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Explicit solutions to fractional Stefan-like problems for Caputo and Riemann–Liouville derivatives

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ABSTRACT

Two fractional two-phase Stefan-like problems are considered by using Riemann-Liouville and Caputo derivatives of order $\alpha \in (0, 1)$ verifying that they coincide with the same classical Stefan problem at the limit case when $\alpha = 1$. For both problems, explicit solutions in terms of the Wright functions are presented. Even though the similarity of the two solutions, a proof that they are different is also given. The convergence when $\alpha \nearrow 1$ of the one and the other solutions to the same classical solution is also given. Numerical examples for the dimensionless version of the problem are presented and analyzed.

Contents

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1. Introduction

This paper deals with Stefan–like problems governed by fractional diffusion equations (FDE). A classical Stefan problem is a problem where a phase-change occurs, usually linked to melting (change from solid to liquid) or freezing (change from liquid to solid). In these problems the diffusion, considered as a heat flow, is expressed in terms of instantaneous local flow of temperature modeled by the Fourier Law. Therefore, the governing equations related to each phase are the well-known heat equations. There is also a latent heat-type condition at the interface that connects the velocity of the free boundary and the heat flux of the temperatures in both phases known as “Stefan condition”. A vast literature on Stefan problems is given in [1–5].

For example, the problem (1) is the mathematical formulation for classical one-dimensional two-phase Stefan problem: *Find the triple* $\{u_1, u_2, s\}$ *such that they have sufficiently regularity and they verify that:*

$$\begin{array}{lll} (i) & \frac{\partial}{\partial t}u_2(x,t)=\lambda_2^2\,\frac{\partial^2}{\partial x^2}u_2(x,t), & 0<x<s(t),\ 0<t<T,\\ (ii) & \frac{\partial}{\partial t}u_1(x,t)=\lambda_1^2\,\frac{\partial^2}{\partial x^2}u_1(x,t), & x>s(t),\ 0<t<T,\\ (iii) & u_1(x,0)=U_i, & 0\le x,\\ (iv) & u_2(0,t)=U_0, & 0<t\le T,\\ (v) & u_1(s(t),t)=u_2(s(t),t)=U_m, & 0<t\le T,\\ (vi) & \rho l\frac{d}{dt}s(t)=k_1\frac{\partial}{\partial x}u_1(s(t),t)-k_2\frac{\partial}{\partial x}u_2(s(t),t), & 0<t\le T,\\ (vii) & s(0)=0, & \end{array} \tag{1}$$

where $U_i < U_m < U_0$, $\lambda_j^2 = \frac{k_j}{\rho c_j}$, $j = 1$ (solid), $j = 2$ (liquid) and we have assumed that the thermophysical properties are constant as well as the free boundary can be represented by an increasing function of time.

Problem (1) is clearly governed by the heat equations $(1-i)$ and $(1-ii)$, and it has a phase-change condition (namely the Stefan condition) given by equation $(1-vi)$.

When the governing equations $(1-i)$ and $(1-ii)$, or the Stefan condition $(1-vi)$ are replaced by other equations involving fractional derivatives in problems like (1), we will refer to them as fractional Stefan-like problems.

For example, the heat equation can be replaced by a fractional diffusion equation (FDE), which is closely linked to the study of anomalous diffusion. A detailed explanation about the relation between anomalous diffusion and randon walk processes can be founded at the work done by Metzler and Klafter [6]. As we know, the diffusion equation is connected to the Brownian motion, where the mean square displacement (msd) of particles is proportional to time. However, in Random Walks the msd is proportional to a power of time. When the exponent of the power law is less than one, the phenomenon is called subdiffusion. It is also interesting the approach given in [7–9] where it is suggested that anomalous diffusion could be caused by heterogeneities in the domain.

For the relation between fractional diffusion equations and their applications, we refer the reader to [10–13] and references therein where applications to the theory of linear viscoelasticity or thermoelasticity, among other, are presented.

In this paper, two approaches leading to subdiffusion are considered. The first one linked to the mathematical interest as generalized operators which interpolates classical derivatives (see [14]), and the second one related to Fourier's generalization laws (see [15]). These two approaches derived in two different formulations for the FDE. In order to present them, let $u = u(x, t)$ be a function of the one-dimensional spatial variable x and time t. A first formulation for the FDE given in terms of fractional integrals (see [16]) is given by:

$$ {}_0I_t^{\alpha} u_{xx}(x,t) = u(x,t) - u(x,0) \tag{2}$$

where, ${}_0I_t^{\alpha}$ is the fractional integral of Riemann–Liouville of order α in the $t-$variable defined as

$$ {}_0I_t^{\alpha} u(x,t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}u(x,\tau)d\tau $$

for every u such that $u(x,\ \cdot) \in L^1(0, T)$ for every $x > 0$. Eq. (2) is also derived in [6], when a fractal time random walk is considered. As it can be seen, no partial derivative in time is part of Eq. (2), but differenciating respect on time to both members we get a second formulation for a FDE

$$ {}_0^{RL}D_t^{1-\alpha} u_{xx}(x,t) = u_t(x,t), \tag{3}$$

where ${}_0^{RL}D_t^{1-\alpha}$ is the fractional derivative of Riemann–Liouville in the $t-$variable defined for every $\alpha \in (0, 1)$ as

$$ {}_0^{RL}D_t^{1-\alpha} u(x,t) = \frac{\partial}{\partial t}\,{}_0I_t^{\alpha} u(x,t) = \frac{1}{\Gamma(\alpha)}\frac{\partial}{\partial t}\int_0^t (t-\tau)^{\alpha-1}u(x,\tau)d\tau $$

for every $u \in AC_t[0,T] = \{u \mid u(x,\cdot)$ is absolutely continuous on $[0,T]$ for every $x \in \mathbb{R}^+\}$.

Nevertheless, when discussing about FDE associated to fractional time derivatives, the reader may retract on the FDE for the Caputo derivative, that is

$$ {}_0^{C}D_t^{\alpha} u(x,t) = u_{xx}(x,t). \tag{4}$$

In Eq. (4), the partial time derivative has been replaced by a fractional derivative in the sense of Caputo respect on time. The Caputo derivative ${}_0^C D_t^\alpha$ is defined for every $\alpha \in (0, 1)$ as

$$ {}_0^C D_t^\alpha u(x,t) = \left[{}_0 I_t^{1-\alpha}(u_t) \right](x,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} u_t(x,\tau) d\tau $$

for every $u \in AC_t[0, T]$.

As we said before, in this paper, problems like (1) governed by equations like (3) or (4) will be studied. The literature on time fractional phase-change problems is rather scant: In [17] a fractional two-phase moving-boundary problem is approximated by a scale Brownian motion model for subdiffusion. In [18] sharp and diffuse interface models of fractional Stefan problems are discussed. In [19] a formulation of a one-phase fractional phase-change problem is given leading to a time dependence on the initial extreme of the fractional derivative. When the starting time considered in the fractional derivative of the governing equation is equal to 0, the mathematical point of view becomes interesting because the problem admit self-similar solutions in terms of the Wright functions (see [20–25]). It is worth noting that in the problems deduced as in [19] and [26] the starting time is not constant, in fact, it depends on the location of the interface due to the assumption that $u \equiv 0$ when $x > s(t)$. See [27] for more details. For a numerical approach we refer to [28] and [29].

It is worth noting the presence of recent publications related to space-fractional Stefan problems modeled by a fractional Laplacian [30,31] or modeled by fractional diffusive fluxes which take the form of Caputo or Riemann-Liouville derivatives [32].

This paper is a continuation of a previous work [33], related to fractional one-phase change problems. In Section 2 some properties of fractional calculus which will be useful later are given. In Section 3, two fractional two-phase Stefan-like problems are considered, both problems admit exact self-similar solutions. Although the two governing equations are equivalent under certain assumptions for boundary-value-problems, when different “fractional Stefan conditions” are considered, the solutions obtained seem to be different. The uniqueness of the self-similar solution for one of the problems is obtained while it is an open problem for the other (see [25]). Finally, numerical examples and plots of the solutions obtained before are presented in Section 4 by considering a dimensionless model.

2. Basic definitions and properties

Proposition 1. *[14] The following properties involving the fractional integrals and derivatives hold:*

1. *The fractional derivative of Riemann–Liouville is a left inverse operator of the fractional integral of Riemann–Liouville of the same order $\alpha \in \mathbb{R}^+$. If $f \in AC[a, b]$, then*

$$ {}_a^{RL} D^\alpha \, {}_a I^\alpha f(t) = f(t) \quad \text{for every } t \in (a, b) $$

2. *The fractional integral of Riemann–Liouville is not, in general, a left inverse operator of the fractional derivative of Riemann–Liouville.*

   In particular, if $0 < \alpha < 1$ then ${}_a I^\alpha ({}^{RL}_a D^\alpha f)(t) = f(t) - \dfrac{{}_a I^{1-\alpha} f(a^+)}{\Gamma(\alpha)(t-a)^{1-\alpha}}$.

3. *If there exist some $\phi \in L^1(a, b)$ such that $f = {}_a I^\alpha \phi$, then*

$$ {}_a I^\alpha \, {}_a^{RL} D^\alpha f(t) = f(t) \quad \text{for every } t \in (a, b). $$

4. *If $f \in AC[a, b]$, then*

$$ {}_a^{RL} D^\alpha f(t) = \frac{f(a)}{\Gamma(1-\alpha)} (t-a)^{-\alpha} + {}_a^C D^\alpha f(t). $$

The fractional integral and derivatives of power functions can be easy calculated (see e.g. [11]). In fact, for every $t \geq a$ we have

$$ {}_a I^\alpha \left((t-a)^\beta \right) = \frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)} (t-a)^{\beta+\alpha}, \qquad \text{for every} \quad \beta > -1, \tag{5} $$

and

$$ {}_a^{RL} D^\alpha \left((t-a)^\beta \right) = \begin{cases} \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} (t-a)^{\beta-\alpha} & \text{if } \beta \neq \alpha - 1, \\ 0 & \text{if } \beta = \alpha - 1. \end{cases} \tag{6} $$

Note that if $\beta > 0$ then ${}_a^{RL} D^\alpha \left((t-a)^\beta \right) = {}_a^C D^\alpha \left((t-a)^\beta \right)$ due to Proposition 1 item 4 and that the Caputo derivative of $(t-a)^\beta$ is not defined for $-1 < \beta < 0$.

Proposition 2. *[34] The following limits hold:*

1. *If we set ${}_a I^0 = Id$ for the identity operator, then for every $f \in L^1(a, b)$,*

$$ \lim_{\alpha \searrow 0} {}_a I^\alpha f(t) = {}_a I^0 f(t) = f(t), \qquad a.e. \text{in } (a, b). $$

2. *For every* $f\in AC[a,b]$, *we have*

$$\lim_{\alpha\nearrow 1} {}_a^C D^\alpha f(t) = f'(t) \quad \text{and} \quad \lim_{\alpha\searrow 1} {}_a^C D^\alpha f(t) = f'(t) - f'(a^+) \quad \text{for all } t\in(a,b).$$

3. *For every* $f\in AC[a,b]$,

$$\lim_{\alpha\nearrow 1} {}_a^{RL} D^\alpha f(t) = f'(t) \quad \text{and} \quad \lim_{\alpha\searrow 1} {}_a^{RL} D^\alpha f(t) = f'(t) \quad a.e. \text{in } (a,b).$$

Definition 1. For every $x\in\mathbb{R}$, the *Wright* function is defined as

$$W(x;\rho;\beta) = \sum_{k=0}^{\infty} \frac{x^k}{k!\Gamma(\rho k+\beta)}, \quad \rho > -1 \text{ and } \beta\in\mathbb{R}. \tag{7}$$

An important particular case of the Wright function is the *Mainardi* function defined by

$$M_\rho(x) = W(-x,-\rho,1-\rho) = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!\Gamma(-\rho n+1-\rho)}, \quad 0<\rho<1.$$

Proposition 3. [12, 35] *Let* $\alpha>0$, $\rho\in(0,1)$ *and* $\beta\in\mathbb{R}$. *Then the next assertions follow:*

1. *For every* $x\in\mathbb{R}$ *we have*

$$\frac{\partial}{\partial x} W(x,\rho,\beta) = W(x,\rho,\rho+\beta).$$

2. *For every* $x>0$ *and* $c>0$,

$${}_0I^\alpha\left[x^{\beta-1}W(-cx^{-\rho},-\rho,\beta)\right] = x^{\beta+\alpha-1}W(-cx^{-\rho},-\rho,\beta+\alpha). \tag{8}$$

Proposition 4. [33, 35] *For every* $\beta\geq 0$, $\rho\in(0,1)$*:*

1. *The Wright function* $W(-\cdot,-\rho,\beta)$ *is positive and strictly decreasing in* $\mathbb{R}^+$.
2. *For every* $x\geq 0$ *the following equality holds*

$$\rho x W(-x,-\rho,\beta-\rho) = W(-x,-\rho,\beta-1) + (1-\beta)W(-x,-\rho,\beta).$$

3. *If, in addition* $0<\rho\leq\mu<\delta$, *then for every* $x>0$ *the following inequality holds*

$$\Gamma(\delta)W(-x,-\rho,\delta) < \Gamma(\mu)W(-x,-\rho,\mu). \tag{9}$$

Proposition 5. [36] *For every* $\beta\geq 0$ *and* $\rho\in(0,1)$ *the following limit holds*

$$\lim_{x\to\infty} W(-x,-\rho,\beta) = 0.$$

Proposition 6. [24, 33]*Let* $x\in\mathbb{R}_0^+$ *be. Then*

$$\lim_{\alpha\nearrow 1} M_{\alpha/2}(2x) = \lim_{\alpha\nearrow 1} W\left(-2x,-\frac{\alpha}{2},1-\frac{\alpha}{2}\right) = M_{1/2}(2x) = \frac{e^{-x^2}}{\sqrt{\pi}}, \tag{10}$$

$$\lim_{\alpha\nearrow 1} W\left(-2x,-\frac{\alpha}{2},\frac{\alpha}{2}\right) = \frac{e^{-x^2}}{\sqrt{\pi}}, \tag{11}$$

$$\lim_{\alpha\nearrow 1}\left[1-W\left(-2x,-\frac{\alpha}{2},1\right)\right] = \text{erf}(x), \tag{12}$$

and

$$\lim_{\alpha\nearrow 1}\left[W\left(-2x,-\frac{\alpha}{2},1\right)\right] = \text{erfc}(x), \tag{13}$$

where erf$(\cdot)$ *is the error function defined by* $\text{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-z^2}dz$ *and* $\text{erfc}(\cdot)$ *is the complementary error function defined by* $\text{erfc}(x) = 1-\text{erf}(x)$. *Moreover, the convergence is uniform over compact sets.*

Proposition 7. *The fractional initial-boundary-value problems for the quarter plane*

$$\begin{array}{lll} (i) & {}_0^C D_t^\alpha u(x,t) = \frac{\partial^2}{\partial x^2}u(x,t), & 0<x,\ 0<t, \\ (ii) & u(x,0) = u_0(x), & 0\leq x, \\ (iii) & u(0,t) = g(t), & 0<t \end{array} \tag{14}$$

and

$$\begin{array}{lll} (i) & \frac{\partial}{\partial t}u(x,t) = {}^{RL}_{0}D^{1-\alpha}_{t}\left(\frac{\partial^2}{\partial x^2}u(x,t)\right), & 0 < x,\ 0 < t, \\ (ii) & u(x,0) = u_0(x), & 0 \le x, \\ (iii) & u(0,t) = g(t), & 0 < t, \end{array} \tag{15}$$

are equivalent, if there exists $\beta > 0$ such that $\beta < \alpha < 1$ and $u_{xx}(x,\ \cdot)$ is an $O(t^{-\beta})$ when $t \to 0^+$.

Proof. Let $u = u(x,t)$ be a function satisfying equation (14–i). Applying ${}^{RL}_{0}D^{1-\alpha}_{t}$ to both sides and using Proposition 1 item 1 we get (15–i).

Let now, for the inverse suppose that u satisfies equation (15-i). Applying ${}_0I^{1-\alpha}_t$ to both sides and using Proposition 1 item 2 yields that

$$ {}^{C}_{0}D^{\alpha}_{t}u(x,t) = \frac{\partial^2}{\partial x^2}u(x,t) - \frac{\lim\limits_{t \searrow 0} {}_0I^{\alpha}\left(\frac{\partial^2}{\partial x^2}u(x,t)\right)}{\Gamma(1-\alpha)t^{\alpha}}, \quad 0 < x, \quad 0 < t. \tag{16}$$

Now, for every x fixed we have that $u_{xx}(x,\ \cdot)$ is an $O(t^{-\beta})$ when $t \to 0^+$, then there exists $\delta > 0$ such that

$$-C\tau^{-\beta} \le u_{xx}(x,\tau) \le C\tau^{-\beta}, \quad 0 < \tau \le t < \delta. \tag{17}$$

Multiplying by $\frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}$ in (17), integrating between 0 and t and applying formula (5) yields that

$$-C\frac{\Gamma(1-\beta)t^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} \le {}_0I^{\alpha}_t\, u_{xx}(x,t) \le C\frac{\Gamma(1-\beta)t^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}, \qquad t < \delta. \tag{18}$$

Taking the limit when t tends to zero in (18) and being $\beta < \alpha$ we conclude that equation (14–i) holds as we wanted to see. □

Remark 1. Equations (14-i) and (15-i) have been treated as equivalent in literature, as it can be seeing at [6,10,15], but the condition

$$\lim_{t \searrow 0} {}_0I^{\alpha}\left(\frac{\partial^2}{\partial x^2}u(x,t)\right) = 0 \tag{19}$$

must be considered and should not be forget it.

Remark 2. It is easy to check that the following functions verify equation (14-i) and (15-i) (we have taken $\lambda = 1$ without loss of generality)

$$w_1(x,t) = x^2 + \frac{2}{\Gamma(\alpha+1)}t^{\alpha}. \tag{20}$$

$$w_2(x,t) = E_{\alpha}(t^{\alpha})\exp\{-x\} \tag{21}$$

and

$$w_3(x,t) = W\left(-\frac{x}{t^{\alpha/2}}, -\frac{\alpha}{2}, 1\right). \tag{22}$$

The condition (19) trivially holds for function w_1 and w_2 and it is no difficult to check it for w_3 (at first differentiating and then using Preposition 3).

3. The fractional Stefan-like problems

In this section, two fractional Stefan-like problems, which have explicit self-similar solutions, will be treated. Before that, some clarifications about the used terminology is presented.

We refer to fractional Stefan Problems when the governing equations are derived from physical assumptions, like considering memory fluxes.

For example, suppose that a process of melting of a semi–infinite slab ($0 \le x < \infty$) of some material is taking place, and the flux involved is a flux with memory. The melt temperature is U_m, and a constant temperature $U_0 > U_m$ is imposed on the fixed face $x = 0$. Let $u_1 = u_1(x,t)$ and $u_2 = u_2(x,t)$ be the temperatures at the solid and liquid phases respectively. Let $J_1 = J_1(x,t)$ and $J_2 = J_2(x,t)$ be the respective functions for the fluxes at position x and time t and let $x = s(t)$ be the function representing the (unknown) position of the free boundary at time t. Suppose further that:

(i) All the thermophysical parameters are constants.
(ii) The function s is an increasing function and consequently, an invertible function.

(iii) J_1 and J_2 are fluxes modeling the material with memory which verifies that "*the weighted sum of the fluxes back in time at the current time, is proportional to the gradient of temperature*", that is, the following equations hold

$$\nu_\alpha \, {}_0I_t^{1-\alpha} J_1(x,t) = -k_1 \frac{\partial u_1}{\partial x}(x,t) \tag{23}$$

and

$$\nu_\alpha \, {}_{h(x)}I_t^{1-\alpha} J_2(x,t) = -k_2 \frac{\partial u_2}{\partial x}(x,t) \tag{24}$$

where the initial time in the fractional integral (24) is given by function h which gives us the time when the phase change occurs. That is,

$$t = h(x) = s^{-1}(x) \qquad (\text{i.e. } \ x = s(t))$$

The number ν_α is a parameter with physical dimension (see(70)) such that

$$\lim_{\alpha \nearrow 1} \nu_\alpha = 1, \tag{25}$$

which has been added in order to preserve the consistency with respect to the units of measurement in equations (23) and (24). Also, the parameter

$$\mu_\alpha = \frac{1}{\nu_\alpha} \tag{26}$$

will be used in the following equations. More details about these parameters are given in Section 4.

Making an analogous reasoning for the two-phase free-boundary problem, in a similar way it was done for one-phase free-boundary problems in [19], the mathematical model for the problem described above is given by

$$\begin{array}{lll}
(i) & \frac{\partial}{\partial t} u_2(x,t) = \lambda_2^2 \, \mu_{\alpha_2} \frac{\partial}{\partial x}\left({}^{RL}_{h(x)}D_t^{1-\alpha}\left(\frac{\partial}{\partial x} u_2(x,t)\right)\right), & 0 < x < s(t), \ 0 < t < T, \\
(ii) & \frac{\partial}{\partial t} u_1(x,t) = \lambda_1^2 \, \mu_{\alpha_1} \frac{\partial}{\partial x}\left({}^{RL}_{0}D_t^{1-\alpha}\left(\frac{\partial}{\partial x} u_1(x,t)\right)\right), & x > s(t), \ 0 < t < T, \\
(iii) & u_1(x,0) = U_i, & 0 \le x, \\
(iv) & u_2(0,t) = U_0, & 0 < t \le T, \\
(v) & u_1(s(t),t) = u_2(s(t),t) = U_m, & 0 < t \le T, \\
(vi) & \rho l \frac{d}{dt} s(t) = k_1 \mu_{\alpha_1} \, {}^{RL}_{0}D_t^{1-\alpha} \frac{\partial}{\partial x} u_1(x,t)\Big|_{(s(t)^+,t)} - k_2 \mu_{\alpha_2} \, {}^{RL}_{h(x)}D_t^{1-\alpha} \frac{\partial}{\partial x} u_2(x,t)\Big|_{(s(t)^-,t)}, & 0 < t \le T. \\
(vii) & s(0) = 0 &
\end{array} \tag{27}$$

where $U_i < U_m < U_0$ and $\mu_{\alpha_j} = \frac{1}{\nu_{\alpha_j}}$ $j = 1, 2$, (note that the parameters μ_{α_j} can be the same in Eqs. (27)–*i* and (27)–*ii*, then from now on we will take $\mu_{\alpha_2} = \mu_{\alpha_1}$ without loss of generality).

Note that self-similar solutions to problem (27) had not been yet founded, due to the difficulty imposed by the variable bottom limit in the fractional derivative for the liquid phase.

Now, as noted at the beginning of this section, this paper deals with Stefan-like problems admitting explicit self-similar solutions. These problems come from the assumption of consider the botton limit $t_0 = 0$ in the fractional time derivatives in the Caputo or Riemann–Liouville sense.

The Stefan-Like Problem for the Caputo derivative. The next problem was treated in [25] and can be obtained by replacing all the time derivatives in (1) by fractional derivatives in the Caputo sense of order $\alpha \in (0, 1)$, i.e.

$$\begin{array}{lll}
(i) & {}^C_0D_t^\alpha u_2(x,t) = \lambda_{\alpha_2}^2 \frac{\partial^2}{\partial x^2} u_2(x,t), & 0 < x < s(t), \ 0 < t < T, \\
(ii) & {}^C_0D_t^\alpha u_1(x,t) = \lambda_{\alpha_1}^2 \frac{\partial^2}{\partial x^2} u_1(x,t), & x > s(t), \ 0 < t < T, \\
(iii) & u_1(x,0) = U_i, & 0 \le x, \\
(iv) & u_2(0,t) = U_0, & 0 < t \le T, \\
(v) & u_1(s(t),t) = u_2(s(t),t) = U_m, & 0 < t \le T, \\
(vi) & \rho l \, {}^C_0D_t^\alpha s(t) = k_{\alpha_1} \frac{\partial}{\partial x} u_1(s(t)^+,t) - k_{\alpha_2} \frac{\partial}{\partial x} u_2(s(t)^-,t), & 0 < t \le T, \\
(vii) & s(0) = 0. &
\end{array} \tag{28}$$

where $U_i < U_m < U_0$, λ_{α_i} are positive parameters named as "subdiffusion coefficients" given by $\lambda_{\alpha_i} = \lambda_i \sqrt{\mu_\alpha}$ for $i = 1, 2$, and k_{α_i} are positive parameters named as "subdiffusion thermal conductivities" given by $k_{\alpha_i} = k_i \mu_\alpha$, $i = 1, 2$.

Definition 2. The triple $\{u_1, u_2, s\}$ is a solution to problem (28) if the following conditions are satisfied

1. u_1 is continuous in the region $\mathcal{R}_T = \{(x,t) : 0 \le x \le s(t), \ 0 < t \le T\}$ and at the point (0, 0), u_1 verifies that

$$0 \le \liminf_{(x,t)\to(0,0)} u_1(x,t) \le \limsup_{(x,t)\to(0,0)} u_1(x,t) < +\infty.$$

2. u_2 is continuous in the region $\{(x, t) \colon x > s(t), 0 < t \le T\}$ and at the point $(0, 0)$, u_2 verifies that

$$0 \le \liminf_{(x,t)\to(0,0)} u_2(x,t) \le \limsup_{(x,t)\to(0,0)} u_2(x,t) < +\infty.$$

3. $u_1 \in C((0,\infty)\times(0,T)) \cap C_x^2((0,\infty)\times(0,T))$, such that $u_1 \in AC_t[0, T]$
4. $u_2 \in C((0,\infty)\times(0,T)) \cap C_x^2((0,\infty)\times(0,T))$, such that $u_2 \in AC_t[0, T]$.
5. $s \in AC[0, T]$.
6. u_1, u_2 and s satisfy (28).

Theorem 1. *[25] A self-similar solution to poblem (28) is given by*

$$\begin{cases} u_2(x,t) = U_0 - \frac{U_0 - U_m}{1 - W\left(-2\xi_\alpha \lambda, -\frac{\alpha}{2}, 1\right)} \left[1 - W\left(-\frac{x}{\lambda_{\alpha_2} t^{\alpha/2}}, -\frac{\alpha}{2}, 1\right)\right] \\ u_1(x,t) = U_i + \frac{U_m - U_i}{W\left(-2\xi_\alpha, -\frac{\alpha}{2}, 1\right)} W\left(-\frac{x}{\lambda_{\alpha_1} t^{\alpha/2}}, -\frac{\alpha}{2}, 1\right) \\ s(t) = 2\xi_\alpha \lambda_{\alpha_1} t^{\alpha/2} \end{cases} \tag{29}$$

*where ξ_α is **a** solution to the equation*

$$\frac{k_{\alpha_2}(U_0 - U_m)\Gamma(1-\frac{\alpha}{2})}{\lambda_{\alpha_2}} F_2(2\lambda x) - \frac{k_{\alpha_1}(U_m - U_i)\Gamma(1-\frac{\alpha}{2})}{\lambda_{\alpha_1}} F_1(2x) = \Gamma\left(1+\frac{\alpha}{2}\right)\lambda_{\alpha_1} \rho l 2x,\ x > 0 \tag{30}$$

where $\lambda = \frac{\lambda_{\alpha_1}}{\lambda_{\alpha_2}} = \frac{\lambda_1\sqrt{\mu_\alpha}}{\lambda_2\sqrt{\mu_\alpha}} = \frac{\alpha_1}{\alpha_2} > 0$, and $F_1 : \mathbb{R}_0^+ \to \mathbb{R}$ and $F_2 : \mathbb{R}_0^+ \to \mathbb{R}$ are the functions defined by

$$F_1(x) = \frac{M_{\alpha/2}(x)}{W\left(-x, -\frac{\alpha}{2}, 1\right)} \quad \text{and} \quad F_2(x) = \frac{M_{\alpha/2}(x)}{1 - W\left(-x, -\frac{\alpha}{2}, 1\right)}. \tag{31}$$

Note 1. The uniqueness of solution to Eq. (30) is still an open problem. However, the uniqueness of similarity solution will be achived for the Riemann–Liouville Stefan–like problem.

The Stefan-Like Problem for the Riemann–Liouville derivative. Consider now the following problem:

$$\begin{array}{lll} (i) & \frac{\partial}{\partial t} w_2(x,t) = \lambda_{\alpha_2}^2 \frac{\partial}{\partial x}\left({}_0^{RL}D_t^{1-\alpha}\left(\frac{\partial}{\partial x} w_2(x,t)\right)\right), & 0 < x < r(t),\ 0 < t < T, \\ (ii) & \frac{\partial}{\partial t} w_1(x,t) = \lambda_{\alpha_1}^2 \frac{\partial}{\partial x}\left({}_0^{RL}D_t^{1-\alpha}\left(\frac{\partial}{\partial x} w_1(x,t)\right)\right), & x > r(t),\ 0 < t < T, \\ (iii) & w_1(x,0) = U_i, & 0 \le x, \\ (iv) & w_2(0,t) = U_0, & 0 < t \le T, \\ (v) & w_1(r(t),t) = w_2(r(t),t) = U_m, & 0 < t \le T, \\ (vi) & \rho l \frac{d}{dt} r(t) = k_{\alpha_1} {}_0^{RL}D_t^{1-\alpha} \frac{\partial}{\partial x} w_1(x,t)\Big|_{(r(t)^+,t)} - k_{\alpha_2} {}_0^{RL}D_t^{1-\alpha} \frac{\partial}{\partial x} w_2(x,t)\Big|_{(r(t)^-,t)}, & 0 < t \le T, \\ (vii) & r(0) = 0. & \end{array} \tag{32}$$

where, as before, $U_i < U_m < U_0$, $\lambda_{\alpha_i} = \lambda_i\sqrt{\mu_\alpha}$ for $i = 1, 2$, and $k_{\alpha_i} = k_i \mu_\alpha$, $i = 1, 2$.

Remark 3. The expression $k_{\alpha_1} {}_0^{RL}D_t^{1-\alpha} \frac{\partial}{\partial x} w_1(x,t)\Big|_{(r(t)^+,t)}$ is equivalent to

$$\lim_{x\to r(t)^+} k_{\alpha_1} {}_0^{RL}D_t^{1-\alpha} \frac{\partial}{\partial x} w_1(x,t), \tag{33}$$

which should not coincide with

$$k_{\alpha_1} {}_0^{RL}D_t^{1-\alpha}\left(\lim_{x\to r(t)^+} \frac{\partial}{\partial x} w_1(x,t)\right). \tag{34}$$

Definition 3. The triple $\{w_1, w_2, r\}$ is a solution to problem (32) if the following conditions are satisfied

1. w_1 is continuous in the region $\mathcal{R}_T = \{(x,t) : 0 \le x \le s(t),\ 0 < t \le T\}$ and at the point $(0, 0)$, u_1 verifies that

$$0 \le \liminf_{(x,t)\to(0,0)} w_1(x,t) \le \limsup_{(x,t)\to(0,0)} w_1(x,t) < +\infty.$$

2. w_2 is continuous in the region $\{(x, t) \colon x > r(t), 0 < t \le T\}$ and at the point $(0, 0)$, w_2 verifies that

$$0 \le \liminf_{(x,t)\to(0,0)} w_2(x,t) \le \limsup_{(x,t)\to(0,0)} w_2(x,t) < +\infty.$$

3. $w_1 \in C((0,\infty)\times(0,T)) \cap C_x^2((0,\infty)\times(0,T))$, such that $w_{1x} \in AC_t(0, T))$.
4. $w_2 \in C((0,\infty)\times(0,T)) \cap C_x^2((0,\infty)\times(0,T))$, such that $w_{2x} \in AC_t[0, T]$.
5. $r \in C^1(0, T)$.

6. There exist ${}_0^{RL}D_t^{1-\alpha}\frac{\partial}{\partial x}w_2(x,t)\Big|_{(s(t)^+,t)}$ and ${}_0^{RL}D_t^{1-\alpha}\frac{\partial}{\partial x}w_1(x,t)\Big|_{(r(t)^-,t)}$ for all $t\in(0,T]$.
7. w_1, w_2 and s satisfy (32).

Theorem 2. *An explicit solution for the two-phase fractional Stefan-like problem (32) is given by*

$$\begin{cases} w_2(x,t)=U_0-\frac{U_0-U_m}{1-W\left(-2\eta_\alpha\lambda,-\frac{\alpha}{2},1\right)}\left[1-W\left(-\frac{x}{\lambda_{\alpha_2}t^{\alpha/2}},-\frac{\alpha}{2},1\right)\right]\\ w_1(x,t)=U_i+\frac{U_m-U_i}{W\left(-2\eta_\alpha,-\frac{\alpha}{2},1\right)}W\left(-\frac{x}{\lambda_{\alpha_1}t^{\alpha/2}},-\frac{\alpha}{2},1\right)\\ r(t)=2\eta_\alpha\lambda_{\alpha_1}t^{\alpha/2}\end{cases}\tag{35}$$

where η_α is the unique positive solution to the equation

$$\frac{k_{\alpha_2}(U_0-U_m)}{\lambda_{\alpha_1}\lambda_{\alpha_2}}G_2(2\lambda x)-\frac{k_{\alpha_1}(U_m-U_i)}{\lambda_{\alpha_1}^2}G_1(2x)=\left(\rho l+\frac{k_{\alpha_1}(U_m-U_i)}{\lambda_{\alpha_1}^2}\right)2x,\tag{36}$$

where $\lambda=\frac{\lambda_{\alpha_1}\sqrt{\mu_\alpha}}{\lambda_{\alpha_2}\sqrt{\mu_\alpha}}=\frac{\lambda_1}{\lambda_2}>0$, $U_i<U_m<U_0$ and $G_1:\mathbb{R}_0^+\to\mathbb{R}$ and $G_2:\mathbb{R}_0^+\to\mathbb{R}$ are the functions defined by

$$G_1(x)=\frac{W\left(-x,-\frac{\alpha}{2},1+\frac{\alpha}{2}\right)}{W\left(-x,-\frac{\alpha}{2},1\right)}\quad\text{and}\quad G_2(x)=\frac{2/\alpha W\left(-x,-\frac{\alpha}{2},\frac{\alpha}{2}\right)}{1-W\left(-x,-\frac{\alpha}{2},1\right)}.\tag{37}$$

Proof. Let the functions

$$\begin{array}{rcl} w_i:\mathbb{R}_0^+\times(0,T) & \to & \mathbb{R}\\ (x,t) & \to & w_i(x,t)=A_i+B_i\left[1-W\left(-\frac{x}{\lambda_{\alpha_i}t^{\alpha/2}},-\frac{\alpha}{2},1\right)\right]\end{array}\tag{38}$$

be the proposed solutions for $i=1,2$. Rewriting expression (8) for the variable t and taking $c=\frac{x}{\lambda_{\alpha_i}}$ gives

$${}_0I_t^\alpha\, t^{\beta-1}W\left(-\frac{x}{\lambda_{\alpha_i}}t^{-\rho},-\rho,\beta\right)=t^{\beta+\alpha-1}W\left(-\frac{x}{\lambda_{\alpha_i}}t^{-\rho},-\rho,\beta+\alpha\right).\tag{39}$$

Then, by using (39) for $\beta=1-\frac{\alpha}{2}$ and Proposition 3 it is easy to check that w_i verifies equations (32-i) and (32-ii) respectively for $i=1,2$.

From condition (32-v) we deduce that $r(t)$ must be proportional to $t^{\alpha/2}$. Therefore we set

$$r(t)=2\eta_\alpha\lambda_{\alpha_1}t^{\alpha/2},\quad t\geq 0\tag{40}$$

where η_α is a constant to be determined and λ_{α_1} was added for simplicity in the next calculations. Now, from conditions $(32-iii)$, $(32-iv)$ and $(32-v)$ it holds that

$$\begin{array}{ll} A_1=U_i+\frac{U_m-U_i}{W\left(-2\eta_\alpha,-\frac{\alpha}{2},1\right)}, & B_1=-\frac{U_m-U_i}{W\left(-2\eta_\alpha,-\frac{\alpha}{2},1\right)}\\ A_2=U_0, & B_2=-\frac{U_0-U_m}{1-W\left(-2\eta_\alpha\lambda,-\frac{\alpha}{2},1\right)}\end{array}$$

As before, by considering (39) for $\beta=1-\frac{\alpha}{2}$ and Proposition 3, it holds that

$$\frac{B_i\alpha/2}{\lambda_{\alpha_1}\lambda_{\alpha_i}t^{1-\alpha/2}}W\left(-\frac{x}{\lambda_{\alpha_i}t^{\alpha/2}},-\frac{\alpha}{2},1+\frac{\alpha}{2}\right)+\frac{B_i\alpha/2}{\lambda_{\alpha_1}\lambda_{\alpha_i}}\frac{x}{t}W\left(-\frac{x}{\lambda_{\alpha_i}t^{\alpha/2}},-\frac{\alpha}{2},1\right),\quad i=1,2.\tag{41}$$

Then replacing (41) and (40) in equation (32-vii), and evaluating the limits following (33) it yields that η_α must verify the next equality

$$\begin{array}{l}\rho l2\eta_\alpha\lambda_{\alpha_1}=-\frac{k_{\alpha_1}(U_m-U_i)}{\lambda_{\alpha_1}^2}\frac{W\left(-2\eta_\alpha,-\frac{\alpha}{2},1+\frac{\alpha}{2}\right)}{W\left(-2\eta_\alpha,-\frac{\alpha}{2},1\right)}-\frac{k_{\alpha_1}(U_m-U_i)}{\lambda_{\alpha_1}^2}2\eta_\alpha-\\ +\frac{k_{\alpha_2}(U_0-U_m)}{\lambda_{\alpha_1}\lambda_{\alpha_2}}\frac{W\left(-2\lambda\eta_\alpha,-\frac{\alpha}{2},1+\frac{\alpha}{2}\right)}{1-W\left(-\lambda2\eta_\alpha,-\frac{\alpha}{2},1\right)}+\frac{k_{\alpha_2}(U_0-U_m)}{\lambda_{\alpha_1}\lambda_{\alpha_2}}\frac{2\lambda\eta_\alpha W\left(-\lambda2\eta_\alpha,-\frac{\alpha}{2},1\right)}{1-W\left(-\lambda_\alpha2\eta_\alpha,-\frac{\alpha}{2},1\right)}.\end{array}\tag{42}$$

which leads to conclude that $\{w_1,w_2,r\}$ is a solution to (32) if and only if η_α is a solution to the equation

$$\frac{k_{\alpha_2}(U_0-U_m)}{\lambda_{\alpha_1}\lambda_{\alpha_2}}\frac{W\left(-\lambda2x,-\frac{\alpha}{2},1+\frac{\alpha}{2}\right)+2\lambda xW\left(-\lambda2x,-\frac{\alpha}{2},1\right)}{1-W\left(-\lambda2x,-\frac{\alpha}{2},1\right)}--k_{\alpha_1}\frac{U_m-U_i}{\lambda_{\alpha_1}^2}\frac{W\left(-2x,-\frac{\alpha}{2},1+\frac{\alpha}{2}\right)}{W\left(-2x,-\frac{\alpha}{2},1\right)}=\left(\rho l+\frac{k_{\alpha_1}(U_m-U_i)}{\lambda_{\alpha_1}^2}\right)2x,\quad x>0.\tag{43}$$

which, by using Proposition 4-2 leads to Eq. (36).

The next step is to prove that Eq. (36) has unique solution. For that purpose we define the function G in $\mathbb{R}^+$ as

$$G(x)=\frac{k_{\alpha_2}(U_0-U_m)}{\lambda_{\alpha_1}\lambda_{\alpha_2}}G_2(2\lambda x)-\frac{k_{\alpha_1}(U_m-U_i)}{\lambda_{\alpha_1}^2}G_1(2x)-\left(\rho l+\frac{k_{\alpha_1}(U_m-U_i)}{\lambda_{\alpha_1}^2}\right)2x.$$

Note that G is a continuous function such that

$$G(0^+) = +\infty. \tag{44}$$

From Proposition 4-3 for every $x > 0$ we have that

$$0 < \frac{W\left(-2x, -\frac{\alpha}{2}, 1+\frac{\alpha}{2}\right)}{W\left(-2x, -\frac{\alpha}{2}, 1\right)} < \frac{1}{\Gamma\left(\frac{\alpha}{2}+1\right)}, \tag{45}$$

then G_1 is bounded. Also, from (45) it holds that

$$-\frac{k_{\alpha_1}(U_i-U_m)}{\lambda_{\alpha_1}^2}\frac{1}{\Gamma\left(\frac{\alpha}{2}+1\right)} + \frac{k_{\alpha_2}(U_0-U_m)}{\lambda_{\alpha_1}\lambda_{\alpha_2}}G_2(2\lambda x) - \left(\rho l + \frac{k_{\alpha_1}(U_m-U_i)}{\lambda_{\alpha_1}^2}\right)2x < G(x) < \frac{k_{\alpha_2}(U_0-U_m)}{\lambda_{\alpha_1}\lambda_{\alpha_2}}G_2(2\lambda x) - \left(\rho l + \frac{k_{\alpha_1}(U_m-U_i)}{\lambda_{\alpha_1}^2}\right)2x, \tag{46}$$

and taking the limit when $x \to \infty$ in (46), by using Proposition 5 we obtain that

$$G(+\infty) = -\infty. \tag{47}$$

Finally, consider the function $K : \mathbb{R}^+ \to \mathbb{R}$ defined as

$$K(x) = -\frac{k_{\alpha_1}(U_m - U_i)}{\lambda_{\alpha_1}^2}[G_1(2x) + 2x] - \rho l 2x. \tag{48}$$

Applying Proposition 3 item 1 and being $\frac{(U_m-U_i)}{\lambda_{\alpha_1}^2} > 0$ it results that K is a strictly decreasing function. On the other hand, from Proposition 4 item 1 we have that G_2 is a strictly decreasing function. Then it can be concluded that G is a strictly decreasing function. Therefore Eq. (36) has a unique positive solution. □

Remark 4. The limits described in Remark 3 are different if we compute them for the functions w_1 and r. In fact, by using the computation made in the previous theorem, we get

$$\lim_{x\to r(t)^+} {}_0^{RL}D_t^{1-\alpha}\frac{\partial}{\partial x}w_1(x,t) = \frac{B_1}{\lambda_{\alpha_1}}\left[\frac{\alpha}{2}t^{\alpha/2-1}W\left(-2\eta_\alpha, -\frac{\alpha}{2}, 1+\frac{\alpha}{2}\right) + \frac{\alpha}{2}2\eta_\alpha t^{\alpha/2-1}W\left(-2\eta_\alpha, -\frac{\alpha}{2}, 1\right)\right]. \tag{49}$$

and from Proposition 4-2, we have:

$$W\left(-2\eta_\alpha, -\frac{\alpha}{2}, 1+\frac{\alpha}{2}\right) + 2\eta_\alpha W\left(-2\eta_\alpha, -\frac{\alpha}{2}, 1\right) = \frac{2}{\alpha}W\left(-\eta_\alpha, -\frac{\alpha}{2}, \frac{\alpha}{2}\right). \tag{50}$$

Then

$$\lim_{x\to r(t)^+} {}_0^{RL}D_t^{1-\alpha}\frac{\partial}{\partial x}w_1(x,t) = \frac{B_1}{\lambda_{\alpha_1}}t^{\alpha/2-1}W\left(-\eta_\alpha, -\frac{\alpha}{2}, \frac{\alpha}{2}\right) \tag{51}$$

whereas

$${}_0^{RL}D_t^{1-\alpha}\left(\lim_{x\to r(t)^+}\frac{\partial}{\partial x}w_1(x,t)\right) = \frac{B_1}{\lambda_{\alpha_1}}t^{\alpha/2-1}\frac{\Gamma\left(1-\frac{\alpha}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right)}M_{\alpha/2}(2\eta_\alpha). \tag{52}$$

And we know that (51) and (52) are different due to Proposition 4-3.

Theorem 3. *If $\lambda = 1$, the explicit solution (35) to problem (32) and the explicit solution (29) to problem (28) are different.*

Proof. Take $U_i = -1$, $U_m = 0$ and $U_0 = 1$. Let $\{u_1, u_2, s\}$ be the solution to problem (28). Then $s(t) = 2\lambda_{\alpha_1}\xi_\alpha t^{\alpha/2}$ where ξ_α is a positive solution to equation

$$\frac{k_{\alpha_2}\Gamma(1-\frac{\alpha}{2})}{\lambda_{\alpha_1}\lambda_{\alpha_2}}\frac{M_{\alpha/2}(2\lambda x)}{1-W\left(-\lambda 2x, -\frac{\alpha}{2}, 1\right)} - \frac{k_{\alpha_1}\Gamma(1-\frac{\alpha}{2})}{\lambda_{\alpha_1}^2}\frac{M_{\alpha/2}(2x)}{W\left(-2x, -\frac{\alpha}{2}, 1\right)} = \Gamma(1+\frac{\alpha}{2})\rho l 2x. \tag{53}$$

On the other hand, let $\{w_1, w_2, r\}$ be the solution to problem (32). Then η_α is the positive solution to equation

$$\frac{k_{\alpha_2}2/\alpha}{\lambda_{\alpha_1}\lambda_{\alpha_2}}\frac{W\left(-2\lambda x, -\frac{\alpha}{2}, \frac{\alpha}{2}\right)}{1-W\left(-\lambda 2x, -\frac{\alpha}{2}, 1\right)} - \frac{k_{\alpha_1}}{\lambda_{\alpha_1}^2}\frac{W\left(-2x, -\frac{\alpha}{2}, 1+\frac{\alpha}{2}\right)}{W\left(-2x, -\frac{\alpha}{2}, 1\right)} = \left(\rho l + \frac{k_{\alpha_1}}{\lambda_{\alpha_1}^2}\right)2x, \tag{54}$$

or equivalently,

$$\frac{k_{\alpha_2}}{\lambda_{\alpha_1}\lambda_{\alpha_2}}\frac{\Gamma(1+\frac{\alpha}{2})2/\alpha W\left(-2\lambda x, -\frac{\alpha}{2}, \frac{\alpha}{2}\right)}{1-W\left(-2\lambda x, -\frac{\alpha}{2}, 1\right)} - -\frac{k_{\alpha_1}\Gamma(1+\frac{\alpha}{2})}{\lambda_{\alpha_1}^2}\frac{W\left(-2x, -\frac{\alpha}{2}, 1+\frac{\alpha}{2}\right) + 2xW\left(-2x, -\frac{\alpha}{2}, 1\right)}{W\left(-2x, -\frac{\alpha}{2}, 1\right)} = \Gamma(1+\frac{\alpha}{2})\rho l 2x. \tag{55}$$

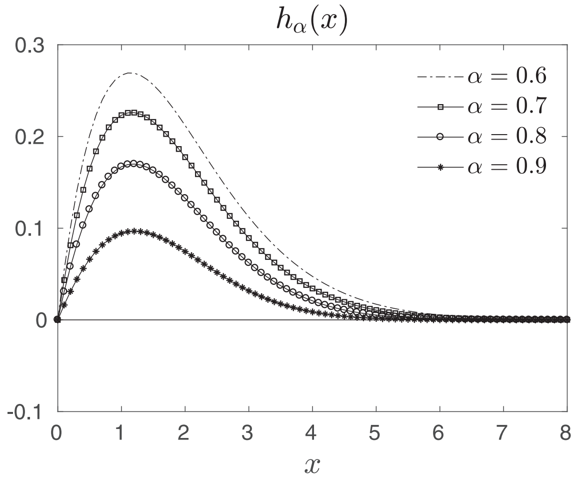

Fig. 1. The function $h_\alpha(x) = \Gamma(\frac{\alpha}{2})W\left(-x, -\frac{\alpha}{2}, \frac{\alpha}{2}\right) - \Gamma(1-\frac{\alpha}{2})M_{\alpha/2}(x)$ for different values of α.

From Proposition 4-2, for every $x > 0$ we have that

$$W\left(-2x, -\frac{\alpha}{2}, 1+\frac{\alpha}{2}\right) + 2xW\left(-2x, -\frac{\alpha}{2}, 1\right) = \frac{2}{\alpha}W\left(-2x, -\frac{\alpha}{2}, \frac{\alpha}{2}\right). \tag{56}$$

Then using the fact that the Gamma function verifies that $\frac{\Gamma(1+\frac{\alpha}{2})}{\frac{\alpha}{2}} = \Gamma(\frac{\alpha}{2})$ and replacing (56) in (55) we deduce that η_α is the unique positive solution to the equation next (Fig. 1)

$$\frac{k_{\alpha_2}}{\lambda_{\alpha_1}\lambda_{\alpha_2}} \frac{\Gamma(\frac{\alpha}{2})W\left(-2\lambda x, -\frac{\alpha}{2}, \frac{\alpha}{2}\right)}{1 - W\left(-2\lambda x, -\frac{\alpha}{2}, 1\right)} - \frac{k_{\alpha_1}}{\lambda^2_{\alpha_1}} \frac{\Gamma(\frac{\alpha}{2})W\left(-2x, -\frac{\alpha}{2}, \frac{\alpha}{2}\right)}{W\left(-2x, -\frac{\alpha}{2}, 1\right)} = \Gamma(1+\frac{\alpha}{2})\rho l 2x,\ x > 0. \tag{57}$$

Then, if we suppose then that $\xi_\alpha = \eta_\alpha$, it results that there exist $\xi_\alpha > 0$ such that

$$\begin{aligned}\frac{k_{\alpha_1}}{\lambda^2_{\alpha_1}} \frac{\Gamma(\frac{\alpha}{2})W\left(-2\xi_\alpha, -\frac{\alpha}{2}, \frac{\alpha}{2}\right)}{W\left(-2\xi_\alpha, -\frac{\alpha}{2}, 1\right)} - \frac{k_{\alpha_1}}{\lambda^2_{\alpha_1}} \frac{\Gamma(1-\frac{\alpha}{2})M_{\alpha/2}(2\xi_\alpha)}{W\left(-2\xi_\alpha, -\frac{\alpha}{2}, 1\right)} &= \frac{k_{\alpha_2}}{\lambda_{\alpha_1}\lambda_{\alpha_2}} \frac{\Gamma(\frac{\alpha}{2})W\left(-2\lambda\xi_\alpha, -\frac{\alpha}{2}, \frac{\alpha}{2}\right)}{1 - W\left(-2\lambda\xi_\alpha, -\frac{\alpha}{2}, 1\right)} \\ &\quad - \frac{k_{\alpha_2}}{\lambda_{\alpha_1}\lambda_{\alpha 2}} \frac{\Gamma(1-\frac{\alpha}{2})M_{\alpha/2}(\lambda 2\xi_\alpha)}{1 - W\left(-2\lambda\xi_\alpha, -\frac{\alpha}{2}, 1\right)}.\end{aligned} \tag{58}$$

By hypothesis $\lambda = 1$, then we conclude that

$$\frac{\frac{k_{\alpha_1}}{\lambda^2_{\alpha_1}}}{W\left(-2\xi_\alpha, -\frac{\alpha}{2}, 1\right)} = \frac{\frac{k_{\alpha_2}}{\lambda_{\alpha_1}\lambda_{\alpha_2}}}{1 - W\left(-2\lambda\xi_\alpha, -\frac{\alpha}{2}, 1\right)}, \tag{59}$$

or equivalently,

$$W\left(-2\xi_\alpha, -\frac{\alpha}{2}, 1\right) = \frac{1}{1 + \frac{k_{\alpha_2}\lambda_{\alpha_2}}{k_{\alpha_1}\lambda_{\alpha_1}}}. \tag{60}$$

Replacing (60) in equation (53) yields that $\rho l \lambda_{\alpha_1} 2\xi_\alpha = 0$ which leads to $\xi_\alpha = 0$, contradicting the fact that $\xi_\alpha > 0$. □

Note 2. It is worth noting that an analogous proof for Theorem 3 but considering $\lambda \neq 1$ does not holds. In fact, if we define the function $h_\alpha : \mathbb{R}^+ \to \mathbb{R}$ as

$$h_\alpha(x) = \Gamma\left(\frac{\alpha}{2}\right)W\left(-x, -\frac{\alpha}{2}, \frac{\alpha}{2}\right) - \Gamma\left(1 - \frac{\alpha}{2}\right)M_{\alpha/2}(x)$$

then equality (58) can be expressed as

$$\frac{k_{\alpha_2}}{\lambda_{\alpha_1}\lambda_{\alpha_2}} \frac{h_\alpha(\lambda 2\xi_\alpha)}{1 - W\left(-\lambda 2\xi_\alpha, -\frac{\alpha}{2}, 1\right)} = \frac{k_{\alpha_1}}{\lambda^2_{\alpha_1}} \frac{h_\alpha(2\xi_\alpha)}{W\left(-2\xi_\alpha, -\frac{\alpha}{2}, 1\right)}. \tag{61}$$

If $\lambda \neq 1$, it is not possible to cancel the espressions $h_\alpha(\lambda 2\xi_\alpha)$ and $h_\alpha(2\xi_\alpha)$ in Eq. (61). Moreover the plots in Fig. 2 lead us to suppose that there exists a positive solution to equation

$$\frac{k_{\alpha_2}}{\lambda_{\alpha_1}\lambda_{\alpha_2}} \frac{h_\alpha(\lambda x)}{1 - W\left(-\lambda x, -\frac{\alpha}{2}, 1\right)} = \frac{k_{\alpha_1}}{\lambda^2_{\alpha_1}} \frac{h_\alpha(x)}{W\left(-x, -\frac{\alpha}{2}, 1\right)},\ x > 0, \tag{62}$$

then, it is not possible to get a contradiction like in (60).

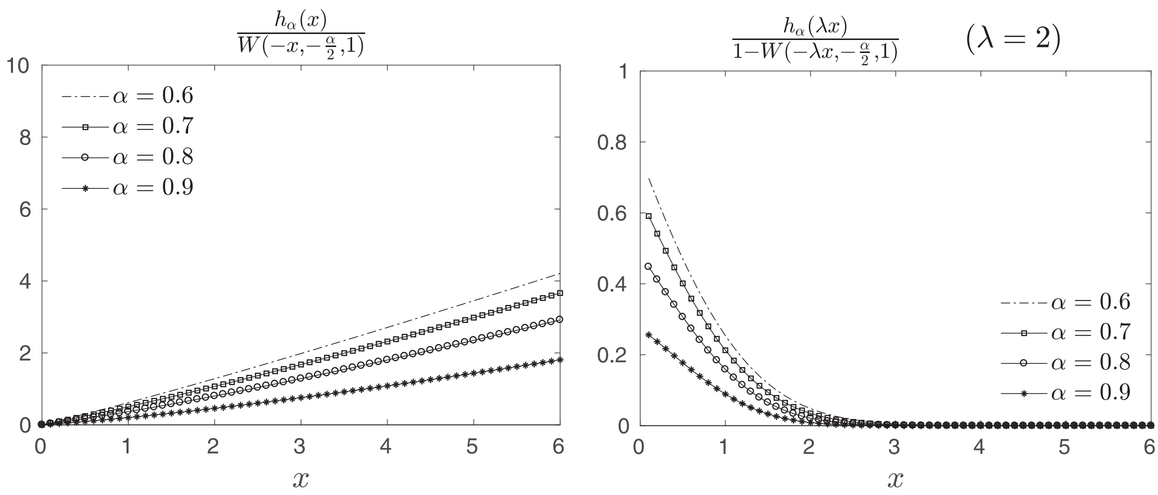

Fig. 2. The left and right quotients of Eq. (62) for different values of α.

However, if we take different values of λ (which are different to 1) and the parameters ξ_α and η_α are estimated numerically for different values of α, we see that they are different and both converge to the same value when $\alpha \nearrow 1$. Numerical examples will be given in the next section.

Theorem 4. *The explicit solution (35) to problem (32) converges, when $\alpha \nearrow 1$, to the unique solution to the classical Stefan problem given by*

$$\begin{array}{lll} (i) & \frac{\partial}{\partial t}u_2(x,t) = \lambda_2^2\,\frac{\partial^2}{\partial x^2}u_2(x,t), & 0<x<s(t),\ 0<t<T,\\ (ii) & \frac{\partial}{\partial t}u_1(x,t) = \lambda_1^2\,\frac{\partial^2}{\partial x^2}u_1(x,t), & x>s(t),\ 0<t<T,\\ (iii) & u_1(x,0)=U_i, & 0\le x,\\ (iv) & u_2(0,t)=U_0, & 0<t\le T,\\ (v) & u_1(s(t),t)=u_2(s(t),t)=U_m, & 0<t\le T,\\ (vi) & \frac{d}{dt}s(t) = k_1\frac{\partial}{\partial x}u_1(s(t),t) - k_2\frac{\partial}{\partial x}u_2(s(t),t), & 0<t\le T,\\ (vii) & s(0)=0 & \end{array} \tag{63}$$

Proof. The unique solution to problem (63) is the Neumann solution given in [37],

$$\begin{cases} z_2(x,t) = U_0 - (U_0-U_m)\dfrac{\operatorname{erf}\left(\frac{x}{2\lambda_2\sqrt{t}}\right)}{\operatorname{erf}(\nu_1\lambda)} \\ z_1(x,t) = U_i + (U_m-U_i)\dfrac{\operatorname{erfc}\left(\frac{x}{2\lambda_1\sqrt{t}}\right)}{\operatorname{erfc}(\nu_1)} w(t) = 2\eta_1\lambda_1\sqrt{t} \end{cases} \tag{64}$$

where η_1 is the unique solution to the equation

$$\frac{k_2(U_0-U_m)}{\lambda_1\lambda_2}\frac{\exp\left\{-\lambda^2x^2\right\}}{\sqrt{\pi}\operatorname{erf}(\lambda x)} - \frac{k_1(U_m-U_i)}{\lambda_1^2}\frac{\exp\left\{-x^2\right\}}{\sqrt{\pi}\operatorname{erfc}(x)} = \rho l x, \quad x>0. \tag{65}$$

Reasoning like in the previous theorem we can state that the solution to problem (32) is given by (35) where η_α is the unique positive solution to the equation

$$\frac{k_{\alpha_2}(U_0-U_m)}{\lambda_{\alpha_1}\lambda_{\alpha_2}\alpha}\frac{W\left(-2\lambda x,-\frac{\alpha}{2},\frac{\alpha}{2}\right)}{1-W\left(-2\lambda x,-\frac{\alpha}{2},1\right)} - \frac{k_{\alpha_1}(U_m-U_i)}{\lambda_{\alpha_1}^2\alpha}\frac{W\left(-2x,-\frac{\alpha}{2},\frac{\alpha}{2}\right)}{W\left(-2x,-\frac{\alpha}{2},1\right)} = \rho l x, \quad x>0. \tag{66}$$

Clearly, if we take $\alpha = 1$ in Eq. (66) we recover Eq. (65). Now, let be the sequence $\{\eta_\alpha\}_\alpha$, where η_α is the unique positive solution to Eq. (66) for each $0<\alpha<1$.

Defining the function

$$f_\alpha(x) = \frac{k_{\alpha_2}(U_0-U_m)}{\rho l\lambda_{\alpha_1}\lambda_{\alpha_2}\alpha}\frac{W\left(-2\lambda x,-\frac{\alpha}{2},\frac{\alpha}{2}\right)}{1-W\left(-2\lambda x,-\frac{\alpha}{2},1\right)} - \frac{k_{\alpha_1}(U_m-U_i)}{\rho l\lambda_{\alpha_1}^2\alpha}\frac{W\left(-2x,-\frac{\alpha}{2},\frac{\alpha}{2}\right)}{W\left(-2x,-\frac{\alpha}{2},1\right)}$$

for every $x\in\mathbb{R}^+$ and $0<\alpha\le 1$, it holds that $f_\alpha(\eta_\alpha)=\eta_\alpha$ for every $\alpha\in(0,1]$.

From [38] we know that f_1 is a strictly decreassing function in $\mathbb{R}^+$. Taking a close interval $[a, b] \subset \mathbb{R}^+$ such that $\eta_1 \in [a, b]$, using the uniform convergence over compact sets of all the positive functions given in Proposition 6 and proceding like in [33, Theorem 2] we can state that

$$\lim_{\alpha \nearrow 1} \eta_\alpha = \eta_1. \tag{67}$$

Finally, by taking the limit when $\alpha \nearrow 1$ in solution (35) and applying Proposition 6, the thesis holds. □

Remark 5. By using the same technique described before, we can improve the result given in [25, Theorem 3.3] by considering the functions g_α defined in $\mathbb{R}^+$ by

$$g_\alpha(x) = \frac{k_{\alpha_2}(U_0 - U_m)}{\rho l \lambda_{\alpha_1} \lambda_{\alpha_2}} \frac{\Gamma(1-\alpha/2)}{\Gamma(1+\alpha/2)} \frac{M_{\alpha/2}(-2\lambda x)}{1 - W\left(-2\lambda x, -\frac{\alpha}{2}, 1\right)} - \frac{k_{\alpha_1}(U_m - U_i)}{\rho l \lambda_{\alpha_1}^2 \alpha} \frac{\Gamma(1-\alpha/2)}{\Gamma(1+\alpha/2)} \frac{M_{\alpha/2}(-2x)}{W\left(-2x, -\frac{\alpha}{2}, 1\right)}$$

and a sequence $\{\xi_\alpha\}_\alpha$ were ξ_α is a solution to equation $g_\alpha(x) = x,\ x > 0$.

4. The dimesionless problems and numerical results

With the aim of giving different plots of the solutions obtained in Section 3, the problems (28) and (32) will be rewritten in their dimensionless form.

First, we give the following table exhibiting the usual heat conduction physical dimensions related to this work. Let us write **T** for temperature, **t** for time, **m** for mass and **X** for position.

$$\begin{array}{ccc} u_1, u_2, w_1, w_2 & \textit{temperatures} & [\mathbf{T}] \\ k_1, k_2 & \textit{thermal conductivities} & \left[\frac{\mathbf{m}\,\mathbf{X}}{\mathbf{T}\mathbf{t}^3}\right] \\ \rho & \textit{mass density} & \left[\frac{\mathbf{m}}{\mathbf{X}^3}\right] \\ c_1, c_2 & \textit{specific heats} & \left[\frac{\mathbf{X}^2}{\mathbf{T}\mathbf{t}^2}\right] \\ \lambda_i^2 = \frac{k_i}{\rho c},\ i=1,2 & \textit{diffusion coefficients} & \left[\frac{\mathbf{X}^2}{\mathbf{t}}\right] \\ l & \textit{latent heat per unit mass} & \left[\frac{\mathbf{X}^2}{\mathbf{t}^2}\right] \end{array} \tag{68}$$

Proposition 8. *For every $\alpha \in (0, 1)$ it holds that*

1. $[{}_0I^\alpha f] = [f]\mathbf{t}^\alpha$ *for every* $f = f(t) \in L^1(0, T)$.
2. $\left[{}_0^{RL}D^\alpha f\right] = \dfrac{[f]}{\mathbf{t}^\alpha}$ *for every* $f = f(t) \in AC[0, T]$.
3. $\left[{}_0^{C}D^\alpha f\right] = \dfrac{[f]}{\mathbf{t}^\alpha}$ *for every* $f = f(t) \in AC[0, T]$.

Recall that the parameters ν_α and μ_α given in (25) where added to preserve the consistency with respect to the units of measurements in equations (23) and (24). That is, being $[J] = [ku_x] = \frac{\mathbf{m}}{\mathbf{t}^3}$ and using Proposition 8, it holds that

$$\left[{}_0I_t^{1-\alpha}J(x,t)\right] = \left[\frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{J(x,\tau)}{(t-\tau)^\alpha} d\tau\right] = \frac{\mathbf{m}}{\mathbf{t}^{2+\alpha}}. \tag{69}$$

Then, replacing (69) in (23) one gets

$$[\nu_\alpha] = \frac{\left[k\frac{\partial u}{\partial x}\right]}{\left[{}_{h(x)}I_t^{1-\alpha}J\right])} = \frac{1}{\mathbf{t}^{1-\alpha}}. \tag{70}$$

Therefore,

$$[\mu_\alpha] = \mathbf{t}^{1-\alpha}. \tag{71}$$

Proposition 9. *Let x_0 be a characteristic position and let U^* be a characteristic temperature. Then, if the following rescaling variable are considered*

$$y = \frac{x}{x_0}, \qquad \tau = \frac{\lambda_1^2}{x_0^2} t \quad \text{and} \qquad \tilde{w} = \frac{w}{U^*}, \tag{72}$$

it holds that

$${}_0I_t^\alpha(w_x(x,t)) = \frac{U^* x_0}{\lambda_1^2} \left(\frac{\lambda_1^2}{x_0^2}\right)^{1-\alpha} {}_0I_\tau^\alpha(\tilde{w}_y(y,\tau)), \tag{73}$$

$$ {}_0I_t^\alpha(w_{xx}(x,t)) = \frac{U^*}{\lambda_1^2}\left(\frac{\lambda_1^2}{x_0^2}\right)^{1-\alpha} {}_0I_\tau^\alpha(\tilde{w}_{yy}(y,\tau)) \tag{74} $$

and

$$ {}_0^{RL}D_t^{1-\alpha}(w_{xx}(x,t)) = \frac{U^*}{x_0^2}\left(\frac{\lambda_1^2}{x_0^2}\right)^{1-\alpha} {}_0^{RL}D_\tau^{1-\alpha}(\tilde{w}_{yy}(y,\tau)). \tag{75} $$

Proof. We prove here Eq. (73). By considering the rescaling (72), we have

$$ \tilde{w}(y,\tau) = \frac{w(x(y),t(\tau))}{U^*}. \tag{76} $$

Then

$$ {}_0I_t^\alpha(w_x(x,t)) = \frac{1}{\Gamma(\alpha)}\int_0^t \frac{w_x(x,z)}{(t-z)^{1-\alpha}}dz = \frac{U^*}{\Gamma(\alpha)}\int_0^t \frac{\frac{1}{x_0}\tilde{w}_y(y,\tau(z))}{(t-z)^{1-\alpha}}dz = $$

$$ = \frac{U^*}{\Gamma(\alpha)}\int_0^{\frac{\lambda_1^2}{x_0^2}t} \frac{\tilde{w}_y(y,v)}{\left(\frac{x_0^2}{\lambda_1^2}\right)^{1-\alpha}(\frac{\lambda_1^2}{x_0^2}t-v)^{1-\alpha}}\frac{x_0}{\lambda_1^2}dv = \frac{U^*}{x_0}\left(\frac{x_0^2}{\lambda_1^2}\right)^\alpha {}_0I_\tau^\alpha(\tilde{w}_y(y,\tau)). $$

□

Now, let us consider problems (28) and (32). By using Proposition 9 it is easy to state that the governing equation (32-i) is equivalent to the following equation

$$ \frac{\partial}{\partial\tau}\tilde{w}_2(y,\tau) = \lambda^2\mu_\alpha\left(\frac{\lambda_1^2}{x_0^2}\right)^{1-\alpha} {}_0^{RL}D_\tau^{1-\alpha}\tilde{w}_{2yy}(y,\tau). \tag{77} $$

Note that $\mu_\alpha = \left(\frac{x_0^2}{\lambda_1^2}\right)^{1-\alpha}$ is an admissible parameter because $[\mu_\alpha] = \mathbf{t}^{1-\alpha}$ and that $\lim_{\alpha\nearrow 1}\mu_\alpha = 1$. Then, the parameter $\mu_\alpha\left(\frac{\lambda_1^2}{x_0^2}\right)^{1-\alpha}$ in Eq. (77) can be omitted.

Analogously, transforming the governing equations, the Stefan conditions, the initial and boundary data, taking $U_m = 0$ and $U^* = |U_i|$ in problems (28) and (32), it follows that the dimensionless associated problems are given by

$$ \begin{array}{lll} (i) & {}_0^CD_\tau^\alpha\tilde{u}_2(y,\tau) = \lambda^2\tilde{u}_{2yy}(y,\tau), & 0<y<\tilde{s}(\tau),\ 0<\tau<\tilde{T}, \\ (ii) & {}_0^CD_\tau^\alpha\tilde{u}_1(y,\tau) = \tilde{u}_{2yy}(y,\tau), & y>\tilde{s}(\tau),\ 0<\tau<\tilde{T}, \\ (iii) & \tilde{u}_1(y,0) = -1, & 0\le x, \\ (iv) & \tilde{u}_2(0,\tau) = \frac{U_0}{|U_i|}, & 0<\tau\le\tilde{T}, \\ (v) & \tilde{u}_1(\tilde{s}(\tau),\tau) = \tilde{u}_1(\tilde{s}(\tau),\tau) = 0, & 0<\tau\le\tilde{T}, \\ (vi) & {}_0^CD_\tau^\alpha\tilde{s}(\tau) = \text{Ste}\left[\tilde{u}_{1y}(\tilde{s}(\tau)^+,\tau) - \frac{k_2}{k_1}\tilde{u}_{2y}(\tilde{s}(\tau)^-,\tau)\right], & 0<\tau\le\tilde{T}, \\ (vii) & \tilde{s}(0) = 0. & \end{array} \tag{78} $$

and

$$ \begin{array}{lll} (i) & \tilde{w}_{2\tau}(y,\tau) = \lambda^2\, {}_0^{RL}D_\tau^{1-\alpha}w_{2yy}(y,\tau), & 0<y<\tilde{r}(\tau),\ 0<\tilde{t}<\tilde{T}, \\ (ii) & \tilde{w}_{1\tau}(y,\tau) = {}_0^{RL}D_\tau^{1-\alpha}w_{1yy}(y,\tau), & y>\tilde{r}(\tau),\ 0<\tau<\tilde{T}, \\ (iii) & \tilde{w}_1(y,0) = -1, & 0\le y, \\ (iv) & \tilde{w}_2(0,t) = \frac{U_0}{|U_i|}, & 0<\tau\le\tilde{T}, \\ (v) & \tilde{w}_1(\tilde{r}(\tau),\tau) = \tilde{w}_2(\tilde{r}(\tau),\tau) = 0, & 0<\tau\le\tilde{T}, \\ (vi) & \frac{d}{dt}\tilde{r}(\tau) = \text{Ste}\left[{}_0^{RL}D_\tau^{1-\alpha}w_{1y}(y,\tau)\Big|_{(\tilde{r}(\tau)^+,\tau)} - \frac{k_2}{k_1}\, {}_0^{RL}D_\tau^{1-\alpha}\tilde{w}_{2y}(y,\tau)\Big|_{(\tilde{r}(\tau)^-,\tau)}\right], & 0<\tau\le\tilde{T}, \\ (vii) & \tilde{r}(0) = 0. & \end{array} \tag{79} $$

where $\lambda = \frac{\lambda_2}{\lambda_1}$ and $\text{Ste} = \frac{|U_i|c_1}{l}$ is the dimensionless Stefan number.

In the following table there are different tests, i.e. sets of parameters for λ, $\frac{k_2}{k_1}$, $U = \frac{U_0}{|U_i|}$ and *Ste*. For each test in Table 1 a correpondig graphic of the comparison between the ξ_α and η_α is given in Fig. 3.

At the end, we present in Figs. 4 and 5 some color maps of temperature for tests 2 and 3, respectively. Three values of α are considered and as expected from Theorem 4, both solutions approach themselves when $\alpha\nearrow 1$.

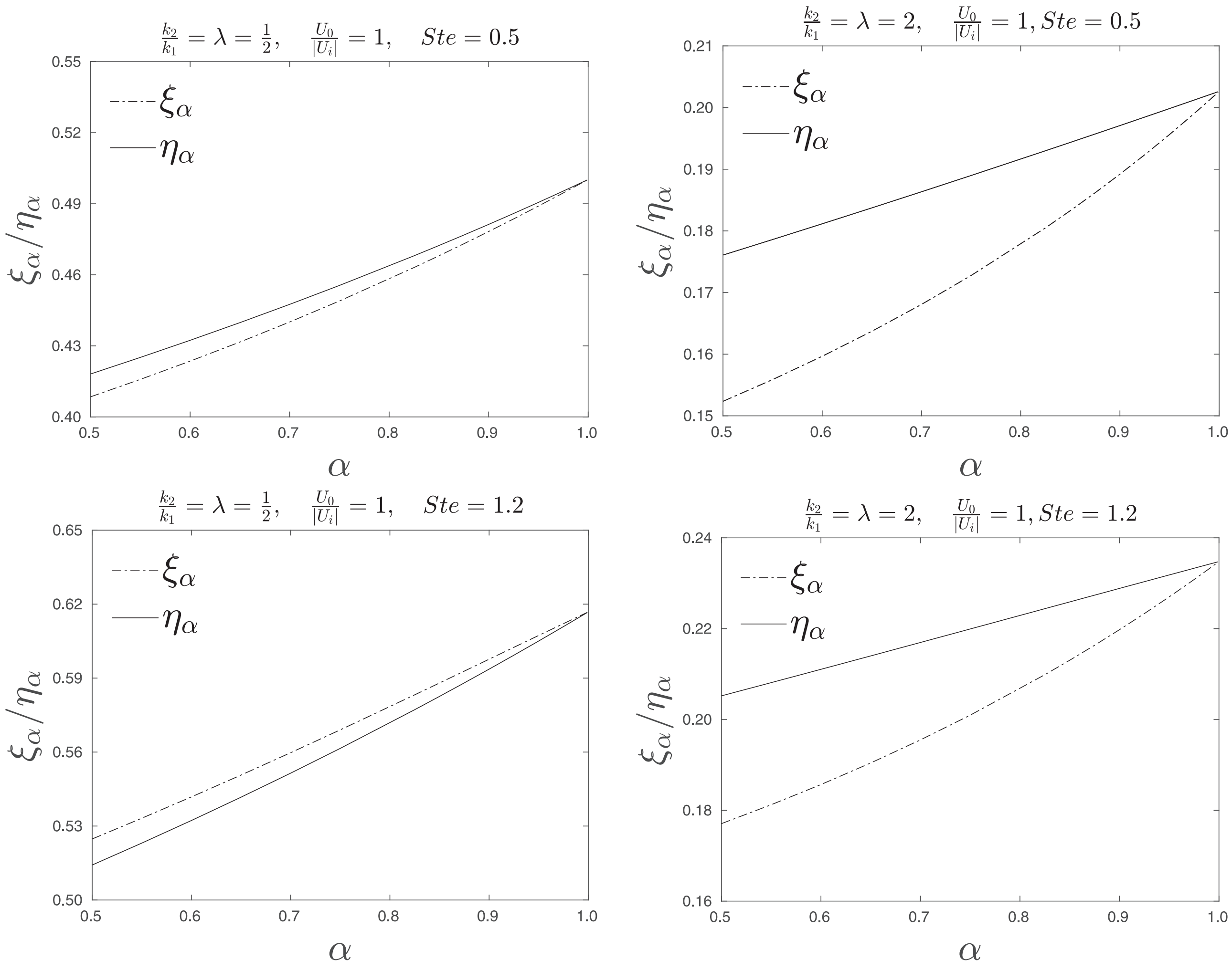

Fig. 3. ξ_α vs. η_α for different values of α.

Table 1
Different set of tests.

	λ	$\frac{k_2}{k_1}$	$U = \frac{U_0}{\lvert U_i \rvert}$	*Ste*
Test 1	0.5	0.5	1.0	0.5
Test 2	2.0	2.0	1.0	0.5
Test 3	0.5	0.5	1.0	1.2
Test 4	2.0	2.0	1.0	1.2

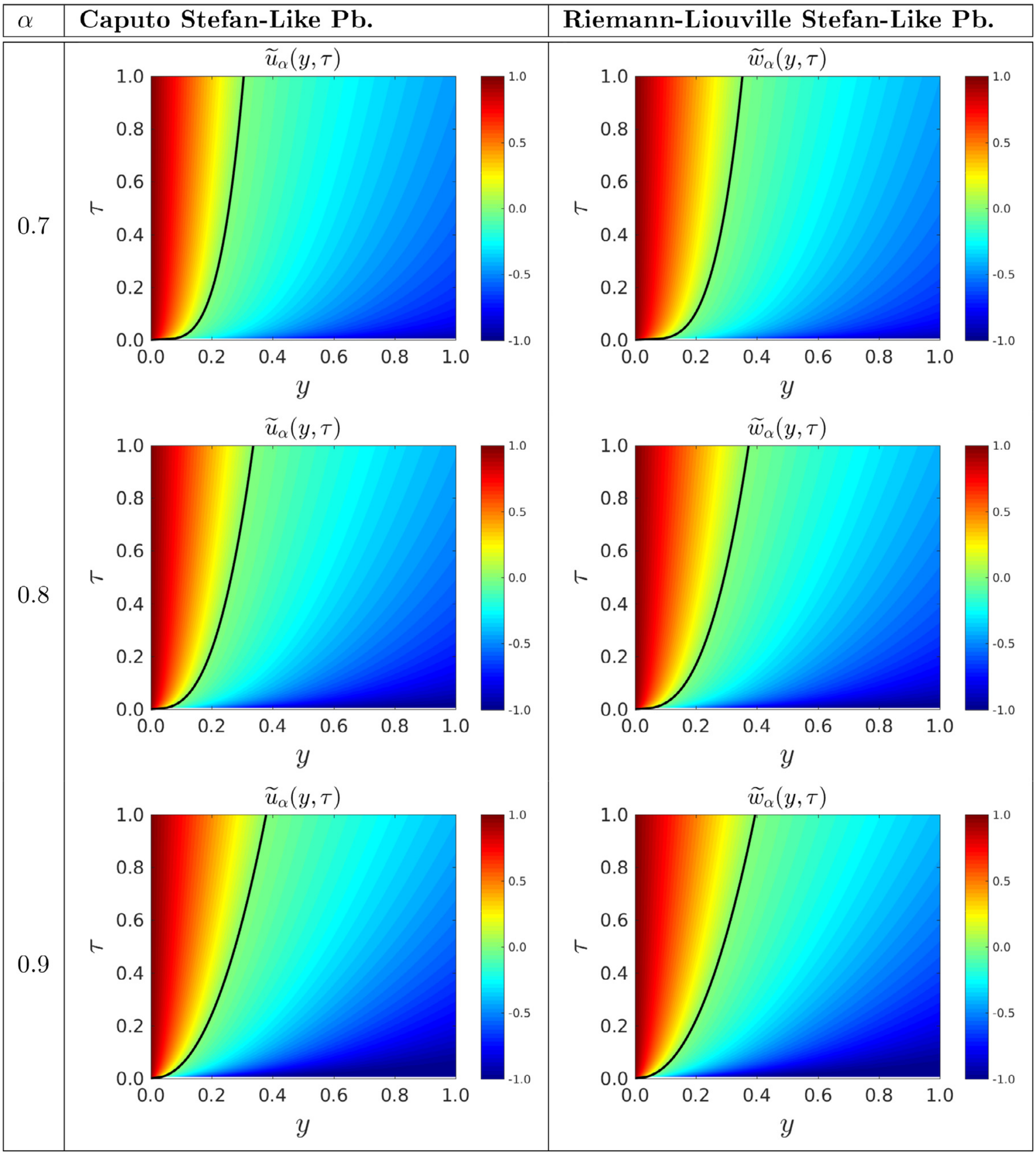

Fig. 4. Caputo's approach Solutions Vs. Riemann-Liouville's aproach Solutions for Test 2

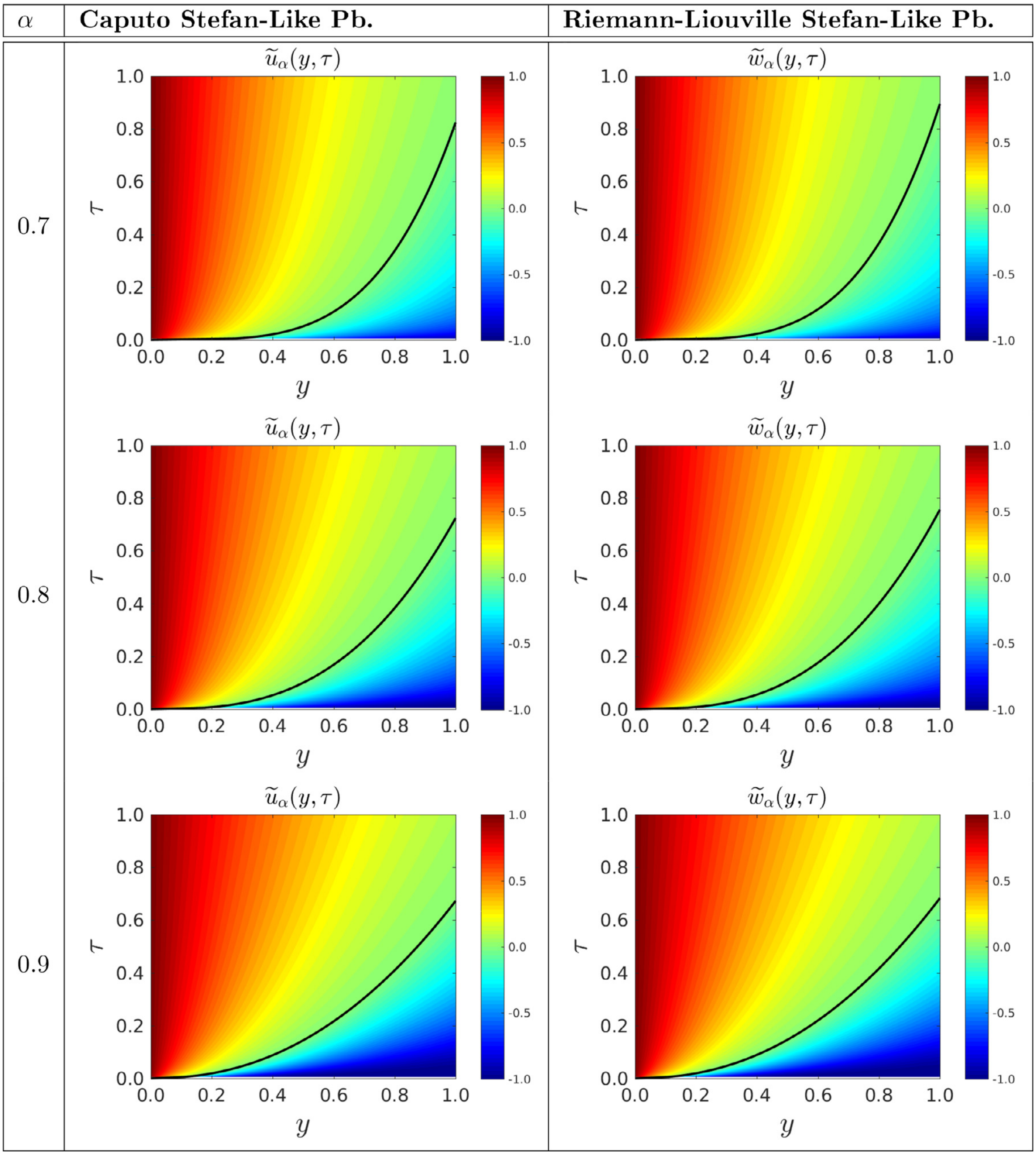

Fig. 5. Caputo's approach Solutions Vs. Riemann-Liouville's aproach Solutions for Test 3.

5. Conclusion

We have presented two different fractional two-phase Stefan-like problems for the Riemann-Liouville and Caputo derivatives of order $\alpha \in (0, 1)$ with the particularity that, if the parameter $\alpha = 1$ is replaced in both problems, we recover the same classical Stefan problem. In both cases, explicit solutions in terms of self-similar variables where given. It was interesting to see that, the role of the different "fractional Stefan conditions" associated to each problem was decisive to conclude that the solutions obtained were different. Also, as expected, we have proved that the two different solutions converge to the same triple of limits functions when α tends to 1, where this "limit solution" is the classical solution to the classical Stefan problem mentioned before.

Finally we would like to comment some open problems related to this research paper: is it possible to find explicit solutions to problem (27) derived in [19]? Which is the best numerical approach for this kind or problems? If we could estimate any solution to problem (27), how different it would be to the explicit solutions obtained in the present work?

Declarations of Competing Interest

None.

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