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Non-classical Stefan problem with nonlinear thermal coefficients and a Robin boundary condition

Adriana C. Briozzo [a,b,*], María Fernanda Natale [b]

[a] *CONICET, Argentina*
[b] *Depto de Matemática, F.C.E., Universidad Austral, Paraguay 1950, S2000FZF Rosario, Argentina*

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ABSTRACT

A non-classical one dimensional Stefan problem with thermal coefficients temperature dependent and a Robin type condition at fixed face $x = 0$ for a semi-infinite material is considered. The source function depends on the evolution the heat flux at the fixed face $x = 0$. Existence of a similarity type solution is obtained and the asymptotic behaviour of free boundary with respect to latent heat fusion is studied. The analysis of several particular cases are given.

1. Introduction

The study of heat transfer problems with phase change such as melting and freezing has attracted growing attention in the last decades due to their wide range of engineering and industrial applications. Stefan problems can be modelled as basic phase-change processes where the location of the interface is a priori unknown. They arise in a broad variety of fields like melting, freezing, drying, friction, lubrication, combustion, finance, molecular diffusion, metallurgy and crystal growth. Due to their importance, they have been largely studied since the last century [1–8].

The Stefan problem is nonlinear even in its simplest form due to the free boundary conditions. When the thermal coefficients are temperature dependent there exists a double nonlinearity of the free boundary problem. Some problems in this area can be seen in [9–15].

We are concerned with the following Stefan problem which is governed by a non-classical and nonlinear heat equation with heat source F, thermal coefficients which depend on the temperature and a convective boundary condition at fixed face $x = 0$. We aim to determine the temperature $T = T(x, t)$ and the free

* Corresponding author at: CONICET, Argentina.
E-mail address: abriozzo@austral.edu.ar (A.C. Briozzo).

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boundary $s = s(t)$ such that the following equations are verified:

$$\rho(T)c(T)T_t = (k(T)T_x)_x - F(Z(t), t) \;, \qquad 0 < x < s(t),\; t > 0 \tag{1}$$

$$k(T(0,t))T_x(0,t) = \frac{h}{\sqrt{t}}(T(0,t) - T^*) \;,\; h > 0 \tag{2}$$

$$T(s(t), t) = T_m \tag{3}$$

$$k\left(T(s(t), t)\right) T_x(s(t), t) = -\rho_0\, l \,\dot{s}\,(t) \tag{4}$$

$$s(0) = 0 \tag{5}$$

where $\rho(T), c(T)$ and $k(T)$ are the density of the material, its specific heat, and its thermal conductivity, respectively; T_m is the phase-change temperature, $\rho_0 > 0$ its the constant density of mass at the melting temperature; $l > 0$ is the latent heat of fusion by unity of mass and $s(t)$ is the position of phase change location. We assume that the temperature of the medium T^* satisfies $T_m < T(0,t) < T^*$, $t > 0$.

We assume that the control function F depends on the evolution of the heat flux at the boundary $x = 0$ as follows

$$Z(t) = T_x(0,t) \;,\;\; F(Z(t), t) = F(T_x(0,t), t) = \frac{\lambda}{\sqrt{t}} T_x(0,t) \tag{6}$$

where λ is a given positive constant, whose physical dimensions are $[\lambda] = m/Tt^{5/2}$.

The non-classical heat conduction problem for a semi-infinite material was motivated by the modelling of a system of temperature regulation in isotropic media and the source term $F(T_x(0,t))$ describes a cooling or heating effect depending on the properties of F which are related to the evolution of the heat flux $T_x(0,t)$ in this article. Some problems of this type were studied in [5,16–20]. Non-classical free boundary problems of the Stefan type were considered in [21,22] from a theoretical point of view by using an equivalent formulation through a system of nonlinear integral equations. In [23] explicit solutions of similarity type to one-phase Stefan problems for non-classical heat equation with constants thermal coefficients and a source F depending on the evolution of the temperature or the heat flux at the fixed face $x = 0$ were obtained. In [24] the existence and uniqueness, local in time, of the solution of a one-phase Stefan problem for a non-classical heat equation with constants thermal coefficients and a convective boundary conditions, was proved. Here the heat source depends on the temperature at $x = 0$. The Friedman–Rubinstein integral representation method and Banach contraction theorem was used.

The analogous Stefan problem to (1)–(5) for a Dirichlet or a Neumann conditions at the fixed face $x = 0$, with a null control function, was considered in [25] where an equivalent integral equation was obtained, but no mathematical justifications are given therein. For this problem, in [26] the existence of an explicit similarity type solution, by using a double fixed point, was given.

The mathematical analysis of two one-phase unidimensional and non-classical Stefan problems with nonlinear thermal coefficients was considered in [27]. Two related cases were studied, one of them has a temperature condition on the fixed face x = 0 and the other one with a flux condition of the type $-q_0/\sqrt{t}$ ($q_0 > 0$). In the first case the source function depends on the heat flux and in the other case it depends on the temperature at the fixed face $x = 0$. In both cases sufficient conditions for data in order to have the existence of an explicit solution of a similarity type were obtained by using a double fixed point.

The goal of this paper is to obtain sufficient conditions on data to prove existence of solution of similarity type to problem (1)–(6).

In Section 2, under certain hypothesis of thermal coefficients, we prove the existence of at least one similarity type solution of a problem (1)–(6) by using a Banach contraction theorem for an integral equation and solving a transcendental equation to prove the existence of the coefficient that characterize the free boundary.

In Section 3 we analyse the behaviour of free boundary with respect to latent heat of fusion l. We prove that for sufficiently large values of l the phase change is a much slower process.

In Section 4 we present some particular cases: the solution of (1)–(6) when it has a null source and then, we give two examples: one of them for constant thermal coefficients and the another example for a linear conductivity related to the temperature.

2. Solving nonclassical Stefan problem

We define the following transformation [27]

$$\theta(x,t)=\frac{T(x,t)-T^{*}}{T_m-T^{*}} \tag{7}$$

then the problem (1)–(6) becomes

$$N(\theta)\theta_t=\alpha_0\left(L(\theta)\theta_x\right)_x-\frac{\lambda}{c_0\rho_0\sqrt{t}}\theta_x(0,t),\ 0<x<s(t),\ t>0 \tag{8}$$

$$k\left((T_m-T^{*})\theta(0,t)+T^{*}\right)\theta_x(0,t)=\frac{h}{\sqrt{t}}\theta(0,t),\ t>0 \tag{9}$$

$$\theta(s(t),t)=1,\ t>0 \tag{10}$$

$$k(T_m)\theta_x(s(t),t)=\frac{\rho_0\, l}{T_m-T^{*}}\overset{\bullet}{s}(t),\ t>0 \tag{11}$$

$$s(0)=0 \tag{12}$$

where

$$N(T)=\frac{\rho(T)c(T)}{\rho_0c_0},\ L(T)=\frac{k(T)}{k_0}, \tag{13}$$

and k_0,ρ_0,c_0 and $\alpha_0=\frac{k_0}{\rho_0c_0}$ are the reference thermal conductivity, density of mass, specific heat and thermal diffusivity respectively.

Now we assume a similarity solution of the type

$$\theta(x,t)=f(\xi)\qquad,\qquad\xi=\frac{x}{2\sqrt{\alpha_0t}} \tag{14}$$

then the conditions (10) and (11) imply that the free boundary $s(t)$ must be

$$s(t)=2\xi_0\sqrt{\alpha_0t} \tag{15}$$

where ξ_0 is a positive parameter to be determined later.

Therefore, the conditions (8)–(11) reduce to the following problem:

$$\left[L(f)f'(\xi)\right]'+2\xi N(f)f'(\xi)=Af'(0),\ 0<\xi<\xi_0 \tag{16}$$

$$L(f(0))f'(0)=pf(0) \tag{17}$$

$$f(\xi_0)=1 \tag{18}$$

$$f'(\xi_0)=M\xi_0 \tag{19}$$

where

$$L(f(x))=\frac{k\big((T_m-T^{*})f(x)+T^{*}\big)}{k_0},\quad N(f(x))=\frac{\rho c\big(\big((T_m-T^{*})f(x)+T^{*}\big)\big)}{\rho_0c_0} \tag{20}$$

and

$$A=\frac{2\lambda}{\sqrt{c_0\rho_0k_0}},\qquad p=2Bi,\qquad M=\frac{2k_0}{k(T_m)Ste}. \tag{21}$$

with $Bi=\frac{\sqrt{\alpha_0}h_0}{k_0}>0$ (generalized Biot number) and $Ste=\frac{c(T^{*}-T_m)}{l}>0$ (Stefan number).

From (16)–(18) we have that f must satisfy the following nonlinear integral equation of Volterra type:

$$f(\xi) = \frac{L(f(0)) + p\Phi\left[\xi, L(f), N(f)\right]}{L(f(0)) + p\Phi\left[\xi_0, L(f), N(f)\right]}, \tag{22}$$

where Φ is given by

$$\Phi\left[\xi, L(f), N(f)\right] := \int_0^{\xi} \frac{1}{G(f)(u)}\, du + A\int_0^{\xi} \frac{w(f)(u)}{G(f)(u)}\, du, \tag{23}$$

$$G(f)(x) := \frac{L(f(x))}{L(f(0))} I(f)(x) \quad , \quad I(f)(x) := \exp\left(\int_0^x 2s\frac{N(f(s))}{L(f(s))}\, ds\right), \tag{24}$$

$$w(f)(x) := \int_0^x \frac{G(f)(u)}{L(f)(u)} du = \frac{1}{L(f(0))}\int_0^x I(f)(u)du \tag{25}$$

The condition (19) for the unknown ξ_0 can be rewritten as

$$\frac{1}{G(f)(\xi_0)} + A\frac{w(f)(\xi_0)}{G(f)(\xi_0)} = M\xi_0\left[\frac{L(f(0))}{p} + \Phi\left[\xi_0, L(f), N(f)\right]\right] \tag{26}$$

In order to prove the existence of solution f and ξ_0 to (22) and (26), we begin by analysing (22) for any given $\xi_0 > 0$.

We consider $C^0\left[0, \xi_0\right]$, the space of continuous real functions defined on $[0, \xi_0]$, with the norm

$$\|f\| = \max_{\xi\in[0,\xi_0]} |f(\xi)| \tag{27}$$

and we define the operator $H : C^0[0, \xi_0] \longrightarrow C^0[0, \xi_0]$ given by

$$H(f)_{(\xi)} = \frac{L(f(0)) + p\Phi\left[\xi, L(f), N(f)\right]}{L(f(0)) + p\Phi\left[\xi_0, L(f), N(f)\right]} \tag{28}$$

By using the Banach fixed point theorem we will demonstrate that for each $\xi_0 > 0$ there exists a unique f such that

$$H(f(\xi)) = f(\xi) \quad , \quad 0 < \xi < \xi_0 \tag{29}$$

which is the solution of (22).

For this, we assume the following hypothesis for dimensionless thermal conductivity and specific heat

$$L_m \leq L(T) \leq L_M \qquad , \qquad N_m \leq N(T) \leq N_M \tag{30}$$

$$|L(g) - L(h)| \leq \widetilde{L}\,\|g-h\| \quad , \quad \forall g, h \in C^0\left(R_0^+\right) \cap L^\infty\left(R_0^+\right) \tag{31}$$

$$|N(g) - N(h)| \leq \widetilde{N}\,\|g-h\| \quad , \quad \forall g, h \in C^0\left(R_0^+\right) \cap L^\infty\left(R_0^+\right) \tag{32}$$

where N_m, N_M, L_m, L_M are positive constants and $\widetilde{L}$ and $\widetilde{N}$ are the Lipschitz constants.

For the convenience of the reader we repeat the relevant material from [27] without proofs, thus making our exposition self-contained.

Lemma 1. *Let ξ_0 be a given positive real number. For all $f,\ f^* \in C^0\left[0, \xi_0\right]$, and $\xi \in (0, \xi_0)$ we have*

$$\frac{L_m}{L_M}\xi_0 \exp\left(-\frac{N_M}{L_m}\xi_0^2\right) \leq \Phi\left[\xi, L(f), N(f)\right] \leq R(\xi_0), \tag{33}$$

$$\left|\Phi\left[\xi, L(f), N(f)\right] - \Phi\left[\xi, L(f^*), N(f^*)\right]\right| \leq C(\xi_0)\left\|f - f^*\right\|. \tag{34}$$

where

$$R(\xi_0) = \frac{L_M}{L_m}\xi_0 + A\frac{\xi_0^3 L_M \exp\left(\frac{N_M}{L_m}\xi_0^2\right)}{2L_m^2}, \quad C(\xi_0) = \xi_0\left(C_1(\xi_0) + AC_2(\xi_0)\right) \tag{35}$$

$$C_1(\xi_0) = \frac{L_M^2}{L_m^2}C_3(\xi_0), \quad C_2(\xi_0) = \frac{\exp\left(\frac{N_M}{L_m}\xi_0^2\right)L_M^2}{L_m^3}\left[\frac{\xi_0^2}{2}C_3(\xi_0) + L_M C_6(\xi_0)\right]$$

$$C_3(\xi_0) = \frac{L_M^2 C_4(\xi_0) + C_5(\xi_0)\exp\left(\frac{N_M \xi_0^2}{L_m}\right)}{L_m^2}, \quad C_4(\xi_0) = \frac{\exp\left(\frac{N_M}{L_m}\xi_0^2\right)\xi_0^2}{L_m^2}\left(\widetilde{N}L_M + N_M\widetilde{L}\right)$$

$$C_5(\xi_0) = 2L_M\widetilde{L}, \quad C_6(\xi_0) = \xi_0\frac{L_M}{L_m^2}\left[\frac{\widetilde{L}\exp\left(\frac{N_M}{L_m}\xi_0^2\right)}{L_m} + C_3(\xi_0)\right]$$

Theorem 1. *Let $\xi_0 > 0$ be a given number. We assume that (30)–(32) hold. If ξ_0 satisfies the inequality*

$$\epsilon(\xi_0) := \frac{p}{L_m^2}\left[C(\xi_0)\left(2L_M + 2pR(\xi_0)\right) + 2R(\xi_0)\widetilde{L}\right] < 1 \tag{36}$$

then there exists a unique solution $f \in C^0\left[0, \xi_0\right]$ of the integral equation (22).

Proof. Assuming the hypothesis given by (30)–(32) we will prove that H is a contraction mapping from $C^0\left[0, \xi_0\right]$ to itself.

From properties of thermal coefficients and (28) we have that the operator H is in fact self mapping on $C^0\left[0, \xi_0\right]$.

Let $f, f^* \in C^0[0, \xi_0]$, then we obtain

$$\left|H(f(\xi)) - H(f^*(\xi))\right| = \left|\frac{L(f(0)) + p\Phi\left[\xi, f\right]}{L(f(0)) + p\Phi\left[\xi_0, f\right]} - \frac{L(f^*(0)) + p\Phi\left[\xi, f^*\right]}{L(f^*(0)) + p\Phi\left[\xi_0, f^*\right]}\right| \tag{37}$$

where we have denoted $\Phi\left[\xi, f\right] = \Phi\left[\xi, L(f), N(f)\right]$.

We have

$$\begin{aligned}
&\left|\frac{L(f(0)) + p\Phi\left[\xi, f\right]}{L(f(0)) + p\Phi\left[\xi_0, f\right]} - \frac{L(f^*(0)) + p\Phi\left[\xi, f^*\right]}{L(f^*(0)) + p\Phi\left[\xi_0, f^*\right]}\right| \\
&\quad = \left|\frac{\left(L(f(0)) + p\Phi\left[\xi, f\right]\right)\left(L(f^*(0)) + p\Phi\left[\xi_0, f^*\right]\right) - \left(L(f^*(0)) + p\Phi\left[\xi, f^*\right]\right)\left(L(f(0)) + p\Phi\left[\xi_0, f\right]\right)}{[L(f(0)) + p\Phi\left[\xi_0, f\right]][L(f^*(0)) + p\Phi\left[\xi_0, f^*\right]]}\right| \\
&\quad \leq \frac{\sum_{i=1}^{6} I_i}{[L(f(0)) + p\Phi\left[\xi_0, f\right]][L(f^*(0)) + p\Phi\left[\xi_0, f^*\right]]}
\end{aligned}$$

where

$$I_1 = pL(f^*(0))\left|\Phi\left[\xi_0, f\right] - \Phi\left[\xi_0, f^*\right]\right| \leq pL_M C(\xi_0)\left\|f - f^*\right\|$$

$$I_2 = p\Phi\left[\xi_0, f^*\right]\left|L(f(0)) - L(f^*(0))\right| \leq p\widetilde{L}R(\xi_0)\left\|f - f^*\right\|$$

$$I_3 = pL(f(0))\left|\Phi\left[\xi, f\right] - \Phi\left[\xi, f^*\right]\right| p \leq L_M C(\xi_0)\left\|f - f^*\right\|$$

$$I_4 = p\Phi\left[\xi, f\right]\left|L(f(0)) - L(f^*(0))\right| \leq p\widetilde{L}R(\xi_0)\left\|f - f^*\right\|$$

$$I_5 = p^2\Phi\left[\xi, f^*\right]\left|\Phi\left[\xi_0, f\right] - \Phi\left[\xi_0, f^*\right]\right| \leq p^2 R(\xi_0)C(\xi_0)\left\|f - f^*\right\|$$

$$I_6 = p^2\Phi\left[\xi_0, f^*\right]\left|\Phi\left[\xi, f\right] - \Phi\left[\xi, f^*\right]\right| \leq p^2 R(\xi_0)C(\xi_0)\left\|f - f^*\right\|$$

Taking into account above inequalities and the fact that

$$L(f(0)) + p\Phi[\xi_0, f] \geq L_m \left[1 + \frac{p}{L_M}\xi_0 exp\left(-\frac{N_M}{L_m}\xi_0^2\right)\right] \geq L_m$$

we have

$$\|H(f) - H(f^*)\| = \max_{\xi\in[0,\xi_0]} |H(f(\xi)) - H(f^*(\xi))| \leq \epsilon(\xi_0)\|f - f^*\| \tag{38}$$

where $\epsilon(\xi_0)$ is defined by (36).

If $\epsilon(\xi_0) < 1$ then the operator H is a contraction mapping from $C^0[0, \xi_0]$ to itself. Therefore, there exists unique fixed point $f \in C^0[0, \xi_0]$ for H, that is to say, there exists a unique solution $f \in C^0[0, \xi_0]$ of the integral equation (22). □

Remark 1. The solution f of (22), depends on the real number $\xi_0 > 0$. We can denote

$$f(\xi) = f_{\xi_0}(\xi) = f(\xi_0, \xi) \quad , \quad 0 < \xi < \xi_0 \quad , \quad \xi_0 > 0. \tag{39}$$

Lemma 2. *There exists a positive number ξ_0^* such that*

$$\epsilon(\xi_0^*) = 1, \quad \epsilon(\xi_0) < 1 \;\; if \;\; 0 \leq \xi_0 < \xi_0^*, \quad \epsilon(\xi_0) > 1 \;\; if \;\; \xi_0 > \xi_0^*.$$

Proof. We have $\epsilon(0) = 0$, $\epsilon(+\infty) = +\infty$ and $\epsilon'(\xi_0) > 0$ for all $\xi_0 > 0$. Then there exists $\xi_0^* > 0$ such that $\epsilon(\xi_0^*) = 1$ and

$$\epsilon(\xi_0) < 1 \;\; if \;\; 0 \leq \xi_0 < \xi_0^* \quad , \;\; \epsilon(\xi_0) > 1 \;\; if \;\; \xi_0 > \xi_0^*. \quad \square$$

Next, we must prove that there exists $0 < \hat{\xi}_0 < \xi_0^*$ such that it satisfies Eq. (26).

We define the function $V = V(x)$, for $x \in [0, \xi_0^*)$ as follows

$$V(x) := \frac{1 + Aw(f_x)(x)}{G(f_x)(x)M\left[\frac{L(f_x(0))}{p} + \Phi(x, f_x)\right]} \tag{40}$$

where we denote f_x to the unique solution of (22) for each $x \in [0, \xi_0^*)$. Then, Eq. (26) is equivalent to

$$V(x) = x \quad , \quad x \in [0, \xi_0^*). \tag{41}$$

We give the following preliminary results:

Lemma 3. *Under hypothesis (30) we have:*

1. $g_1(x) \leq \frac{\int_0^x I(f_x)(u)du}{I(f_x(x))} \leq g_2(x)$
2. $g_3(x) \leq \frac{1}{G(f_x)(x)} \leq g_4(x)$
3. $g_3(x) + \frac{A}{L_M}g_1(x) \leq \frac{1+Aw(f_x)(x)}{G(f_x)(x)} \leq g_4(x) + \frac{A}{L_m}g_2(x)$

where

$$g_1(x) := \sqrt{\frac{L_M}{N_m}} f_1\left(\sqrt{\frac{N_m}{L_M}}x\right), \qquad g_2(x) := \sqrt{\frac{L_m}{N_M}} f_1\left(\sqrt{\frac{N_M}{L_m}}x\right),$$

function f_1 is the Dawson's function defined by [23]

$$f_1(x) = exp(-x^2)\int_0^x exp(z^2)dz$$

and

$$g_3(x) := \frac{L_m}{L_M} exp\left(-\frac{N_M}{L_m}x^2\right), \qquad g_4(x) := \frac{L_M}{L_m} exp\left(-\frac{N_m}{L_M}x^2\right).$$

Lemma 4. *For $x \in [0, \xi_0^*)$ we have:*

$$V_2(x) \le V(x) \le V_1(x),$$

where

$$V_1(x) = \frac{p}{ML_m}\left[g_4(x) + \frac{A}{L_m} g_2(x)\right], \quad x \ge 0 \tag{42}$$

$$V_2(x) = \frac{p}{ML_M}\left[g_3(x) + \frac{A}{L_M} g_1(x)\right], \quad x \ge 0 \tag{43}$$

which satisfy the following properties

$$V_1(0) = \frac{pL_M}{ML_m^2}, \qquad V_1(+\infty) = 0, \qquad V_1'(x) < 0$$
$$V_2(0) = \frac{pL_m}{ML_M^2}, \qquad V_2(+\infty) = 0, \qquad V_2'(x) < 0.$$

Theorem 2. *(a) There exist unique $x_1 > 0$ and $x_2 > 0$ solutions of equations $V_1(x) = x$ and $V_2(x) = x$, respectively.*

(b) If

$$x_1 < \xi_0^* \tag{44}$$

then Eq. (26) has at least one solution $\hat{\xi}_0 \in (x_2, x_1)$.

Proof. It follows easily from the previous lemmas. □

Remark 2. The inequality (44) is equivalent to the following condition for the latent heat fusion

$$l > \hat{l} := \frac{pB(\xi_0^*)k(T_m)(T^* - T_m)}{2\rho_0\alpha_0 L_m^2} \tag{45}$$

where $B = B(x)$ is a decreasing function given by:

$$B(x) = \frac{L_M exp\left(-\frac{N_M}{L_m}x^2\right) + A\sqrt{\frac{L_m}{N_M}} f_1\left(\sqrt{\frac{N_M}{L_m}}x\right)}{x}$$

Finally, we can enunciate the following theorem of existence of the solution.

Theorem 3. *If N and L verify the conditions (30)–(32) and data satisfies (45) then there exists at least one solution of the problem (1)–(6) where the free boundary is given by*

$$s(t) = 2\hat{\xi}_0\sqrt{\alpha_0 t}$$

with $\hat{\xi}_0$ established by Theorem 2, the temperature is given by

$$T(x,t) = (T_m - T^*) f_{\hat{\xi}_0}(\xi) + T^*,$$

with $\xi = x/2\sqrt{\alpha_0 t}$ and $f_{\hat{\xi}_0}$ is the unique solution of the integral equation (22) on the interval $[0, \hat{\xi}_0]$.

3. Asymptotic behaviour of free boundary with respect to latent heat fusion

From previous section we have that the existence of solution of problem (1)–(6) is given for sufficiently large latent heat of fusion.

We will analyse the behaviour of the free boundary $s(t) = 2\hat{\xi}_0\sqrt{\alpha_0 t}$ with respect to the latent heat.

We consider Eq. (26) for the coefficient $\hat{\xi}_0$ which characterizes the free boundary. Taking into account definition (21) we note that coefficient M is directly proportional to latent heat l and we can write $M = M(l)$. Moreover the notation $V = V_l(x)$ and $\hat{\xi}_0(l)$ is adopted in order to emphasize the dependence of the solution of (26) on l. This fact is going to facilitate the subsequent analysis of the asymptotic behaviour of the free boundary, that is

$$V_l(x) = \frac{1 + Aw(f_x)(x)}{G(f_x)(x)M(l)\left[\frac{L(f_x(0))}{p} + \Phi(x, f_x)\right]}, x \in [0, \xi_0^*) \tag{46}$$

for latent heat $l > \hat{l}$ and $\hat{\xi}_0(l)$ is a solution of equation

$$V_l(x) = x$$

Thus, we obtain the following result:

Lemma 5. *If $\hat{l} < l_1 < l_2$ we have*

$$\hat{\xi}_0(l_1) > \hat{\xi}_0(l_2), \qquad s_{l_1}(t) > s_{l_2}(t),$$

for each $t > 0$. Moreover

$$\lim_{l \to +\infty} \hat{\xi}_0(l) = 0.$$

Proof. Let l_1, l_2 be such that $\hat{l} < l_1 < l_2$. From (46) we have for each $x \in [0, \xi_0^*)$:

$$V_{l_2}(x) < V_{l_1}(x)$$

then $\hat{\xi}_0(l_1) > \hat{\xi}_0(l_2)$ and $s_{l_1}(t) > s_{l_2}(t)$, for each $t > 0$.

From Theorem 2 we have that if $l > \hat{l}$ then

$$x_2(l) < \hat{\xi}_0(l) < x_1(l)$$

where $x_1(l)$ and $x_2(l)$ are the unique solutions of equations $V_{1l}(x) = x$ and $V_{2l}(x) = x$ respectively, where V_{1l} and V_{2l} are given by (42) and (43). Therefore, taking into account properties of V_{il} we have

$$\lim_{l \to +\infty} \hat{\xi}_0(l) = 0. \quad \square$$

Corollary 1. *For each $t > 0$ we have*

$$\lim_{l \to +\infty} s_l(t) = 0$$

that is, when l grows up the phase change is a much slower process.

4. Particular cases

4.1. Stefan problem with null source

We now turn to the case $\lambda = 0$, that is, we have that (1)–(6) is a Stefan problem with null source. The solution to (16)–(19), for $A = 0$, must satisfies the integral equation

$$f(\xi) = \frac{L(f(0)) + p\Phi_0\left[\xi, L(f), N(f)\right]}{L(f(0)) + p\Phi_0\left[\xi_0, L(f), N(f)\right]},$$

with

$$\Phi_0\left[\xi, L(f), N(f)\right] := \int_0^{\xi} \frac{1}{G(f)(u)}\, du = L(f)(0) \int_0^{\xi} \frac{1}{L(f)(u)I(f)(u)}\, du \tag{47}$$

and the condition for the coefficient ξ_{00} which characterizes the free boundary is

$$\frac{1}{G(f)(\xi_{00})} = M\xi_{00}\left[\frac{L(f(0))}{p} + \Phi_0\left[\xi_{00}, L(f), N(f)\right]\right] \tag{48}$$

In this case the condition (45) that guarantees the existence of at least a solution $\hat{\xi}_{00}$ of (48) turns out

$$l > \hat{l}_{00} := \frac{pB_0(\xi^*_{00})k(T_m)(T^* - T_m)}{2\rho_0\alpha_0 L_m^2} \tag{49}$$

where

$$B_0(x) = \frac{L_M exp\left(-\frac{N_M}{L_m}x^2\right)}{x} \tag{50}$$

and ξ^*_{00} satisfies

$$\epsilon_0(\xi^*_{00}) = \frac{L_M p\xi^*_{00}}{L_m^2}\left[C_1(\xi^*_{00})\left(3 + \frac{2p\xi^*_{00}}{L_m}\right) + \frac{2\widetilde{L}}{L_m}\right] = 1. \tag{51}$$

4.2. Constant thermal coefficients

For the particular case of constant thermal coefficients we have $L = N = 1$ and the explicit solution to (16)–(19) is given by

$$f(\xi) = \frac{1 + p\sqrt{\frac{\pi}{2}}\left[erf(\xi) + \frac{2A}{\sqrt{\pi}}\int_0^{\xi} f_1(x)\, dx\right]}{1 + p\sqrt{\frac{\pi}{2}}\left[erf(\hat{\xi}_0) + \frac{2A}{\sqrt{\pi}}\int_0^{\hat{\xi}_0} f_1(x)\, dx\right]} \tag{52}$$

with $\hat{\xi}_0$ the unique solution of

$$exp\left(-\xi_0^2\right) + Af_1(\xi_0) = M\xi_0\left[\frac{1}{p} + \sqrt{\frac{\pi}{2}}erf(\xi_0) + A\int_0^{\xi_0} f_1(x)\, dx\right] \tag{53}$$

and $M = \frac{2}{Ste}$.

4.3. Linear conductivity

The case $\rho(T) = \rho_0$, $c(T) = c_0$ and $k(T) = k_0\left[1 + \eta\frac{T-T^*}{T_m-T^*}\right] = k_0\left[1 + \eta f\right]$ implies $N(f) = 1$ and $L(f) = 1 + \eta f$. The problem (16)–(19) becomes

$$((1 + \eta f)f')' + 2\xi f' = Af'(0). \tag{54}$$

$$(1 + \eta f(0))f'(0) = pf(0) \tag{55}$$

$$f(\xi_0) = 1 \tag{56}$$

$$f'(\xi_0) = \frac{2\xi_0}{(1 + \eta)Ste} \tag{57}$$

which was studied in [11] for $A = 0$. A result on existence and uniqueness of solution of nonlinear boundary problem of second order (54)–(57) was proved and this solution was defined as a *Generalized Modified Error* (GME) function. Therefore the existence of similarity type solution to (1)–(6) was proved. Moreover it was shown that the solution to problem with Dirichlet boundary condition in place of the convective one, can be obtained at the limit case of the solution to problem (1)–(6) when the coefficient h that characterizes the heat transfer at $x = 0$ goes to infinity. GME function is a non-negative bounded analytic function which is increasing and concave, just as the classical error is. Finally, it was proposed a strategy to obtain explicit approximations for the GME.

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