{V}^r(\lambda) = \mathcal{V}^r(f_\lambda, \lambda) := \frac{\text{Ste}\, G(f)(0)}{2} E(f)(\lambda). \tag{5.11}$$

We can now state the following results. The proofs are omitted due to the fact that they are obtained analogously to the results presented in the previous sections.

Lemma 5.2. *Assume that (2.1)–(2.3) and (5.6)–(5.7) hold. Then for all $\lambda \in (0, \overline{\lambda}_r)$ we have that*

$$0 \le \mathcal{V}^r(\lambda) \le \mathcal{V}_2^r(\lambda) \tag{5.12}$$

where $\mathcal{V}_2^r$ is given by

$$\mathcal{V}_2^r(\lambda) = \frac{\text{Ste}\,(2\text{Bi} + rT^{*4})}{2} \exp\left(2\overline{\lambda}_r \tfrac{\mu_M}{L_m} - \lambda^2 \tfrac{N_m}{L_M}\right), \quad \lambda > 0. \tag{5.13}$$

Moreover, if

$$\mathcal{V}_2^r(\overline{\lambda}_r) < \overline{\lambda}_r \tag{5.14}$$

there exists a unique solution $0 < \lambda_{2r} < \overline{\lambda}_r$ to the equation

$$\mathcal{V}_2^r(\lambda) = \lambda, \quad \lambda > 0. \tag{5.15}$$

Theorem 5.3. *Assume that (2.1)–(2.3), (5.6)–(5.7)* and *(5.14) hold. Then, there exists at least one solution $\widetilde{\lambda}_r \in (0, \lambda_{2r})$ to Eq. (5.10).*

We return the original problem (1.1a), (1.1b††), (1.1d)–(1.1e). Notice that conditions (5.6) and (5.7) can be rewritten as

$$\frac{\left(2\text{Bi}+r(T^{*4}-T_m^4)\right)\sqrt{k_0\rho_0 c_0 k_M}}{k_m\sqrt{\gamma_m}}\sqrt{\pi}\exp\left(\frac{v_M^2 k_M}{k_m^2\gamma_m}\right) < 1, \qquad \frac{(2\text{Bi}+rD_5)\sqrt{\rho_0 c_0 k_0}}{v_M} < 1, \tag{5.16}$$

and condition (5.14) is equivalent to the following inequality for the latent heat

$$\ell > \frac{(T^* - T_m)c_0\,(2\text{Bi} + rT^{*4})}{2} \frac{\exp\left(2\overline{\lambda}_r \frac{\mu_M}{L_m} - \overline{\lambda}_r^2 \frac{N_m}{L_M}\right)}{\overline{\lambda}_r} \tag{5.17}$$

Theorem 5.4. *Assume that (2.1)–(2.3), (5.16)* and *(5.17) hold. Then, there exists at least one solution to the Stefan problem (1.1a), (1.1b††), (1.1d)–(1.1e), where the free boundary is given by*

$$s(t) = 2\widetilde{\lambda}_r\sqrt{\alpha_0 t}, \qquad t > 0, \tag{5.18}$$

with $\widetilde{\lambda}_r$ defined by Theorem 5.3, and the temperature is given by

$$T(x,t) = (T_m - T^*)f_{\widetilde{\lambda}_r}(\xi) + T^*, \qquad 0 \le \xi \le \widetilde{\lambda}_r \tag{5.19}$$

being $\xi = \frac{x}{2\sqrt{\alpha_0 t}}$ the similarity variable and $f_{\widetilde{\lambda}_r}$ the unique solution of the integral equation (5.2) which was established in Theorem 5.1.

6. Particular cases

6.1. Constant thermal coefficients

In this section we are going to recover the particular case analysed in [5] that arises when we consider constant thermal coefficients,

$$\rho(T) = \rho_0, \qquad c(T) = c_0, \qquad k(T) = k_0 \tag{6.1}$$

and a velocity given by $v(T) = \frac{\mu(T)}{\sqrt{t}}$ with

$$\mu(T) = \rho_0 c_0 \sqrt{\alpha_0}\,\text{Pe}. \tag{6.2}$$

where Pe denotes the Peclet number.

Replacing those values in (2.10) and (2.12) we get that $L^* = N^* = 1$ and $\mu^* = \text{Pe}$. Then Φ, E, U and I defined by (2.17), (2.15) and (2.16), respectively become

$$U(f)(z) = \exp(2z\text{Pe}), \qquad I(f)(z) = \exp(z^2),$$
$$E(f)(z) = \exp(2z\text{Pe} - z^2) \qquad \Phi(f)(\xi) = \tfrac{\sqrt{\pi}\exp(\text{Pe}^2)}{2}\left(\text{erf}(\text{Pe}) - \text{erf}(\text{Pe} - \xi)\right). \tag{6.3}$$

As a consequence we get that the explicit solution to the problem with a Dirichlet condition at the fixed face governed by (1.1a)–(1.1e), is obtain trough the solution to the ordinary differential problem (2.13a)–(2.13d), given by

$$f(\xi) = \frac{\text{erf}(\text{Pe}) - \text{erf}(\text{Pe} - \xi)}{\text{erf}(\text{Pe}) - \text{erf}(\text{Pe} - \lambda)}, \qquad 0 \le \xi \le \lambda, \tag{6.4}$$

where $\lambda = \lambda(\text{Pe})$ is the unique solution to the following equation

$$\text{Ste} = \sqrt{\pi}\;\; \lambda\left(\text{erf}(\text{Pe}) - \text{erf}(\text{Pe} - \lambda)\right)\exp\left((\text{Pe} - \lambda)^2\right). \tag{6.5}$$

In a similar way we get that the problem with a Neumann condition governed by (1.1a),(1.1b$\star$),(1.1c)–(1.1e) is equivalent to the ordinary differential problem (3.3a)–(3.3d) whose explicit solution (see [5]) turns out to be

$$f(\xi) = q^*\frac{\sqrt{\pi}\exp\left(\text{Pe}^2\right)}{2}\left(\text{erf}(\text{Pe} - \xi) - \text{erf}(\text{Pe} - \lambda)\right), \qquad 0 \le \xi \le \lambda, \tag{6.6}$$

where λ is a solution to the following equation

$$\frac{q}{\rho_0 \ell\sqrt{\alpha_0}} = \lambda\exp\left(\lambda^2 - 2\lambda\text{Pe}\right). \tag{6.7}$$

Notice that for $\text{Pe} \le \sqrt{2}$ it is obtained not only existence but also uniqueness of solution to Eq. (6.7).

6.2. Linear thermal coefficients

In this subsection we analyse the case where the thermal coefficients are given by

$$\rho(T) = \rho_0, \qquad c(T) = c_0\left(1 + \alpha\frac{T - T^*}{T_m - T^*}\right), \qquad k(T) = k_0\left(1 + \beta\frac{T - T^*}{T_m - T^*}\right) \tag{6.8}$$

with α and β given positive constants. In addition, we consider a velocity given by $v(T) = \frac{\mu(T)}{\sqrt{t}}$ with

$$\mu(T) = \rho_0 c(T)\sqrt{\alpha_0}\,\text{Pe}, \tag{6.9}$$

where Pe denotes the Peclet number.

This particular case appears in [10], where it is considered the problem with a convective condition (1.1a),(1.1b†),(1.1c)–(1.1e).

From (2.10) and (2.12) we get that

$$L^*(f) = 1 + \beta f, \qquad N^*(f) = 1 + \alpha f, \qquad \mu^*(f) = \text{Pe}\,(1 + \alpha f).$$

Notice that as $f \in C^0[0,\lambda]$ with $0 \le f \le 1$, we get that L^*, N^* and μ^* verify

$$1 \le L^*(f) \le 1 + \beta, \qquad 1 \le N^*(f) \le 1 + \alpha, \qquad \text{Pe} \le \mu^*(f) \le \text{Pe}(1 + \alpha)$$

In addition L^*, N^* and μ^* satisfy hypothesis (2.23)–(2.25) with $L_m = 1$, $L_M = 1 + \beta$, $\widetilde{L} = \beta$; $N_m = 1$, $N_M = 1 + \alpha$, $\widetilde{N} = \alpha$, $\mu_m = \text{Pe}$, $\mu_M = \text{Pe}(1 + \alpha)$ and $\widetilde{\mu} = \text{Pe}\,\alpha$.

Moreover the function Φ defined by (2.17) becomes

$$\Phi(f)(\xi) = \int_0^\xi \frac{\exp\left(2\displaystyle\int_0^z (\text{Pe} - z)\frac{1+\alpha f(z)}{1+\beta f(z)}\right)}{1 + \beta f(z)}\text{d}z. \tag{6.10}$$

Taking into account the hypothesis assumed in Theorem 2.9, the coefficients α, β and the numbers Ste, Pe must satisfy the following condition

$$2(1+\beta)\beta < 1, \tag{6.11}$$

that is, $0 < \beta < \frac{\sqrt{3}-1}{2}$ and $\mathcal{E}(\lambda_2) < 1$ where $\mathcal{E}$ is given by

$$\mathcal{E}(\lambda) = 2(1+\beta)\exp((1+\alpha)\lambda^2)D_4(\lambda) \tag{6.12}$$

with $D_4(\lambda)$ given by (2.34) and λ_2 defined by (2.44).

Those hypothesis are sufficient conditions in order to guarantee the existence of solution when a Dirichlet or Robin condition are imposed at the fixed face.

In case we consider $\mathrm{Pe} = 0$ and $\alpha = 0$ we recover the problem studied in [14] and [24] governed by

$$\left((1+\beta f)f'(\xi)\right)' + 2f'(\xi)\xi = 0, \qquad 0 < \xi < \lambda \tag{6.13}$$

$$(1+\beta f(0))f'(0) = 2\mathrm{Bi}\, f(0), \tag{6.14}$$

$$f(\lambda) = 1, \tag{6.15}$$

$$f'(\lambda) = \frac{2\lambda}{(1+\beta)\mathrm{Ste}}, \tag{6.16}$$

where the existence of solution is obtained trough the Generalized Modified Error Function.

7. Conclusions

We have studied four different one-phase Stefan problems for a semi-infinite domain, with the special feature of involving a moving phase change material as well as temperature dependent thermal coefficients. All the problems that we have analysed were governed by the diffusion–convection equation, where the uniform speed that appears in the convective term not only depends on the temperature but also on time. We have proved existence of at least one similarity solution imposing Dirichlet, Neumann, Robin or radiative–convective boundary condition at the fixed face. In each case, we have obtained an equivalent ordinary differential problem from where it was formulated an integral equation coupled with a condition for the parameter that characterizes the free boundary. The system obtained was solved though a double-fixed point analysis. Moreover, we have provided the solutions to some particular problems that arise when we set the thermal coefficients to be constant or linear functions of the temperature.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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