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# Integral formulation for a Stefan problem with spherical symmetry

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**Abstract.** A one-dimensional Stefan problem with spherical symmetry corresponding to the evaporation process of a droplet is considered. An equivalent integral formulation is obtained, and through a fixed point theorem, the existence and uniqueness of the solution are proved.

**Mathematics Subject Classification.** 80A22, 45D05, 35K05.

**Keywords.** Stefan problem, Spherical symmetry, Integral equation, Fixed point Banach's theorem.

## 1. Introduction

In this paper, the evaporation process of an spherical droplet is studied. We consider a liquid fuel droplet, initially at a variable temperature $T_0 = T_0(r)$ and of radius $R_0$ immersed into a gas at constant temperature $T_g > T_0(r)$. This evaporation process can be mathematically modelled, in a similar way as in [10], through a Stefan-like problem which consists in finding the droplet's temperature $T = T(r,t)$ and the droplet's radius, the free boundary, $R = R(t)$ such that

$$\overline{\rho} c T_t = \frac{k}{r^2}(r^2 T_r)_r, \qquad 0 < r < R(t)\ ,\ 0 < t < t_e, \tag{1.1a}$$

$$T_r(0,t) = 0, \qquad 0 < t < t_e, \tag{1.1b}$$

$$kT_r(R(t),t) + h(t)(T_s - T_g) = \overline{\rho} L \dot{R}(t), \qquad 0 < t < t_e, \tag{1.1c}$$

$$T(r,0) = T_0(r), \qquad 0 \le r \le R(0), \tag{1.1d}$$

$$T(R(t),t) = T_s, \qquad 0 < t < t_e, \tag{1.1e}$$

$$R(0) = R_0, \tag{1.1f}$$

where $c$ is the specific heat capacity, $k$ is the thermal conductivity and $\overline{\rho}$ is the density of the liquid fuel. We assume that $T_s$ is the droplet's boiling point with $T_0(r) \le T_s < T_g$ and the initial temperature $T_0 \in C^1[0, R_0]$ satisfies $T_0(R_0) = T_s$. In the Stefan-like condition (1.1c), $L$ denotes the specific heat of evaporation and the continuous function $h(t) > 0$ represents the convective heat transfer coefficient. At the surface of the droplet, evaporation and convection are assumed to be the dominant cooling and heating mechanisms, and the radius of the droplet $R(t)$ is expected to decrease with time. The time where $R(t)$ reaches zero, i.e., the time taken for the droplet to evaporate completely, is called "extinction time" and it is denoted by $t_e$. In [16], a numerical scheme to obtain an approximate solution to the problem (1.1a)–(1.1f) has been developed assuming not only that $h$ depends on $R$ but also that a particular condition on $\dot{R}(t)$ holds. In this manuscript, we assume $h$ to be independent of the unknown function $R(t)$ as it was taken in [14].

Several articles regarding Stefan problems for spheres can be found in the solidification of an ice ball [8], in the solidification of a molten material [11,15] or in the course of drug diffusion [9]. In [6] the melting of a metal sphere immersed into its own liquid is studied. In this problem, the solid shell freezes around the sphere which first grows and then melts again. Using the Green's function, an integral equation is obtained and solved numerically.

The goal of this paper is to prove existence and uniqueness of the solution to the free boundary problem (1.1a)–(1.1f) through its integral formulation by solving a nonlinear integral equation through a fixed point theorem.

This paper is divided into four sections. Section 1 gives a brief introduction with the background and statement of the problem. In Sect. 2 we rewrite the problem in a nondimensional form considering that the domain of the Stefan problem is up to a time $\sigma < t_e$. We also summarize the most relevant preliminary results about the fundamental solution of equation (2.2a). Section 3 is devoted to obtaining an integral formulation to the nondimensional problem (2.2a)–(2.2f) whose solution depends on the solution of an integral equation of Volterra type. Finally, in Sect. 4, our main result is stated and proved. We use the Banach fixed point theorem to prove the existence and uniqueness local in time of the solution of the Volterra integral equation.

## 2. Nondimensionalization of the problem and preliminaries

Let us study the Stefan problem (1.1a)–(1.1f) for $0 \leq t < \sigma$, being $\sigma \leq t_e$. In order to obtain its integral formulation, it will be convenient to proceed in non-dimensional variables. For this purpose, we introduce the following function $v$ and the variables $x$ and $\tau$:

$$v(x,\tau) = \frac{T(r,t) - T_s}{T_g - T_s}, \qquad x = \frac{r}{R_0}, \qquad \tau = \frac{\alpha}{R_0^2} t, \tag{2.1}$$

where we have chosen the time scale $\frac{R_0^2}{\alpha}$ from the heat conduction equation (1.1a), with $\alpha = \frac{k}{\rho c}$.

Under (2.1), the governing dimensionless equations for the problem (1.1a)–(1.1f) become:

$$v_\tau = \frac{1}{x^2}(x^2 v_x)_x, \qquad 0 < x < X(\tau)\ ,\ 0 < \tau < \overline{\sigma}, \tag{2.2a}$$

$$v_x(0,\tau) = 0, \qquad 0 < \tau < \overline{\sigma}, \tag{2.2b}$$

$$v_x(X(\tau),\tau) - \overline{H}(\tau) = \frac{1}{\text{Ste}} \dot{X}(\tau), \qquad 0 < \tau < \overline{\sigma}, \tag{2.2c}$$

$$v(x,0) = v_0(x), \qquad 0 \leq x \leq X(0), \tag{2.2d}$$

$$v(X(\tau),\tau) = 0, \qquad 0 < \tau < \overline{\sigma}, \tag{2.2e}$$

$$X(0) = 1, \tag{2.2f}$$

where $\text{Ste} = \frac{c(Tg - Ts)}{L}$ is the Stefan number and

$$\overline{\sigma} = \frac{\alpha}{R_0^2}\sigma, \qquad \overline{H}(\tau) = \frac{h(t)R_0}{k}, \tag{2.3}$$

$$X(\tau) = \frac{R(t)}{R_0}, \qquad v_0(x) = \frac{T_0(r) - T_s}{T_g - T_s}. \tag{2.4}$$

In order to give an integral representation of the free boundary problem (2.2a)–(2.2f), we will study the fundamental solution of the heat-conduction equation in spherical coordinates (2.2a).

Let us consider the fundamental solution $\overline{K}$ to the heat equation in Cartesian coordinates, the Green function $\overline{G}$ and the Neumann function $\overline{N}$ given by

$$\overline{K}(x,\tau,\rho,\eta)=\begin{cases}\frac{1}{2\sqrt{\pi(\tau-\eta)}}\exp\left(\frac{-(x-\rho)^2}{4(\tau-\eta)}\right) & \text{if } \eta<\tau,\\ 0 & \text{if } \eta\geq\tau,\end{cases}$$
$$\overline{G}(x,\tau,\rho,\eta)=\overline{K}(x,\tau,\rho,\eta)-\overline{K}(-x,\tau,\rho,\eta),$$
$$\overline{N}(x,\tau,\rho,\eta)=\overline{K}(x,\tau,\rho,\eta)+\overline{K}(-x,\tau,\rho,\eta).$$

Then, we define the following functions [5], for $x,\rho\neq 0$:

$$K(x,\tau,\rho,\eta)=\frac{\overline{K}(x,\tau,\rho,\eta)}{x\rho},\quad G(x,\tau,\rho,\eta)=\frac{\overline{G}(x,\tau,\rho,\eta)}{x\rho}, \tag{2.5}$$

$$N(x,\tau,\rho,\eta)=\frac{\overline{N}(x,\tau,\rho,\eta)}{x\rho}. \tag{2.6}$$

Taking into account the known properties available in the literature of the functions $\overline{K}$, $\overline{G}$ and $\overline{N}$ (see [4,7]), we can determine the following results which will be useful for obtaining the integral formulation for the Stefan problem (2.2a)–(2.2f).

**Lemma 2.1.** *We have the following properties*

i. $K_\tau-\frac{1}{x^2}\left(x^2K_x\right)_x=0,\qquad K_\eta+\frac{1}{\rho^2}(\rho^2K_\rho)_\rho=0.$

ii. $K_\tau(x,\tau,\rho,\eta)=\frac{1}{x\rho}\overline{K}_\tau(x,\tau,\rho,\eta),\qquad K_\eta(x,\tau,\rho,\eta)=\frac{1}{x\rho}\overline{K}_\eta(x,\tau,\rho,\eta).$

iii. $K_x(x,\tau,\rho,\eta)=-\frac{1}{x^2\rho}\overline{K}(x,\tau,\rho,\eta)+\frac{1}{x\rho}\overline{K}_x(x,\tau,\rho,\eta).$

iv. $K_{xx}(x,\tau,\rho,\eta)=\frac{2}{x^3\rho}\overline{K}(x,\tau,\rho,\eta)-\frac{2}{x^2\rho}\overline{K}_x(x,\tau,\rho,\eta)+\frac{1}{x\rho}\overline{K}_{xx}(x,\tau,\rho,\eta).$

v. *Functions $G$ and $N$ satisfy properties i)-iv).*

vi. *The derivative $G_{\rho x}$ is given by*

$$G_{\rho x}(x,\tau,\rho,\eta)=\frac{1}{x^2\rho^2}\overline{G}(x,\tau,\rho,\eta)-\frac{1}{x\rho^2}\overline{G}_x(x,\tau,\rho,\eta)$$
$$+\frac{1}{x^2\rho}\overline{N}_x(x,\tau,\rho,\eta)+\frac{1}{x\rho}\overline{N}_\eta(x,\tau,\rho,\eta).$$

vii. *For every $\varphi$ continuous in $[0,\tau]$, we have the following* ***Jump formulas:***

$$\lim_{x\to X(\tau)^\pm}\int_0^\tau\varphi(\eta)K_x(x,\tau,X(\eta),\eta)\mathrm{d}\eta=\mp\frac{\varphi(\tau)}{2X^2(\tau)}$$
$$+\int_0^\tau\varphi(\eta)K_x(X(\tau),\tau,X(\eta),\eta)\mathrm{d}\eta.$$

$$\lim_{x\to X(\tau)^\pm}\int_0^\tau\varphi(\eta)X^2(\eta)K_x(x,\tau,X(\eta),\eta)\mathrm{d}\eta=\mp\frac{\varphi(\tau)}{2}$$
$$+\int_0^\tau\varphi(\eta)X^2(\eta)K_x(X(\tau),\tau,X(\eta),\eta)\mathrm{d}\eta.$$

*Proof.* Properties $i)-vi)$ follow immediately from definitions (2.5)–(2.6) and properties of functions $\overline{K},\overline{G}$ and $\overline{N}$ available in [4]. The jump formulas given in $vii)$ arise from following relation [7]

$$\lim_{x\to X(\tau)^\pm}\int_0^\tau\varphi(\eta)\overline{K}_x(x,\tau,X(\eta),\eta)\mathrm{d}\eta=\mp\frac{\varphi(\tau)}{2}+\int_0^\tau\varphi(\eta)\overline{K}_x(X(\tau),\tau,X(\eta),\eta)\mathrm{d}\eta.$$

□

The next inequality will be also useful in the following sections.

**Lemma 2.2.** *For* $\beta, x > 0$ , $\tau > \eta$ , $n \in \mathbb{N}$ *we have*

$$\frac{\exp\left(\frac{-x^2}{\beta(\tau-\eta)}\right)}{(\tau-\eta)^{\frac{n}{2}}} \leq \left(\frac{n\beta}{2ex^2}\right)^{\frac{n}{2}} . \tag{2.7}$$

## 3. Integral formulation

In this section, we will give an integral formulation of the free boundary problem (2.2a)–(2.2f). The following lemmas will allow us to prove our main results.

**Lemma 3.1.** *If* $v = v(x,\tau), X = X(\tau)$ *satisfy* (2.2a)–(2.2f) *we have*

$$X(\tau) = 1 - \mathrm{Ste}\left(\int_0^\tau \overline{H}(\eta)\mathrm{d}\eta - \int_0^\tau v_x(X(\eta),\eta)\mathrm{d}\eta\right), \qquad 0 \leq \tau \leq \overline{\sigma}. \tag{3.1}$$

*Proof.* It is immediately obtained from (2.2c) and (2.2f). □

**Lemma 3.2.** *Let* $M > 0$ *such that* $|v_x(X(\tau),\tau)| \leq M, \quad \forall \tau \in [0,\overline{\sigma}]$. *Then,* $X = X(\tau)$, *given by* (3.1), *satisfies*

$$|X(\tau) - X(\eta)| \leq \overline{A}(M,\overline{\sigma})\,|\tau-\eta|\,, \quad \forall \tau,\eta \in [0,\overline{\sigma}]\,, \tag{3.2}$$

*where* $\overline{A}(M,\overline{\sigma}) := \mathrm{Ste}\left\{M + \|\overline{H}\|_{\overline{\sigma}}\right\}$ *and* $\|\overline{H}\|_{\overline{\sigma}} = \max\limits_{\tau\in[0,\overline{\sigma}]} |\overline{H}(\tau)|$.

*Moreover, if*

$$\overline{A}(M,\overline{\sigma})\overline{\sigma} \leq \frac{n-1}{n}, \tag{3.3}$$

*with* $n \in \mathbb{N}, \;\; n \geq 2$, *then we have*

$$\frac{1}{n} \leq X(\tau) \leq 1. \tag{3.4}$$

*Proof.* Formula (3.2) follows from the integral representation of $X(\tau)$ given by (3.1) and the definition of $\overline{A}(M,\overline{\sigma})$.

In addition, taking into account that $X(\tau)$ is a decreasing function in the variable $\tau$, we get $X(\tau) < 1$, $\forall \tau > 0$. Considering formula (3.1) and assuming (3.3) gives

$$0 < \mathrm{Ste}\left(\int_0^\tau \overline{H}(\eta)\mathrm{d}\eta - \int_0^\tau v_x(X(\eta),\eta)\mathrm{d}\eta\right) \leq \overline{A}(M,\overline{\sigma})\overline{\sigma},$$

from where it follows that

$$1 \geq X(\tau) \geq 1 - \overline{A}(M,\overline{\sigma})\overline{\sigma} \geq 1 - \frac{n-1}{n} = \frac{1}{n}.$$

□

Then, we are able to obtain the integral formulation of the free boundary problem (2.2a)–(2.2f) which is established in the next theorem.

**Theorem 3.3.** *The solution $v = v(x,\tau)$, $X = X(\tau)$ , $0 \le x \le X(\tau)$, $0 \le \tau \le \overline{\sigma}$ of the free boundary problem (2.2a)–(2.2f) has the following integral representation*

$$v(x,\tau) = \int_0^1 \rho^2 v_0(\rho) G(x,\tau,\rho,0)\mathrm{d}\rho + \int_0^\tau X^2(\eta) w(\eta) G(x,\tau,X(\eta),\eta)\mathrm{d}\eta. \tag{3.5}$$

$$X(\tau) = 1 - \mathrm{Ste}\left(\int_0^\tau \overline{H}(\eta)\mathrm{d}\eta - \int_0^\tau w(\eta)\mathrm{d}\eta\right), \qquad 0 \le \tau \le \overline{\sigma}. \tag{3.6}$$

*where $w(\eta) = v_x(X(\eta),\eta)$ if and only if $w$ such that $|w(\eta)| \le M$, $\forall \eta \in [0,\overline{\sigma}]$ with $0 \le \overline{A}(M,\overline{\sigma})\overline{\sigma} \le \frac{n-1}{n}$ satisfies the Volterra integral equation*

$$w(\tau) = 2\left\{\int_0^1 \rho^2 v_0(\rho) G_x(X(\tau),\tau,\rho,0)\mathrm{d}\rho \right. \\ \left. + \int_0^\tau X^2(\eta) w(\eta) G_x(X(\tau),\tau,X(\eta),\eta)\mathrm{d}\eta\right\}. \tag{3.7}$$

*Proof.* Let $v = v(x,\tau)$ be the solution of problem (2.2a)–(2.2f). If we integrate the Green identity

$$\left[\rho^2 v(\rho,\eta) G(x,\tau,\rho,\eta)\right]_\eta - \left[\rho^2\left(v_\rho(\rho,\eta)G(x,\tau,\rho,\eta) - v(\rho,\eta)G_\rho(x,\tau,\rho,\eta)\right)\right]_\rho = 0, \tag{3.8}$$

over the domain $D_{\tau,\epsilon} = \{(\rho,\eta) : 0 < \rho < X(\eta), \epsilon < \eta < \tau - \epsilon\}$ we get:

$$0 = \int_0^{X(\epsilon)} \rho^2 v(\rho,\epsilon) G(x,\tau,\rho,\epsilon)\,\mathrm{d}\rho \\ + \int_\epsilon^{\tau-\epsilon} X'(\eta) X^2(\eta) v(X(\eta),\eta)) G(x,\tau,X(\eta),\eta)\,\mathrm{d}\eta \\ + \int_\epsilon^{\tau-\epsilon} X^2(\eta)\left[v_\rho(X(\eta),\eta)G(x,\tau,X(\eta),\eta) - v(X(\eta),\eta)G_\rho(x,\tau,X(\eta),\eta)\right]\,\mathrm{d}\eta \\ - \int_0^{X(\tau-\epsilon)} \rho^2 v(\rho,\tau-\epsilon) G(x,\tau,\rho,\tau-\epsilon)\,\mathrm{d}\rho. \tag{3.9}$$

Taking $\epsilon \to 0$, by using the Poisson formula, we get (3.5).

If we differentiate (3.5) with respect to variable $x$ and we take $x \to X(\tau)^-$, by the jump formula given by Lemma 2.1, we obtain the Volterra integral equation (3.7) for $w$. Formula (3.6) for the free boundary $X(\tau)$ follows immediately from Lemma 3.1.

Conversely, by elementary computations we can verify that if $w$ satisfies (3.7) then $v$ and $X$ given by (3.5) and (3.6), respectively, satisfy (2.2a)–(2.2d).

In order to prove (2.2e), we define $\phi(\tau) = v(X(\tau),\tau), \tau > 0$ and we take $M$ such that $|w(\tau)| \le M$, $\forall \tau \in [0,\overline{\sigma}]$. If we integrate the Green identity (3.8) over the domain $D_{\tau,\epsilon}$ and we let $\epsilon \to 0$ in (3.9), we obtain that

$$v(x,\tau) = \int_0^1 \rho^2 v_0(\rho) G(x,\tau,\rho,0)\mathrm{d}\rho$$
$$+ \,\mathrm{Ste} \int_0^\tau X^2(\eta) v(X(\eta),\eta) G(x,\tau,X(\eta),\eta) \left[w(\eta) - \overline{H}(\eta)\right] d\eta$$
$$+ \int_0^\tau X^2(\eta) \left[w(\eta) G(x,\tau,X(\eta),\eta) - v(X(\eta),\eta) G_\rho(x,\tau,X(\eta),\eta)\right] \mathrm{d}\eta. \tag{3.10}$$

Then, if we compare this last expression with (3.5), we deduce that

$$\int_0^\tau \phi(\eta) X^2(\eta) \left[\mathrm{Ste}\left(w(\eta) - \overline{H}(\eta)\right) G(x,\tau,X(\eta),\eta) - G_\rho(x,\tau,X(\eta),\eta)\right] \mathrm{d}\eta = 0. \tag{3.11}$$

Then, if we let $x \to X(\tau)^-$ and apply Lemma 2.1, we obtain

$$\frac{\phi(\tau)}{2} = \int_0^\tau \frac{\phi(\eta)}{X(\tau)} \Big[\mathrm{Ste} X(\eta) \left(w(\eta) - \overline{H}(\eta)\right) \overline{G}(X(\tau),\tau,X(\eta),\eta)$$
$$-\overline{G}(X(\tau),\tau,X(\eta),\eta) - X(\eta) \overline{G}_\rho(X(\tau),\tau,R(\eta),\eta)\Big] \mathrm{d}\eta.$$

By using Lemmas 2.1 and 2.2 and taking into account [3], we have

$$|\overline{G}(X(\tau),\tau,X(\eta),\eta)| \le \frac{1}{\sqrt{\pi(\tau-\eta)}}, \tag{3.12}$$

$$|\overline{G}_\rho(X(\tau),\tau,X(\eta),\eta)| \le \frac{\overline{A}(M,\overline{\sigma})}{4\sqrt{\pi(\tau-\eta)}} + \frac{n^3}{2\sqrt{\pi}} \left(\frac{3}{4e}\right)^{3/2}, \tag{3.13}$$

and

$$m(\eta) := \Big| \frac{1}{X(\tau)} \Big[\mathrm{Ste} X(\eta) \left(w(\eta) - \overline{H}(\eta)\right) \overline{G}(X(\tau),\tau,X(\eta),\eta)$$
$$-\overline{G}(X(\tau),\tau,X(\eta),\eta) - X(\eta) \overline{G}_\rho(X(\tau),\tau,X(\eta),\eta)\Big]\Big|$$
$$\le n \left\{ \frac{5\overline{A}(M,\overline{\sigma}) + 4}{4\sqrt{\pi(\tau-\eta)}} + \frac{n^3}{2\sqrt{\pi}} \left(\frac{3}{4e}\right)^{3/2} \right\}. \tag{3.14}$$

Therefore, we have

$$|\phi(\tau)| \le \int_0^\tau |\phi(\eta)| m(\eta) \mathrm{d}\eta, \tag{3.15}$$

where $m \in L^1(0,\overline{\sigma}), m \ge 0$. As a consequence, if we take into account Gronwall's inequality, we get $\phi(\tau) = 0$, for all $0 < \tau < \overline{\sigma}$, i.e., $\phi$ vanishes identically in $[0,\overline{\sigma}]$. □

## 4. Existence and uniqueness of solution

We will use the Banach fixed point theorem to prove existence and uniqueness of the solution $w \in C^0[0,\overline{\sigma}]$ for the Volterra integral equation (3.7), where $\overline{\sigma}$ is a positive number to be determined (see references [1,2]).

We consider the following closed set of the Banach space $C^0[0,\overline{\sigma}]$ given by

$$\mathcal{C}_{\overline{\sigma},M} = \{w \in C^0[0,\overline{\sigma}] : \|w\|_{\overline{\sigma}} \leq M\},$$

with the norm $\|w\|_{\overline{\sigma}} = \max\limits_{\tau\in[0,\overline{\sigma}]} |w(\tau)|$ where $M$ is a constant to be determined.

In the remainder of this paper, unless otherwise stated, we will write $\|w\|$ to denote $\|w\|_{\overline{\sigma}}$.

We define on $\mathcal{C}_{\overline{\sigma},M}$ the operator $\Psi$ given by

$$\Psi(w)(\tau) = 2\left\{\int_0^1 \rho^2 v_0(\rho) G_x(X(\tau),\tau,\rho,0)\mathrm{d}\rho + \int_0^\tau X^2(\eta)w(\eta)G_x(X(\tau),\tau,X(\eta),\eta)\mathrm{d}\eta\right\}. \tag{4.1}$$

**Remark 4.1.** It should be noticed that in the definition of the operator $\Psi$, the function $X(\tau)$ depends on $w(\tau)$ by formula (3.6).

We must prove that there exist $\xi > 0$, $\overline{\sigma} > 0$ and $M > 0$ such that

$$i) \quad \Psi(w) \in \mathcal{C}_{\overline{\sigma},M}, \qquad \forall w \in \mathcal{C}_{\overline{\sigma},M}, \tag{4.2}$$

$$ii) \quad \|\Psi(w_1) - \Psi(w_2)\| \leq \xi\|w_1 - w_2\|, \qquad \xi < 1, \quad \forall w_1, w_2 \in \mathcal{C}_{\overline{\sigma},M}. \tag{4.3}$$

To this end, we will need the following preliminary lemmas

**Lemma 4.2.** *The operator $\Psi$ can be rewritten as*

$$\Psi(w)(\tau) = 2\left\{-\int_0^1 \left[\frac{\rho v_0(\rho)}{X^2(\tau)}\overline{G}(X(\tau),\tau,\rho,0) + \frac{v_0(\rho)+\rho v_0'(\rho)}{X(\tau)}\overline{N}(X(\tau),\tau,\rho,0)\right]\mathrm{d}\rho + \int_0^\tau X^2(\eta)w(\eta)G_x(X(\tau),\tau,X(\eta),\eta)\mathrm{d}\eta\right\}. \tag{4.4}$$

*Proof.* If we replace $G_x$ by the formula given in Lemma 2.1 and use the identity $\overline{G}_x = -\overline{N}_\rho$, then we can integrate by parts. Taking into account that $v_0(X_0) = 0$, we get:

$$\int_0^1 \rho^2 v_0(\rho)G_x(X(\tau),\tau,\rho,0)\mathrm{d}\rho = -\int_0^1 \frac{\rho v_0(\rho)}{X^2(\tau)}\overline{G}(X(\tau),\tau,\rho,0)\mathrm{d}\rho + \int_0^1 \frac{(v_0(\rho)+\rho v_0'(\rho))}{X(\tau)}\overline{N}(X(\tau),\tau,\rho,0)\mathrm{d}\rho.$$

□

**Lemma 4.3.** *Let $w_1, w_2 \in \mathcal{C}_{\overline{\sigma},M}$ and $X_i$ be defined by (3.6) corresponding to $w_i$, $i = 1,2$. Then, the following inequalities hold*

(i) $|X_1(\tau) - X_2(\tau)| \leq \text{Ste}\,\overline{\sigma}\,\|w_1 - w_2\|, \quad \tau > 0,$

(ii) $\left|\frac{1}{X_1(\tau)} - \frac{1}{X_2(\tau)}\right| \leq n^2\text{Ste}\,\overline{\sigma}\|w_1 - w_2\|, \quad \tau > 0,$

(iii) $\left|\frac{1}{X_1^2(\tau)} - \frac{1}{X_2^2(\tau)}\right| \leq 2n^4\text{Ste}\,\overline{\sigma}\|w_1 - w_2\|, \quad \tau > 0,$

(iv) $\left|\frac{X_1(\eta)}{X_1(\tau)} - \frac{X_2(\eta)}{X_2(\tau)}\right| \leq n(1+n)\text{Ste}\,\overline{\sigma}\|w_1 - w_2\|, \quad \tau > 0,$

(v) $\left|\frac{X_1(\eta)}{X_1^2(\tau)} - \frac{X_2(\eta)}{X_2^2(\tau)}\right| \leq 3n^4 \text{Ste}\, \overline{\sigma} \|w_1 - w_2\|, \quad \tau > 0.$

*Proof.* It follows immediately from the integral representation (3.6) of the free boundary and Lemma 3.2. □

**Lemma 4.4.** *If we assume* (3.3) *and* $\overline{\sigma} \leq 1$*, then we have the following inequalities*

$$\left| \int_0^1 \left[ \frac{\rho v_0(\rho)}{X^2(\tau)} \overline{G}(X(\tau), \tau, \rho, 0) + \frac{v_0(\rho) + \rho v_0'(\rho)}{X(\tau)} \overline{N}(X(\tau), \tau, \rho, 0) \right] \mathrm{d}\rho \right| \leq \overline{C}_1, \tag{4.5}$$

$$\left| \int_0^\tau X^2(\eta) w(\eta) G_x(X(\eta), \tau, X(\eta), \eta) \mathrm{d}\eta \right| \leq \overline{C}_2 \sqrt{\overline{\sigma}}, \tag{4.6}$$

*where*

$$\overline{C}_1 = n\left((n+1)\|v_0\| + \|v_0'\|\right),$$
$$\overline{C}_2 = M \left\{ \frac{n}{2\sqrt{\pi}} \left( \overline{A}(M,1) + n^3 \left(\tfrac{3}{2e}\right)^{3/2} \right) + \frac{2n^2}{\sqrt{\pi}} \right\}$$

*with* $\|v_0\| = \max\limits_{0 \leq \rho \leq 1} |v_0(\rho)|$.

*Proof.* Taking into account that

$$\left| \int_0^1 \left[ \frac{\rho v_0(\rho)}{X^2(\tau)} \overline{G}(X(\tau), \tau, \rho, 0) + \frac{v_0(\rho) + \rho v_0'(\rho)}{X(\tau)} \overline{N}(X(\tau), \tau, \rho, 0) \right] \mathrm{d}\rho \right|$$
$$\leq n^2 \|v_0\| \int_0^1 \left| \overline{G}(X(\tau), \tau, \rho, 0) \right| \mathrm{d}\rho$$
$$+ n\left(\|v_0\| + \|v_0'\|\right) \int_0^1 \left| \overline{N}(X(\tau), \tau, \rho, 0) \right| \mathrm{d}\rho,$$

and

$$\int_0^1 \left| \overline{G}(X(\tau), \tau, \rho, 0) \right| \mathrm{d}\rho \leq 1, \qquad \int_0^1 \left| \overline{N}(X(t), \tau, \rho, 0) \right| \mathrm{d}\rho \leq 1,$$

we obtain (4.5).

From Lemma 2.1, we have

$$X^2(\eta) G_x(X(\tau), \tau, X(\eta), \eta) = X(\eta) \left[ -\frac{1}{X^2(\tau)} \overline{G}(X(\tau), \tau, X(\eta), \eta) \right.$$
$$\left. + \frac{1}{X(\tau)} \overline{G}_x(X(\tau), \tau, X(\eta), \eta) \right].$$

Since

$$\left| \overline{G}(X(\tau), \tau, X(\eta), \eta) \right| \leq \frac{1}{\sqrt{\pi(\tau - \eta)}}, \tag{4.7}$$

by Lemma 3.2, we obtain

$$\left| \int_0^\tau \frac{X(\eta) w(\eta)}{X^2(\tau)} \overline{G}(X(\tau), \tau, X(\eta), \eta) \mathrm{d}\eta \right| \leq \frac{2n^2 M}{\sqrt{\pi}} \sqrt{\overline{\sigma}}.$$

By using [3], we can find that

$$\left| \int_0^\tau \frac{X(\eta) w(\eta)}{X(\tau)} \overline{G}_x(X(\tau), \tau, X(\eta), \eta) \mathrm{d}\eta \right| \leq \frac{nM}{2\sqrt{\pi}} \left( \overline{A}(M,1) + n^3 \left(\tfrac{3}{2e}\right)^{3/2} \right) \sqrt{\overline{\sigma}}. \tag{4.8}$$

Therefore, we get (4.6). □

**Lemma 4.5.** *Let*

$$M = 2\overline{C}_1 + 1 \tag{4.9}$$

*If we assume* (3.3) *and* $\overline{\sigma} \leq 1$ *such that*

$$2\overline{C}_2\,\sqrt{\overline{\sigma}} \leq 1, \tag{4.10}$$

*then we have that* $\Psi(w) \in \mathcal{C}_{\overline{\sigma},M}$, $\forall w \in \mathcal{C}_{\overline{\sigma},M}$, *i.e.,* (4.2) *is verified.*

*Proof.* Due to the definitions of $M$ and Lemma 4.4, we can observe that

$$|\Psi(w)(\tau)| \leq (M-1) + 2\overline{C}_2\,\sqrt{\overline{\sigma}}.$$

Therefore, if (4.10) holds then $|\Psi(w)(\tau)| \leq M$, $\forall \tau \in [0, \overline{\sigma}]$. The continuity of $\Psi(w)$ arises immediately from definition (4.1). □

Let $w_i \in \mathcal{C}_{\overline{\sigma},M}$ and $X_i$ defined by (3.6) corresponding to $w_i$, $i = 1, 2$, respectively. Then, we can state the following lemma.

**Lemma 4.6.** *If we assume (3.3) and* $\overline{\sigma} \leq 1$ *such that*

$$2M \text{ Ste } \overline{A}(M, \overline{\sigma})\, \overline{\sigma} \leq 1, \tag{4.11}$$

*then we have the following inequalities*

$$\int_0^1 |\rho v_0(\rho)| \left| \frac{\overline{G}\,(X_1(\tau), \tau, \rho, 0)}{X_1^2(\tau)} - \frac{\overline{G}\,(X_2(\tau), \tau, \rho, 0)}{X_2^2(\tau)} \right| \mathrm{d}\rho \leq \overline{D}_1 \sqrt{\overline{\sigma}} \|w_1 - w_2\|, \tag{4.12}$$

$$\int_0^1 |v_0(\rho) + \rho v_0'(\rho)| \left| \frac{\overline{N}(X_1(\tau), \tau, \rho, 0)}{X_1(\tau)} - \frac{\overline{N}(X_2(\tau), \tau, \rho, 0)}{X_2(\tau)} \right| \mathrm{d}\rho \leq \overline{D}_2 \sqrt{\overline{\sigma}} \|w_1 - w_2\|, \tag{4.13}$$

$$\begin{aligned} &\int_0^\tau \left| X_1^2(\eta) w_1(\eta) G_x(X_1(\tau), \tau, X_1(\eta), \eta) - X_2^2(\eta) w_2(\eta) G_x(X_2(\tau), \tau, X_2(\eta), \eta) \right| \mathrm{d}\eta \\ &\leq \overline{D}_3 \sqrt{\overline{\sigma}} \|w_1 - w_2\|, \end{aligned} \tag{4.14}$$

*where*

$$\begin{aligned} \overline{D}_1 &= 2 \text{ Ste } n^2\, \|v_0\| \left( \tfrac{1}{\sqrt{\pi}} + n^2 \right), \\ \overline{D}_2 &= (\|v_0\| + \|v_0'\|)\, n \text{ Ste} \left( \tfrac{2}{\sqrt{\pi}} + n \right), \\ \overline{D}_3 &= \left( M n^2 \overline{P}_1 + \overline{P}_2 + M n \overline{P}_3 + n \overline{P}_4 \left(1 + \text{Ste } M(1+n)\right) \right), \end{aligned}$$

*with*

$$\begin{aligned} \overline{P}_1 &= \tfrac{\text{Ste}}{\sqrt{\pi}} \left( \overline{A}(M,1) + \left(\tfrac{3}{2e}\right)^{3/2} n^3 \right), \qquad \overline{P}_2 = \tfrac{2n^2}{\sqrt{\pi}} \left(1 + 3Mn^2 \text{ Ste}\right), \\ \overline{P}_3 &= \tfrac{\text{Ste}}{\sqrt{\pi}} \left( \tfrac{1}{2} + \overline{A}^2(M,1) + n^2 \left( \left(\tfrac{5}{2e}\right)^{5/2} n^3 + \tfrac{1}{2e} \right) \right), \\ \overline{P}_4 &= \tfrac{\overline{A}(M,1)}{2\sqrt{\pi}} + \tfrac{12 n^{3/2}}{(2e)^{3/2}}. \end{aligned}$$

*Proof.* We can observe that

$$\begin{aligned} \left| \tfrac{\overline{G}(X_1(\tau),\tau,\rho,0)}{X_1^2(\tau)} - \tfrac{\overline{G}(X_2(\tau),\tau,\rho,0)}{X_2^2(\tau)} \right| &\leq \tfrac{1}{X_1^2(\tau)} \left| \overline{G}\,(X_1(\tau), \tau, \rho, 0) - \overline{G}\,(X_2(\tau), \tau, \rho, 0) \right| \\ &\quad + \left| \overline{G}\,(X_2(t), \tau, \rho, 0) \right| \left| \tfrac{1}{X_1^2(\tau)} - \tfrac{1}{X_2^2(\tau)} \right|. \end{aligned}$$

By using [3], Lemma 3.2 and Lemma 4.3 iii), we obtain (4.12). Repeating similar arguments, we get (4.13).

In the following reasoning, we will provide a detailed proof of (4.14). Taking into account Lemma 2.1 iii), v) and adding and subtracting suitable terms, we get

$$\left|X_1^2(\eta)w_1(\eta)G_x(X_1(\tau),\tau,X_1(\eta),\eta)-X_2^2(\eta)w_2(\eta)G_x(X_2(\tau),\tau,X_2(\eta),\eta)\right|$$
$$\leq M_1+M_2+M_3+M_4,$$

where

$$\begin{aligned}
M_1&=\left|\tfrac{X_1(\eta)w_1(\eta)}{X_1^2(\tau)}\left[\overline{G}(X_1(\tau),\tau,X_1(\eta),\eta)-\overline{G}(X_2(\tau),\tau,X_2(\eta),\eta)\right]\right|,\\
M_2&=|\overline{G}(X_2(\tau),\tau,X_2(\eta),\eta)|\left|\tfrac{X_1(\eta)w_1(\eta)}{X_1^2(\tau)}-\tfrac{X_2(\eta)w_2(\eta)}{X_2^2(\eta)}\right|,\\
M_3&=\left|\tfrac{X_1(\eta)w_1(\eta)}{X_1(\tau)}\left[\overline{G}_x(X_1(\tau),\tau,X_1(\eta),\eta)-\overline{G}_x(X_2(\tau),\tau,X_2(\eta),\eta)\right]\right|,\\
M_4&=|\overline{G}_x(X_2(\tau),\tau,X_2(\eta),\eta)|\left|\tfrac{X_1(\eta)w_1(\eta)}{X_1(\tau)}-\tfrac{X_2(\eta)w_2(\eta)}{X_2(\tau)}\right|.
\end{aligned}$$

We are going to bound each integral $\int\limits_0^\tau M_i \mathrm{d}\eta$.

First, applying the definition of $\overline{G}$ we immediately obtain

$$\begin{aligned}
&\left|\overline{G}(X_1(\tau),\tau,X_1(\eta),\eta)-\overline{G}(X_2(\tau),\tau,X_2(\eta),\eta)\right|\\
&\quad\leq\left|\overline{K}(X_1(\tau),\tau,X_1(\eta),\eta)-\overline{K}(X_2(\tau),\tau,X_2(\eta),\eta)\right|\\
&\qquad+\left|\overline{K}(-X_1(\tau),\tau,X_1(\eta),\eta)-\overline{K}(-X_2(\tau),\tau,X_1(\eta),\eta)\right|.
\end{aligned}\tag{4.15}$$

Using the mean value theorem, we get, by following [2,13], that

$$\begin{aligned}
S_1&:=\left|\overline{K}(X_1(\tau),\tau,X_1(\eta),\eta)-\overline{K}(X_2(\tau),\tau,X_2(\eta),\eta)\right|\\
&\leq\tfrac{1}{2\sqrt{\pi(\tau-\eta)}}\tfrac{|-c|}{2(\tau-\eta)}\exp\left(\tfrac{-c^2}{4(\tau-\eta)}\right)\left(|X_2(\tau)-X_1(\tau)|+|X_1(\eta)-X_2(\eta)|\right)
\end{aligned}$$

where $c\leq\max\limits_{i=1,2}|X_i(\tau)-X_i(\eta)|$. Then, by Lemmas 3.2 and 4.3 we find that

$$S_1\leq\tfrac{\overline{A}(M,\overline{\sigma})\,\mathrm{Ste}\,\overline{\sigma}}{2\sqrt{\pi}}\tfrac{\|w_1-w_2\|}{\sqrt{\tau-\eta}}.\tag{4.16}$$

In a similar way, we get

$$\begin{aligned}
S_2&:=\left|\overline{K}(-X_1(\tau),\tau,X_1(\eta),\eta)-\overline{K}\left(-X_2(\tau),\tau,X_2(\eta),\eta\right)\right|\\
&\leq\tfrac{1}{2\sqrt{\pi(\tau-\eta)}}\tfrac{|-c^*|}{2(\tau-\eta)}\exp\left(\tfrac{-c^{*2}}{4(\tau-\eta)}\right)|X_2(\tau)+X_2(\eta)-X_1(\tau)-X_1(\eta)|,
\end{aligned}$$

where $c^*$ is between $X_1(\tau)+X_1(\eta)$ and $X_2(\tau)+X_2(\eta)$. Then, due to Lemma 3.2, we can claim that $\frac{2}{n}\leq c^*\leq 2$. In consequence, by formula (2.7), it follows

$$\frac{\exp\left(\frac{-c^{*2}}{4(\tau-\eta)}\right)}{(\tau-\eta)^{3/2}}\leq\frac{\exp\left(\frac{-1}{n^2(\tau-\eta)}\right)}{(\tau-\eta)^{3/2}}\leq\left(\frac{3n^2}{2e}\right)^{3/2}=\left(\frac{3}{2e}\right)^{3/2}n^3.$$

Thus, we get

$$S_2\leq\left(\tfrac{3}{2e}\right)^{3/2}\tfrac{n^3\,\mathrm{Ste}}{\sqrt{\pi}}\,\overline{\sigma}\|w_1-w_2\|.\tag{4.17}$$

Taking into account (4.15), (4.16) and (4.17), we obtain, by just integrating, that

$$\int\limits_0^\tau\left|\overline{G}\left(X_1(\tau),\tau,X_1(\eta),\eta\right)-\overline{G}\left(X_2(\tau),\tau,X_2(\eta),\eta\right)\right|\mathrm{d}\eta\leq\overline{P}_1\sqrt{\overline{\sigma}}\|w_1-w_2\|,$$

with $\overline{P}_1 = \frac{\text{Ste}}{\sqrt{\pi}}\left(\overline{A}(M,1) + \left(\frac{3}{2e}\right)^{3/2} n^3\right)$. Therefore, by Lemma 3.2 we get

$$\int_0^\tau M_1 \mathrm{d}\eta \leq M n^2 \overline{P}_1 \sqrt{\sigma}\, \|w_1 - w_2\|.$$

From (4.7) and Lemmas 3.2, 4.3, it follows that

$$\begin{aligned}
&|\overline{G}(X_2(\tau),\tau,X_2(\eta),\eta)| \left|\tfrac{X_1(\eta)}{X_1^2(\tau)} w_1(\eta) - \tfrac{X_2(\eta)}{X_2^2(\tau)} w_2(\eta)\right| \\
&\quad \leq \tfrac{1}{\sqrt{\pi(\tau-\eta)}} \left(\tfrac{X_1(\eta)}{X_1^2(\tau)} \|w_1 - w_2\| + \|w_2\| \left|\tfrac{X_1(\eta)}{X_1^2(\tau)} - \tfrac{X_2(\eta)}{X_2^2(\tau)}\right|\right).
\end{aligned}$$

Therefore,

$$\int_0^\tau M_2 \mathrm{d}\eta \leq \overline{P}_2 \sqrt{\sigma}\, \|w_1 - w_2\|,$$

with $\overline{P}_2 = \frac{2n^2}{\sqrt{\pi}}\left(1 + 3Mn^2 \text{ Ste}\right)$.

In a similar way to what we have done for $M_1$, we obtain that

$$\begin{aligned}
&\left|\overline{G}_x(X_1(\tau),\tau,X_1(\eta),\eta) - \overline{G}_x(X_2(\tau),\tau,X_2(\eta),\eta)\right| \\
&\quad \leq \left|\overline{K}_x(X_1(\tau),\tau,X_1(\eta),\eta) - \overline{K}_x(X_2(\tau),\tau,X_2(\eta),\eta)\right| \\
&\qquad + \left|\overline{K}_x(-X_1(\tau),\tau,X_1(\eta),\eta) - \overline{K}_x(-X_2(\tau),\tau,X_2(\eta),\eta)\right|.
\end{aligned}$$

On the one hand, we can check that

$$\begin{aligned}
&\left|\overline{K}_x(X_1(\tau),\tau,X_1(\eta),\eta) - \overline{K}_x(X_2(\tau),\tau,X_2(\eta),\eta)\right| \\
&\quad = \frac{1}{2|\tau-\eta|}\left|\overline{K}(X_1(\tau),\tau,X_1(\eta),\eta)(X_1(\tau) - X_1(\eta))\right. \\
&\qquad \left. - \overline{K}(X_2(\tau),\tau,X_2(\eta),\eta)(X_2(\tau) - X_2(\eta))\right| \\
&\quad \leq \frac{\left|\overline{K}(X_1(\tau),\tau,X_1(\eta),\eta)\right|}{2|\tau-\eta|}\left\{|X_1(\tau) - X_1(\eta) - X_2(\tau) + X_2(\eta)|\right. \\
&\qquad \left. + |X_2(\tau) - X_2(\eta)|\left|1 - \frac{\overline{K}(X_2(\tau),\tau,X_2(\eta),\eta)}{\overline{K}(X_1(\tau),\tau,X_1(\eta),\eta)}\right|\right\},
\end{aligned} \tag{4.18}$$

and we can observe that

$$\tfrac{\overline{K}(X_2(\tau),\tau,X_2(\eta),\eta)}{\overline{K}(X_1(\tau),\tau,X_1(\eta),\eta)} = \exp(f(\tau,\eta)),$$

with

$$\begin{aligned}
f(\tau,\eta) &= \tfrac{-(X_2(\tau)-X_2(\eta))^2 + (X_1(\tau)-X_1(\eta))^2}{4(\tau-\eta)} \\
&= \tfrac{(X_1(\tau)-X_1(\eta)+X_2(\tau)-X_2(\eta))(X_1(\tau)-X_1(\eta)-X_2(\tau)+X_2(\eta))}{4(\tau-\eta)}.
\end{aligned}$$

Taking into account Lemmas 3.2 and 4.3, we can obtain that

$$|f(\tau,\eta)| \leq \text{Ste}\, \overline{A}(M,\overline{\sigma})\, \overline{\sigma}\, \|w_1 - w_2\| \leq 2M\, \overline{A}(M,\overline{\sigma}) \text{ Ste}\, \overline{\sigma}.$$

We know that if $|y| \leq 1$ then $|1 - \exp(y)| \leq 2|y|$. Therefore, if we assume (4.11) and calling $y = f(\tau,\eta)$ we get

$$|1 - \exp(f(\tau,\eta))| \leq 2|f(\tau,\eta)| \leq 2\overline{A}(M,\overline{\sigma}) \text{ Ste}\, \overline{\sigma}\|w_1 - w_2\|.$$

In consequence,

$$\begin{aligned}&\left|\overline{K}_x(X_1(\tau),\tau,X_1(\eta),\eta)-\overline{K}_x(X_2(\tau),\tau,X_2(\eta),\eta)\right|\\&\leq\frac{\mathrm{Ste}}{\sqrt{4\pi(\tau-\eta)}}\left(\frac{1}{2}+\overline{A}^2(M,\overline{\sigma})\overline{\sigma}\right)\|w_1-w_2\|.\end{aligned}\tag{4.19}$$

On the other hand, by the mean value theorem we can ascertain that

$$\begin{aligned}&\left|\overline{K}_x(-X_1(\tau),\tau,X_1(\eta),\eta)-\overline{K}_x(-X_2(\tau),\tau,X_2(\eta),\eta)\right|\\&\leq|\overline{K}_{xx}(g(\tau,\eta),\tau,0,\eta)||X_1(\tau)+X_1(\eta)-X_2(\tau)-X_2(\eta)|,\end{aligned}$$

with $g(\tau,\eta)$ between $X_2(\tau)+X_2(\eta)$ and $X_1(\tau)+X_1(\eta)$.

From Lemma 3.2, we have $\frac{2}{n}\leq g(\tau,\eta)\leq 2$. Then, by formula (2.7) we can derive the following inequality

$$\begin{aligned}\overline{K}_{xx}(g(\tau,\eta),\tau,0,\eta)&\leq\frac{\exp\left(-\frac{g^2(\tau,\eta)}{4(\tau-\eta)}\right)}{2\sqrt{\pi(\tau-\eta)}}\left(\frac{g^2(\tau,\eta)}{4(\tau-\eta)^2}+\frac{1}{2(\tau-\eta)}\right)\\&\leq\frac{\exp\left(-\frac{1}{n^2(\tau-\eta)}\right)}{2\sqrt{\pi(\tau-\eta)}}\left(\frac{1}{(\tau-\eta)^2}+\frac{1}{2(\tau-\eta)}\right)\\&\leq\frac{1}{2\sqrt{\pi}}\left(\frac{5n^2}{2e}\right)^{5/2}+\frac{1}{4\sqrt{\pi}}\frac{n^2}{e}\\&\leq\frac{n^2}{2\sqrt{\pi}}\left(\left(\frac{5}{2e}\right)^{5/2}n^3+\frac{1}{2e}\right).\end{aligned}\tag{4.20}$$

Due to (4.19) and (4.20), by integrating, we obtain

$$\int_0^\tau|\overline{G}_x(X_1(\tau),\tau,X_1(\eta),\eta)-\overline{G}_x(X_2(\tau),\tau,X_2(\eta),\eta)|\mathrm{d}\eta\leq\overline{P}_3\sqrt{\overline{\sigma}}\|w_1-w_2\|,$$

with $\overline{P}_3=\frac{\mathrm{Ste}}{\sqrt{\pi}}\left(\frac{1}{2}+\overline{A}^2(M,1)+n^2\left(\left(\frac{5}{2e}\right)^{5/2}n^3+\frac{1}{2e}\right)\right)$.

Then,

$$\int_0^\tau M_3\mathrm{d}\eta\leq Mn\overline{P}_3\sqrt{\overline{\sigma}}\|w_1-w_2\|.\tag{4.21}$$

Following similar arguments, we deduce that

$$\int_0^\tau|\overline{G}_x(X_2(\tau),\tau,X_2(\eta),\eta)|\mathrm{d}\eta\leq\overline{P}_4\sqrt{\overline{\sigma}},$$

with $\overline{P}_4=\frac{\overline{A}(M,1)}{2\sqrt{\pi}}+\frac{12n^{3/2}}{(2e)^{3/2}}$ and therefore

$$\int_0^\tau M_4\mathrm{d}\eta\leq\overline{P}_4n\left(1+\mathrm{Ste}\,M(1+n)\right)\sqrt{\overline{\sigma}}\|w_1-w_2\|.\tag{4.22}$$

From the above inequalities, we conclude that (4.14) holds. □

**Lemma 4.7.** *Let $M$ given by (4.9) and*

$$F:=2(\overline{D}_1+\overline{D}_2+\overline{D}_3),\tag{4.23}$$

*where $\overline{D}_i$, $i=1,2,3$ are given in Lemma 4.6. If we assume (3.3), (4.11) and $\overline{\sigma}\leq 1$ such that*

$$F\sqrt{\overline{\sigma}}\leq 1,\tag{4.24}$$

*then we have that $\Psi$ is a contraction mapping, i.e., (4.3) is verified.*

*Proof.* From Lemma 4.6, we can observe that

$$\|\Psi(w_1) - \Psi(w_2)\| \leq F\sqrt{\overline{\sigma}}\|w_1 - w_2\|, \quad \forall w_1, w_2 \in \mathcal{C}_{\overline{\sigma},M}.$$

Therefore, by condition (4.24), $\Psi$ turns out to be a contraction mapping. □

Therefore, we can obtain the following main result:

**Theorem 4.8.** *For $M$ given by (4.9), if we choose $\overline{\sigma} \leq 1$ such that the conditions (3.3), (4.10), (4.11) and (4.24) are satisfied, then the Volterra integral equation (3.7) has a unique solution $w$ on $\mathcal{C}_{\overline{\sigma},M}$. Then, there exists a unique solution to the free boundary problem (2.2a)–(2.2f) given by $v = v(x,\tau)$, $X = X(\tau)$ , $0 < x < X(\tau)$, $\tau \in [0, \overline{\sigma}]$ whose integral representation is given by (3.5)–(3.6).*

*Proof.* The proof of the theorem follows from the results obtained in the previous sections applying the methodology of the integral representation of Friedman-Rubinstein [7,12]. □

Once we have proved the existence and uniqueness of solution to the problem (2.2a)–(2.2f), let us return to the original problem (1.1a)–(1.1f) and state the following main result:

**Theorem 4.9.** *Let us define $M$ by*

$$M = \frac{2n}{(T_g - T_s)}\left[(n+1)\|T_0 - T_s\|_{R_0} + \|T_0'\|_{R_0}\right], \tag{4.25}$$

*where $T_0'$ denotes the derivative of the function $T_0 = T_0(r)$ with respect to $r$. If we choose the supremum of $\sigma$ such that $\sigma \leq \frac{R_0^2}{\alpha}$ and the following conditions hold*

$$A(M,\sigma)\sigma \leq \min\left\{\frac{R_0^2(n-1)}{\alpha n}; \frac{R_0^2}{2\alpha M \text{Ste}}\right\}, \tag{4.26}$$

$$2M\left\{\frac{n}{2\sqrt{\pi}}\left(A\left(M, \tfrac{R_0^2}{\alpha}\right) + n^3\left(\tfrac{3}{2e}\right)^{3/2}\right) + \frac{2n^2}{\sqrt{\pi}}\right\}\sqrt{\sigma} \leq \tfrac{R_0}{\sqrt{\alpha}} \tag{4.27}$$

$$2\left(D_1 + D_2 + D_3\right)\sqrt{\sigma} \leq \tfrac{R_0}{\sqrt{\alpha}}, \tag{4.28}$$

*where $n \in \mathbb{N}$, $n \geq 2$,*

$$A(M,\sigma) := \text{Ste}\left\{M + \frac{R_0}{k}\|h\|_\sigma\right\}, \tag{4.29}$$

*and*

$$D_1 = \tfrac{2\,\text{Ste}\,n^2\|T_0 - T_s\|_{R_0}}{(T_g - T_s)}\left(\tfrac{1}{\sqrt{\pi}} + n^2\right),$$
$$D_2 = \left(\|T_0 - T_s\|_{R_0} + \|T_0'\|_{R_0}\right)\tfrac{n\,\text{Ste}}{(T_g - T_s)}\left(\tfrac{2}{\sqrt{\pi}} + n\right),$$
$$D_3 = \left(Mn^2\,P_1 + P_2 + MnP_3 + nP_4\left(1 + \text{Ste}\,M\,(1+n)\,\right)\right),$$

*with*

$$P_1 = \tfrac{\text{Ste}}{\sqrt{\pi}}\left(A(M, \tfrac{R_0^2}{\alpha}) + \left(\tfrac{3}{2e}\right)^{3/2} n^3\right), \qquad P_2 = \tfrac{2n^2}{\sqrt{\pi}}\left(1 + 3Mn\,\text{Ste}\,\right),$$
$$P_3 = \tfrac{\text{Ste}}{\sqrt{\pi}}\left(\tfrac{1}{2} + A^2(M, \tfrac{R_0^2}{\alpha}) + n^2\left(\left(\tfrac{5}{2e}\right)^{5/2} n^3 + \tfrac{1}{2e}\right)\right),$$
$$P_4 = \tfrac{A(M, \frac{R_0^2}{\alpha})}{2\sqrt{\pi}} + \tfrac{12n^{3/2}}{(2e)^{3/2}},$$

*then, there exist a unique solution to problem* (1.1a)–(1.1f) *given by* $T = T(r,t)$, $R = R(t)$, $0 \leq r \leq R(t)$, $t \in [0,\sigma]$ *whose integral representation is given by:*

$$T(r,t) = (T_g - T_s)\left\{\int_0^{R_0} z^2 \left(\tfrac{T_0(z)-T_s}{T_g-T_s}\right) G(r,\alpha t, z, 0)\ dz \right.$$
$$\left. + \tfrac{\alpha}{R_0}\int_0^t R^2(\nu)\, W(\nu)\, G(r,\alpha t, R(\nu), \alpha\nu)\, \mathrm{d}\nu\right\} + T_s, \tag{4.30}$$

$$R(t) = R_0\left\{1 - \frac{\alpha \mathrm{Ste}}{R_0}\left(\frac{1}{k}\int_0^t h(\nu)\,\mathrm{d}\nu - \frac{1}{R_0}\int_0^t W(\nu)\mathrm{d}\nu\right)\right\} \tag{4.31}$$

*where* $W(t)$ *is the unique solution of Volterra integral equation*

$$W(t) = 2\left\{R_0\int_0^{R_0}\left(\tfrac{T_0(z)-T_s}{T_g-T_s}\right) z^2 G_x(R(t),\alpha t, z, 0)dz\right.$$
$$\left. + \alpha\int_0^t R^2(\nu)\, W(\nu)\, G_x(R(t),\alpha t, R(\nu), \nu)\mathrm{d}\nu\right\}. \tag{4.32}$$

**Remark 4.10.** In the special case that the extinction time $t_e$ is smaller than $R_0^2/\alpha$, the assumption $\sigma \leq R_0^2/\alpha$ is not longer necessary for the proof of Theorem 4.9.

**Remark 4.11.** (Extension of the solution) In order to extend the solution for all times, following [7], for a sufficiently small $\eta$, we can consider the problem defined on $\sigma - \eta < t < \sigma + \epsilon$ with $\epsilon > 0$ that consists of finding $T^1 = T^1(x,t)$ and $R^1 = R^1(t)$ such that:

$$\overline{\rho} c T^1_t = \frac{k}{r^2}(r^2 T^1_r)_r, \qquad 0 < r < R^1(t)\ ,\ \sigma - \eta < t < \sigma + \varepsilon, \tag{4.33a}$$
$$T^1_r(0,t) = 0, \qquad \sigma - \eta < t < \sigma + \varepsilon, \tag{4.33b}$$
$$kT^1_r(R^1(t),t) + h(t)(T_s - T_g) = \overline{\rho} L \dot{R}^1(t), \qquad \sigma - \eta < t < \sigma + \varepsilon, \tag{4.33c}$$
$$T^1(r,\sigma-\eta) = T(r,\sigma-\eta), \qquad 0 \leq r \leq R(\sigma-\eta), \tag{4.33d}$$
$$T^1(R^1(t),t) = T_s, \qquad \sigma - \eta < t < \sigma + \varepsilon, \tag{4.33e}$$
$$R^1(\sigma-\eta) = R(\sigma-\eta), \tag{4.33f}$$

where $T, R$ is the solution to the problem (1.1a)–(1.1f).

Setting the following change of variables $t^* = t - (\sigma - \eta)$, $T^*(r,t^*) = T^1(r,t)$ and $R^*(t^*) = R^1(t)$, we obtain:

$$\overline{\rho} c T^*_{t^*} = \frac{k}{r^2}(r^2 T^*_r)_r, \qquad 0 < r < R^*(t^*)\ ,\ 0 < t^* < \eta + \varepsilon, \tag{4.34a}$$
$$T^*_r(0,t^*) = 0, \qquad 0 < t < \eta + \varepsilon, \tag{4.34b}$$
$$kT^*_r(R^*(t^*),t^*) + h^*(t^*)(T_s - T_g) = \overline{\rho} L \dot{R}^*(t^*), \qquad 0 < t^* < \eta + \varepsilon, \tag{4.34c}$$
$$T^*(r,0) = T^1_0(r), \qquad 0 \leq r \leq R^1_0, \tag{4.34d}$$
$$T^*(R^*(t^*),t^*) = T_s, \qquad 0 < t^* < \eta + \varepsilon, \tag{4.34e}$$
$$R^*(0) = R^1_0, \tag{4.34f}$$

with $h^*(t^*) = h(t)$, $T^1_0(r) = T(r,\sigma-\eta)$ and $R^1_0 = R(\sigma-\eta)$.

Repeating the prior analysis that led us to Theorem 4.9,, we can prove existence and uniqueness of the solution $T_1$, $R_1$ for the problem (4.33a)–(4.33f) for $\sigma-\eta<t<\sigma+\epsilon$, which coincides with $T$, $R$ in the common interval of existence $(\sigma-\eta,\sigma)$. Then, we can repeat this process until we reach the extinction time $t_e$.

**Remark 4.12.** (*Extinction time*) From equation (4.31), we have that $R(t)=R_0V(t)$ where

$$V(t)=1-\frac{\alpha\mathrm{Ste}}{R_0}\left(\frac{1}{k}\int\limits_0^t h(\nu)\,\mathrm{d}\nu-\frac{1}{R_0}\int\limits_0^t W(\nu)\mathrm{d}\nu\right)$$

satisfies $V(0)=1$, $V'(t)<0$, and $\lim\limits_{t\to t_e^-}V(t)=0$.

Then, there exists the inverse function $V^{-1}:(0,1]\mapsto[0,t_e)$ that satisfies

$$t_e=\lim_{z\to 0^+}V^{-1}(z).$$

## 5. Conclusion

In this paper,, we have studied the evaporation process of a spherical droplet through a one-dimensional Stefan problem with spherical symmetry. An equivalent integral formulation for the corresponding dimensionless problem has been obtained and we have proved existence and uniqueness of the solution, applying the methodology of Friedman–Rubinstein. Then, we have returned to the original problem obtaining also its integral formulation and proving existence and uniqueness of solution.

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