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On a two-phase Stefan problem with convective boundary condition including a density jump at the free boundary

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We consider a two-phase Stefan problem for a semi-infinite body $x > 0$, with a convective boundary condition including a density jump at the free boundary with a time-dependent heat transfer coefficient of the type $h/\sqrt{t}$, $h > 0$ whose solution was given in D. A. Tarzia, PAMM. Proc. Appl. Math. Mech. 7, 1040307–1040308 (2007). We demonstrate that the solution to this problem converges to the solution to the analogous one with a temperature boundary condition when the heat transfer coefficient $h \to +\infty$. Moreover, we analyze the dependence of the free boundary respecting to the jump density.

KEYWORDS
two-phase Stefan problem, density jump, asymptotic behavior, phase-change process

MSC CLASSIFICATION
35R35; 80A22; 35C05

1 | INTRODUCTION

Many heat diffusion processes that involve phase change are the so-called Stefan problems.[1-10] These are characterized by the presence of an unknown moving surface that separates both phases.

In the classical Stefan problem and in many others, it is assumed that there is no density difference between both phases during the phase transition. However, two types of density change are relevant in phase-change processes.[1] One is due to the dependence of density on temperature arising in heat transfer process and the other is due to the difference between the solid and liquid densities at the melting temperature. According to the bibliography, there are also several papers[11-14] that analyze free boundary problems including a density change and the induced motion caused by the phase transition.

In reality, melting and solidification processes are always affected by changes in the density, which translate physically to the shrinkage or expansion of one of the phases. For instance, in countries with cold climates, pipe bursting is a recurrent problem. As water freezes, the molecules crystallize into a hexagonal form, which takes up more space than molecules in the liquid form thus causing the pipe to burst. The changes in the mathematical model when the density jump between phases is introduced are significant.

We consider the two-phase Stefan problem for a semi-infinite material $x > 0$, taking into account a density jump under the change of phase studied in previous studies.[15,16] The free boundary $s = s(t) > 0$, defined for $t > 0$, and the temperatures $\theta_i(x,t)$, $i = 1, 2$ satisfying the following conditions (problem P_1):

$$\alpha_1 \theta_{1xx} = \theta_{1t} \quad , \quad 0 < x < s(t) \quad , \quad t > 0 \quad , \tag{1}$$

$$\alpha_2 \theta_{2xx} + \frac{\rho_1 - \rho_2}{\rho_2} \dot{s}(t) \theta_{2x} = \theta_{2t} \ , \ x > s(t) \ , \quad t > 0 \quad , \tag{2}$$

$$\theta_1 (s(t), t) = \theta_2 (s(t), t) = 0 \ , \quad t > 0 \quad , \tag{3}$$

$$k_1\theta_{1x}(s(t),t) - k_2\theta_{2x}(s(t),t) = \rho_1 l \dot{s}(t)\ , \quad t>0\ , \tag{4}$$

$$\theta_2(x,0) = \theta_0\ , \quad x>0\ , \tag{5}$$

$$k_1\theta_{1x}(0,t) = \frac{h}{\sqrt{t}}\left[\theta_1(0,t) - \theta^*\right]\ , \quad t>0\ , \tag{6}$$

$$s(0) = 0 \tag{7}$$

where $\theta_0 > 0$ is the initial constant temperature, $\alpha_1 = a_1^2 = \frac{k_1}{\rho_1 c_1}$ is the thermal diffusivity of the solid phase, $\alpha_2 = a_2^2 = \frac{k_2}{\rho_2 c_2}$ is the thermal diffusivity of the liquid phase, ρ_1 is the mass density of solid phase, ρ_2 is the mass density of the liquid phase, $\theta_1(x,t)$ is the solid phase temperature, $\theta_2(x,t)$ is the liquid phase temperature, θ^* is the bulk temperature of the medium with $\theta^* < \theta_1(0,t) < 0$, l is the latent heat of fusion of the medium, and $h > 0$ is the heat transfer coefficient. Moreover, without loss of generality, we take a null phase-change temperature. From now on, we assume that $\rho_1 \neq \rho_2$; for most materials, they differ by up to 10% and in extreme cases by up to 30%.[1]

In previous studies,[15,16] an explicit solution of similarity type for the temperature of both phases and the solid-liquid interface was obtained. The explicit solution to the particular case $\rho_1 = \rho_2$ was given in Zubair and Chaudhry.[17] In Tarzia,[18] the relationship between Neumann solutions for two-phase Stefan problems with convective and temperature boundary conditions was studied for the case with null density jump.

In Báncora and Tarzia,[19] an analogous problem to P_1 was considered with an outward heat flux $q(t) = q_0/\sqrt{t}\,(q_0 > 0)$ at the fixed face $x = 0$. If the constant q_0 satisfies a certain inequality, there exists an explicit solution of the Neumann type for the two-phase Stefan problem, which generalizes Tarzia[20] considering the jump density in the phase-change problem for a semi-infinite body. Following Neumann's idea for the two-phase Stefan problem,[4,21-23] the explicit solution to P_1 with a constant temperature condition at the fixed face $x = 0$ is obtained.

In several manuscripts,[24-30] the behavior of the solution to a free boundary problem in respect of the heat transfer coefficient h was studied.

In Naaktgeboren,[31] the classical one-phase Stefan problem is presented in dimensionless form with a time-varying heat-power boundary condition. The asymptotic behavior of the solution for the generalized form of the Biot number $Bi \to 0$ was studied from a physical point of view. In Briozzo and Tarzia,[32] the mathematical analysis of this asymptotic behavior of the solution with respect to the heat transfer coefficient was considered, and an order of convergence was also obtained.

In Section 2, we wish to investigate the asymptotic behavior of solution to problem P_1 when $h \to +\infty$. We prove that the solution to P_1 converges to the solution to P_2, which is analogous to the Stefan problem[32] with a temperature condition on the fixed face $x = 0$, and it is given by (1) -(5), (7), and

$$\theta_1(0,t) = \theta^* < 0\ , \quad t>0\ , \tag{8}$$

instead of (6).

In Section 3,we study the free boundary $s = s(t)$ and its dependence on the jump density where several results are given.

2 | ASYMPTOTIC BEHAVIOR OF THE SOLUTION TO P_1 WHEN $h \to +\infty$

Taking into account the results proved in Tarzia and Tarzia,[15,16] if the heat transfer coefficient h satisfies

$$h > h_0 = \frac{k_2\theta_0}{a_2\sqrt{\pi}(-\theta^*)}, \tag{9}$$

then Problem P_1 has a unique similarity type solution given by

$$\theta_1(x,t) = \frac{-h\sqrt{\pi}a_1\theta^*\left[\operatorname{erf}\left(\frac{x}{2a_1\sqrt{t}}\right) - \operatorname{erf}\left(\frac{\gamma}{a_1}\right)\right]}{k_1 + h\sqrt{\pi}a_1\operatorname{erf}\left(\frac{\gamma}{a_1}\right)}, \quad 0<x<s(t),\ t>0, \tag{10}$$

$$\theta_2(x,t) = \frac{\theta_0\left[\operatorname{erf}\left(\frac{\gamma}{a_2}\epsilon + \frac{x}{2a_2\sqrt{t}}\right) - \operatorname{erf}\left(\frac{\gamma}{a_2}(\epsilon+1)\right)\right]}{\operatorname{erfc}\left(\frac{\gamma}{a_2}(\epsilon+1)\right)}, \quad x>s(t),\ t>0, \tag{11}$$

where

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-r^2)\, dr, \quad \operatorname{erfc}(x) = 1 - \operatorname{erf}(x), \quad \epsilon = \frac{|\rho_1 - \rho_2|}{\rho_2}, \tag{12}$$

and

$$s(t) = 2\gamma\sqrt{t}\ ,\ t > 0 \tag{13}$$

is the free boundary with γ as the unique solution to equation

$$F(x) = \sqrt{\pi} a_1 a_2 \rho_1 l x. \tag{14}$$

Function F is defined by

$$F(x) = F_1(x) - F_2(x), \tag{15}$$

where F_i for $i = 1, 2$ are given by

$$F_1(x) = \frac{-\theta^* k_1 a_1 a_2 h \sqrt{\pi} \exp\left(-\frac{x^2}{a_1^2}\right)}{k_1 + h\sqrt{\pi} a_1 \operatorname{erf}\left(\frac{x}{a_1}\right)}, \tag{16}$$

$$F_2(x) = \frac{\theta_0 k_2 a_1 \exp\left(-\frac{x^2}{a_2^2}(\epsilon+1)^2\right)}{\operatorname{erfc}\left(\frac{x}{a_2}(\epsilon+1)\right)}, \tag{17}$$

which satisfy the following properties:

$$F_1(0) = -\theta^* a_1 a_2 h \sqrt{\pi}, \quad F_1(+\infty) = 0, \quad F_1'(x) < 0, \tag{18}$$

$$F_2(0) = \theta_0 k_2 a_1, \quad F_2(+\infty) = +\infty, \quad F_2'(x) > 0. \tag{19}$$

Moreover, it was proved that if $h \le h_0$, then there is no solution to problem P_1, just obtaining a problem of heat conduction in the initial liquid phase.

Now we will study the behavior of solution to (10), (11), and (13) when the heat transfer coefficient $h \to +\infty$.[33]

From now on, we assume that $\theta_i = \theta_{ih}(x, t), i = 1, 2$ and $s = s_h(t)$ are the solution to the problems (1) -(6) for each $h > h_0$. We will prove that θ_{ih}, s_h converges to $u_i = u_i(x, t)$, $y = y(t)$ for $i = 1, 2$, respectively, which is the solution to the following two-phase parabolic free boundary problem P_2:

$$\alpha_1 u_{1xx} = u_{1t}\quad ,\quad 0 < x < y(t)\ ,\ \ t > 0\ , \tag{20}$$

$$\alpha_2 u_{2xx} + \frac{\rho_1 - \rho_2}{\rho_2} \dot{y}(t) u_{2x} = u_{2t}\ ,\ x > y(t)\ ,\ \ t > 0\ , \tag{21}$$

$$u_1\,(y(t), t) = u_2\,(y(t), t) = 0\ ,\ \ t > 0\ , \tag{22}$$

$$k_1 u_{1x}\,(y(t), t) - k_2 u_{2x}\,(y(t), t) = \rho_1 l\,\dot{y}(t)\ ,\ \ t > 0\ , \tag{23}$$

$$u_2(x, 0) = \theta_0\ ,\ \ t > 0\ , \tag{24}$$

$$u_1\,(0, t) = \theta^* < 0\ ,\ \ t > 0\ , \tag{25}$$

$$y(0) = 0. \tag{26}$$

The explicit solution to the problems (20) -(26)[19] is given by

$$u_1(x, t) = \frac{-\theta^* \left[\operatorname{erf}\left(\frac{x}{2a_1\sqrt{t}}\right) - \operatorname{erf}\left(\frac{\gamma^*}{a_1}\right)\right]}{\operatorname{erf}\left(\frac{\gamma^*}{a_1}\right)}, 0 \le x \le y(t), \tag{27}$$

$$u_2(x, t) = \frac{\theta_0 \left[\operatorname{erf}\left(\frac{\gamma^* \epsilon}{a_2} + \frac{x}{2a_2\sqrt{t}}\right) - \operatorname{erf}\left(\frac{\gamma^*}{a_2}(\epsilon+1)\right)\right]}{\operatorname{erfc}\left(\frac{\gamma^*}{a_2}(\epsilon+1)\right)}, x \ge y(t), \tag{28}$$

where the free boundary is

$$y(t) = 2\gamma^* \sqrt{t}\ ,\ t > 0, \tag{29}$$

and γ^* is the unique solution of

$$G(x) = \rho_1 l a_1 a_2 \sqrt{\pi} x\ ,\ x > 0, \tag{30}$$

with

$$G(x) = \frac{-\theta^* a_2 k_1 \exp\left(-\frac{x^2}{a_1^2}\right)}{\operatorname{erf}\left(\frac{x}{a_1}\right)} - F_2(x). \tag{31}$$

In order to get the uniform convergence, it will be necessary to prove some preliminary results.
First, we will prove that γ_h converges to γ^* when $h \to +\infty$.

Lemma 1. *The sequence $\{\gamma_h\}$ is increasing and bounded. Moreover,*

$$\lim_{h \to +\infty} \gamma_h = \gamma^*. \tag{32}$$

Proof. From properties of $F = F_h(x)$ defined by (15), we have

a) $h_0 < h_1 \le h_2 \Rightarrow F_{h_1}(x) \le F_{h_2}(x), \quad \forall x \ge 0.$

b) $F_h(x) \le G(x), \quad \forall x \ge 0, \quad h > h_0.$

c) $\lim_{h \to +\infty} F_h(x) = G(x) \quad \forall x \ge 0.$

Then the sequence $\{\gamma_h\}$ is increasing and bounded, then (32) holds. □

Corollary 1. *For each $t > 0$, the sequence $\{s_h(t)\}$ is monotonically increasing, and $\lim_{h \to +\infty} s_h(t) = y(t)$.*

Remark 1. Taking into account that for $h > h_0$ there exists a unique $\gamma = \gamma_h$ solution to (14), we have the following relation:

$$h = P(\gamma_h), \tag{33}$$

where function $P = P(x) \in C^1(0, \gamma^*)$ is defined by

$$P(x) = \frac{k_1 R(x)}{-\theta^* k_1 a_1 a_2 h \sqrt{\pi} \exp\left(-\frac{x^2}{a_1^2}\right) - \sqrt{\pi} a_1 \operatorname{erf}\left(\frac{x}{a_1}\right) R(x)}, \tag{34}$$

with $R(x) = F_2(x) + \sqrt{\pi} a_1 a_2 \rho_1 l x$.

Moreover, P satisfies $P(0^+) = h_0, \quad P(\gamma^*) = +\infty, \quad P'(x) > 0.$

Now we consider the family of functions $\{\theta_{1h}\}$. We first show that $\{\theta_{1h}(x,t)\}$ converges pointwise to $u_1(x,t)$ on $[0, y(t)]$ when $h \to +\infty$. To this end, we define an extension $\tilde{\theta}_{1h} = \tilde{\theta}_{1h}(x,t) \in C^1\,[0, y(t)]$ of θ_{1h} as follows:

$$\tilde{\theta}_{1h}(x,t) = \begin{cases} \theta_{1h}(x,t) & if \quad 0 \le x < s_h(t) \\ \dfrac{-h\theta^* \exp\left(-\frac{\gamma_h^2}{a_1^2}\right)}{k_1 + h\sqrt{\pi} a_1 \operatorname{erf}\left(\frac{\gamma_h}{a_1}\right)} \dfrac{(x - s_h(t))}{\sqrt{t}} & if \quad s_h(t) \le x \le y(t). \end{cases} \tag{35}$$

Lemma 2. *For each $t > 0$ and $x \in [0, y(t)]$, we have*

$$\lim_{h \to +\infty} \tilde{\theta}_{1h}(x,t) = u_1(x,t).$$

Proof. Let $t > 0$ and $x \in [0, y(t))$. By Corollary 1, there exists $h^* = h^*(x) > h_0$ such that $x \in [0, s_h(t))$ for all $h \ge h^*$, then

$$\lim_{h \to +\infty} \tilde{\theta}_{1h}(x,t) = \lim_{h \to +\infty} \theta_{1h}(x,t) =$$

$$\lim_{h\to+\infty} \frac{-h\sqrt{\pi}a_1\theta^*}{k_1 + h\sqrt{\pi}a_1 \mathrm{erf}\left(\frac{\gamma_h}{a_1}\right)} \left[\mathrm{erf}\left(x/2a_1\sqrt{t}\right) - \mathrm{erf}\left(\gamma_h/a_1\right)\right].$$

Taking into account Lemma 1 and (27), we obtain that the sequence $\{\tilde{\theta}_{1h}(x,t)\}$ converges to $u_1(x,t)$.

If $x = y(t)$, then

$$\lim_{h\to+\infty} \tilde{\theta}_{1h}(y(t),t) = \lim_{h\to+\infty} \frac{-h\theta^* \exp\left(-\frac{\gamma_h^2}{a_1^2}\right)}{k_1 + h\sqrt{\pi}a_1 \mathrm{erf}(\gamma_h/a_1)} \frac{(y(t) - s_h(t))}{\sqrt{t}}$$
$$= \lim_{h\to+\infty} \frac{-h\theta^* \exp\left(-\frac{\gamma_h^2}{a_1^2}\right)}{k_1 + h\sqrt{\pi}a_1 \mathrm{erf}(\gamma_h/a_1)} 2\left(\gamma^* - \gamma_h\right) = 0.$$

Hence, the sequence $\{\tilde{\theta}_{1h}(x,t)\}$ converges to $u_1(x,t)$ pointwise on $[0, y(t)]$ for each $t > 0$. □

Lemma 3. *Functions $\tilde{\theta}_{1h} \in C^1\,[0, y(t)]$ satisfy $\left|\frac{\partial\tilde{\theta}_{1h}}{\partial x}\right| \le \frac{M}{\sqrt{t}}$ on $[0, y(t)]$ for all $h > h_0$, $t > 0$.*

Proof. Let $h > h_0$, $t > 0$ and $x \in [0, y(t)]$.

If $x \in [s_h(t), y(t)]$, then

$$\frac{\partial\tilde{\theta}_{1h}(x,t)}{\partial x} = \frac{-h\theta^*}{k_1 + h\sqrt{\pi}a_1 \mathrm{erf}\left(\frac{\gamma_h}{a_1}\right)} \frac{\exp\left(-\frac{\gamma_h^2}{a_1^2}\right)}{\sqrt{t}}.$$

We define

$$J(h) = \frac{-h\theta^*}{k_1 + h\sqrt{\pi}a_1 \mathrm{erf}\left(\frac{\gamma_h}{a_1}\right)} = \frac{-h\theta^*}{k_1 + h\sqrt{\pi}a_1 \mathrm{erf}\left(\frac{P^{-1}(h)}{a_1}\right)},$$

where P^{-1} is the inverse function of P, which is given by (34). For $h > h_0$, J is a positive and continuous function, it satisfies

$$\lim_{h\to h_0^+} J(h) = \frac{-\theta^* h_0}{k_1}, \quad \lim_{h\to+\infty} J(h) = \frac{-\theta^*}{\sqrt{\pi}a_1 \mathrm{erf}\left(\frac{\gamma^*}{a_1}\right)}. \tag{36}$$

Therefore J is uniformly bounded on $(h_0, +\infty)$, and there exists $M > 0$ such that $0 < J(h) \le M$ for all $h > h_0$.

Otherwise, if $x \in [0, s_h(t))$, we have

$$\frac{\partial\tilde{\theta}_{1h}(x,t)}{\partial x} = \frac{-h\theta^*}{k_1 + h\sqrt{\pi}a_1 \mathrm{erf}\left(\frac{\gamma_h}{a_1}\right)} \frac{\exp\left(-\frac{x^2}{4a_1^2 t}\right)}{\sqrt{t}}.$$

Then for $x \in [0, y(t)]$ and $h > h_0$, we have $\left|\frac{\partial\tilde{\theta}_{1h}}{\partial x}(x,t)\right| \le \frac{M}{\sqrt{t}}$; this is to say that $\{\tilde{\theta}_{1h}(x,t)\}$ are uniformly bounded on $[0, y(t)]$, and this is precisely the assertion of the lemma. □

Corollary 2. *For any $t > 0$, the functions $\{\tilde{\theta}_{1h}(x,t)\}$ are equicontinuous on $[0, y(t)]$.*

Theorem 1. *For each $t > 0$, we have the family of functions $\{\tilde{\theta}_{1h}\}$ converges uniformly to u_1 for $h \to +\infty$ on $[0, y(t)]$.*

Proof. From Corollary 2, Lemma 2, and by using Ascoli-Arzela lemma, we obtain the uniform convergence of $\{\tilde{\theta}_{1h}\}$ to u_1 on $[0, y(t)]$. □

We now apply a similar argument to prove that the family of functions $\{\theta_{2h}\}$ converges uniformly to u_2 for $h \to +\infty$ on $[y(t), +\infty)$.

Lemma 4. *For each $t > 0$ and $x \in [y(t), +\infty)$, we have*

$$\lim_{h\to+\infty} \theta_{2h}(x,t) = u_2(x,t).$$

Proof. Let $t > 0$ and $x > y(t)$. We have

$$\lim_{h\to+\infty} \theta_{2h}(x,t) = \lim_{h\to+\infty} \frac{\theta_0 \left[\mathrm{erf}\left(\frac{\gamma_h}{a_2}\epsilon + \frac{x}{2a_2\sqrt{t}}\right) - \mathrm{erf}\left(\frac{\gamma_h}{a_2}(\epsilon+1)\right)\right]}{\mathrm{erfc}\left(\frac{\gamma_h}{a_2}(\epsilon+1)\right)}.$$

Taking into account Lemma 1 and (28), we obtain that the sequence $\{\theta_{2h}(x,t)\}$ converges to $u_2(x,t)$.

If $x = y(t)$, then

$$\lim_{h\to+\infty} \theta_{2h}(y(t),t) = \lim_{h\to+\infty} \frac{\theta_0 \left[\mathrm{erf}\left(\frac{\gamma_h}{a_2}\epsilon + \frac{\gamma^*}{a_2}\right) - \mathrm{erf}\left(\frac{\gamma_h}{a_2}(\epsilon+1)\right)\right]}{\mathrm{erfc}\left(\frac{\gamma_h}{a_2}(\epsilon+1)\right)} = 0.$$

Therefore, the sequence $\{\theta_{2h}(x,t)\}$ converges to $u_2(x,t)$ pointwise on $[y(t), +\infty]$ for each $t > 0$. □

Lemma 5. *The functions $\theta_{2h} \in C^1[y(t), +\infty)$ satisfy $\left|\frac{\partial\theta_{2h}}{\partial x}\right| \le \frac{N}{\sqrt{t}}$ on $[y(t), +\infty)$ for all $h > h_0$, $t > 0$.*

Proof. Let $t > 0$ and $x \in [y(t), +\infty)$. We have

$$\frac{\partial\theta_{2h}}{\partial x}(x,t) = \frac{\theta_0 \exp\left(-\left(\frac{\gamma_h}{a_2}\epsilon + \frac{x}{2a_2\sqrt{t}}\right)^2\right)}{a_2\sqrt{\pi}\mathrm{erfc}\left(\frac{\gamma_h}{a_2}(\epsilon+1)\right)\sqrt{t}},$$

and taking into account $\gamma_h \le \gamma^*$, we obtain

$$\frac{\partial\theta_{2h}}{\partial x}(x,t) \le \frac{\theta_0}{a_2\sqrt{\pi}\mathrm{erfc}\left(\frac{\gamma^*}{a_2}(\epsilon+1)\right)\sqrt{t}}.$$

Thus, for $x \in [y(t), +\infty)$ and $h > h_0$, we have $\left|\frac{\partial\theta_{2h}}{\partial x}(x,t)\right| \le \frac{N}{\sqrt{t}}$. □

Theorem 2. *For each $t > 0$, we have the family of functions $\{\theta_{2h}\}$, which converges uniformly to u_2 for $h \to +\infty$ on $[y(t), +\infty)$.*

Proof. By the above lemma, for any $t > 0$, the functions $\theta_{2h}(x,t)$ are equicontinuous on $[y(t), +\infty)$, and from Lemma 4, they converge pointwise to $u_2(x,t)$ for $h \to +\infty$. Then, by Ascoli-Arzela lemma, we obtain their uniform convergence on $[y(t), +\infty)$. □

3 | MONOTONE DEPENDENCE OF THE FREE BOUNDARY WITH RESPECT TO THE DENSITY JUMP

For the two- phase Stefan problem P_1 with a density jump given by $\rho_2 - \rho_1$, we have that for $h > h_0$, the solution given by (10)-(11) and (13) depends of $\epsilon = \frac{|\rho_1-\rho_2|}{\rho_2}$. Next, we will study the behavior of the solution when the jump density goes to 0. Moreover we analyze the free boundary when the density jump increases, that is to say, ϵ increases. From now on, we will denote $s_\epsilon = s_\epsilon(t) = 2\gamma_\epsilon\sqrt{t}$ by the corresponding free boundary for the density jump determined by ϵ, where γ_ϵ corresponds to a unique solution to (14).

Following what was studied in Tarzia[18] for Stefan problem with convective condition at fixed face $x = 0$ and a null density jump, that is, $\epsilon = 0$, we denote γ_0 by the coefficient of the free boundary $s_0(t) = 2\gamma_0\sqrt{t}$, which satisfies the equation

$$\frac{-\theta^* k_1 a_1 a_2 h \exp\left(-\frac{x^2}{a_1^2}\right)}{k_1 + h\sqrt{\pi}a_1\mathrm{erf}\left(\frac{x}{a_1}\right)} - \frac{\theta_0 k_2 a_1 \exp\left(-\frac{x^2}{a_2^2}\right)}{\mathrm{erfc}\left(\frac{x}{a_2}\right)} = \sqrt{\pi}a_1a_2\rho_1 l x, \tag{37}$$

which is equivalent to

$$F_1(x) - \frac{\theta_0 k_2 a_1 \exp\left(-\frac{x^2}{a_2^2}\right)}{\operatorname{erfc}\left(\frac{x}{a_2}\right)} = \sqrt{\pi} a_1 a_2 \rho_1 l x. \tag{38}$$

In order to undertake our studies, we rewrite (14) as follows:

$$M_1(x) = F_{2\epsilon}(x), \tag{39}$$

where

$$M_1(x) = F_1(x) - \sqrt{\pi} a_1 a_2 \rho_1 l x, \tag{40}$$

and we denote $F_{2\epsilon}(x) = F_2(x)$, which is defined by (17) to remark its dependence on ϵ.

We note then that we can rewrite function $F_{2\epsilon}$ as follows:

$$F_{2\epsilon}(x) = \theta_0 k_2 a_1 \sqrt{\pi} H\left(\frac{x}{a_2}(\epsilon + 1)\right), \tag{41}$$

with

$$H(x) = \frac{x}{Q(x)}, \quad x > 0 \quad \text{and} \quad Q(x) = \sqrt{\pi} x \exp(x^2) \operatorname{erfc}(x). \tag{42}$$

The functions above have the following properties:

Lemma 6. *Functions Q, H, and M_1 satisfy*

a) $Q(0) = 0, \quad Q(+\infty) = 1, \quad Q'(x) = \frac{Q(x)}{x}(2x^2 + 1) - 2x > 0, x > 0.$

b) $H(0) = \frac{1}{\sqrt{\pi}}, \quad H(+\infty) = +\infty, \quad H'(x) > 0, x > 0.$

c) $M_1(0) = -\theta^* a_1 a_2 h \sqrt{\pi}, \qquad M_1'(x) < 0, \quad x > 0$

$M_1(\gamma_0) = \sqrt{\pi} \theta_0 k_2 a_1 H\left(\frac{\gamma_0}{a_2}\right),$ *and* $M_1(x) \geq \theta_0 k_2 a_1, \quad 0 \leq x \leq \gamma_0.$

Proof.

a) See Briozzo et al.[34]

b) Taking into account the definitions given by (42), we have

$$\lim_{x \to 0^+} H(x) = \lim_{x \to 0^+} \frac{x}{Q(x)} = \lim_{x \to 0^+} \frac{1}{\sqrt{\pi} \exp(x^2) \operatorname{erfc}(x)} = \frac{1}{\sqrt{\pi}}.$$

From properties of Q immediately follow that $H(+\infty) = +\infty$ and

$$H'(x) = \frac{Q(x) - xQ'(x)}{Q^2(x)} = \frac{2x^2(1 - Q(x))}{Q^2(x)} > 0.$$

c) By using (18), it is easy to see that $M_1(0) = -\theta^* a_1 a_2 h \sqrt{\pi}$ and $M_1'(x) < 0$ for $x > 0$.

Taking into account that γ_0 is the solution to (38) and from definition of H, we have

$$M_1(\gamma_0) = F_1(\gamma_0) - \sqrt{\pi} a_1 a_2 \rho_1 l \gamma_0 = \frac{\theta_0 k_2 a_1 \exp\left(-\frac{\gamma_0^2}{a_2^2}\right)}{\operatorname{erfc}\left(\frac{\gamma_0}{a_2}\right)} = \sqrt{\pi} \theta_0 k_2 a_1 H\left(\frac{\gamma_0}{a_2}\right).$$

From (b), we have $H(x) \geq \frac{1}{\sqrt{\pi}}$ for all $x > 0$; moreover, M is a decreasing function, therefore we obtain

$$M_1(x) \geq M_1(\gamma_0) = \sqrt{\pi} \theta_0 k_2 a_1 H\left(\frac{\gamma_0}{a_2}\right) \geq \theta_0 k_2 a_1, \quad 0 \leq x \leq \gamma_0.$$

□

For $x \in (0, \gamma_0]$, we can define the function N_1 as the following:

$$N_1(x) = \frac{a_2}{x} H^{-1}\left(\frac{M_1(x)}{\sqrt{\pi}\theta_0 k_2 a_1}\right) - 1. \tag{43}$$

We have

Lemma 7. *Function N_1 defined by (43) satisfies*

$$N_1(0) = +\infty, \quad N_1(\gamma_0) = 0, \quad N_1'(x) < 0. \tag{44}$$

Lemma 8. *If $h > h_0$, we obtain the following relationship between ϵ and γ_ϵ*

$$\gamma_\epsilon = N_1^{-1}(\epsilon), \tag{45}$$

where N_1 is defined by (43).

Moreover,

$$\lim_{\epsilon \to 0^+} \gamma_\epsilon = \gamma_0, \tag{46}$$

and if $\epsilon_1 < \epsilon_2$, we have $\gamma_{\epsilon_1} > \gamma_{\epsilon_2}$.

Proof. From (39)-(41), the definition (43), and the properties given in the lemma above, we obtain the relationship (45). Furthermore,

$$\lim_{\epsilon \to 0^+} \gamma_\epsilon = \lim_{\epsilon \to 0^+} N_1^{-1}(\epsilon) = \gamma_0.$$

Moreover, let $\epsilon_1 < \epsilon_2$ be, if $h > h_0$, there exist γ_{ϵ_1} and γ_{ϵ_2}, the corresponding solutions to (39). From the properties of N_1, we obtain that $N_1^{-1}(\epsilon_1) > N_1^{-1}(\epsilon_2)$. This is $\gamma_{\epsilon_1} > \gamma_{\epsilon_2}$. □

Theorem 3. *If $h > h_0$ and $\epsilon_1 < \epsilon_2$, we have*

$$s_{\epsilon_1}(t) > s_{\epsilon_2}(t),$$

for each $t > 0$, and

$$\lim_{\epsilon \to 0^+} s_\epsilon(t) = s_0(t). \tag{47}$$

Proof. It arises immediately from the previous lemma. □

In a similar way to what was done for P_1, we are going to prove problem P_2, the relation of the free boundary $y_\epsilon = y_\epsilon(t) = 2\gamma_\epsilon^* \sqrt{t}$ with respect to the jump density determined by ϵ. Taking into account that γ_ϵ^* is the solution to (30), it can be expressed as follows:

$$M_2(\gamma_\epsilon^*) = F_{2\epsilon}(\gamma_\epsilon^*), \tag{48}$$

where

$$M_2(x) = \frac{-\theta^* a_2 k_1 \exp\left(-\frac{x^2}{a_1^2}\right)}{\operatorname{erf}(x/a_1)} - \rho_1 l a_1 a_2 \sqrt{\pi} x, \tag{49}$$

satisfying $M_2(0) = +\infty$, $M_2(+\infty) = -\infty$, and $M_2'(x) < 0$.

It is easy to see that if $\epsilon_1 < \epsilon_2$ then $\gamma_{\epsilon_1}^* > \gamma_{\epsilon_2}^*$ and therefore $y_{\epsilon_1}(t) > y_{\epsilon_2}(t)$ for all $t > 0$. Moreover, we obtain

$$\gamma_\epsilon^* = N_2^{-1}(\epsilon), \tag{50}$$

with

$$N_2(x) = \frac{a_2}{x} H^{-1}\left(\frac{M_2(x)}{\sqrt{\pi}\theta_0 k_2 a_1}\right) - 1, \quad x \in (0, \gamma_0^*], \tag{51}$$

where γ_0^* denotes the coefficient of the free boundary $y_0(t)$ given in Tarzia[18] for Stefan problem with temperature condition at fixed face $x = 0$ and a null density jump.

Furthermore,

$$\lim_{\epsilon\to 0^+} \gamma_\epsilon^* = \lim_{\epsilon\to 0^+} N_2^{-1}(\epsilon) = \gamma_0^*, \tag{52}$$

and

$$\lim_{\epsilon\to 0^+} y_\epsilon(t) = y_0(t). \tag{53}$$

4 | CONCLUSIONS

The asymptotic behavior of the solution to a two-phase Stefan problem, with a convective boundary condition including a density jump at the free boundary, when the heat transfer coefficient $h \to +\infty$, was studied. The convergence of the solution to this problem to the solution to the analogous one with a temperature boundary condition was proved. Moreover, the behavior of the free boundary when the density goes to 0 is studied, and the monotone dependence of a density jump of the coefficient that characterize the free boundary was obtained.

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CONFLICT OF INTEREST

The authors declare that they have no conflict of interest.

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