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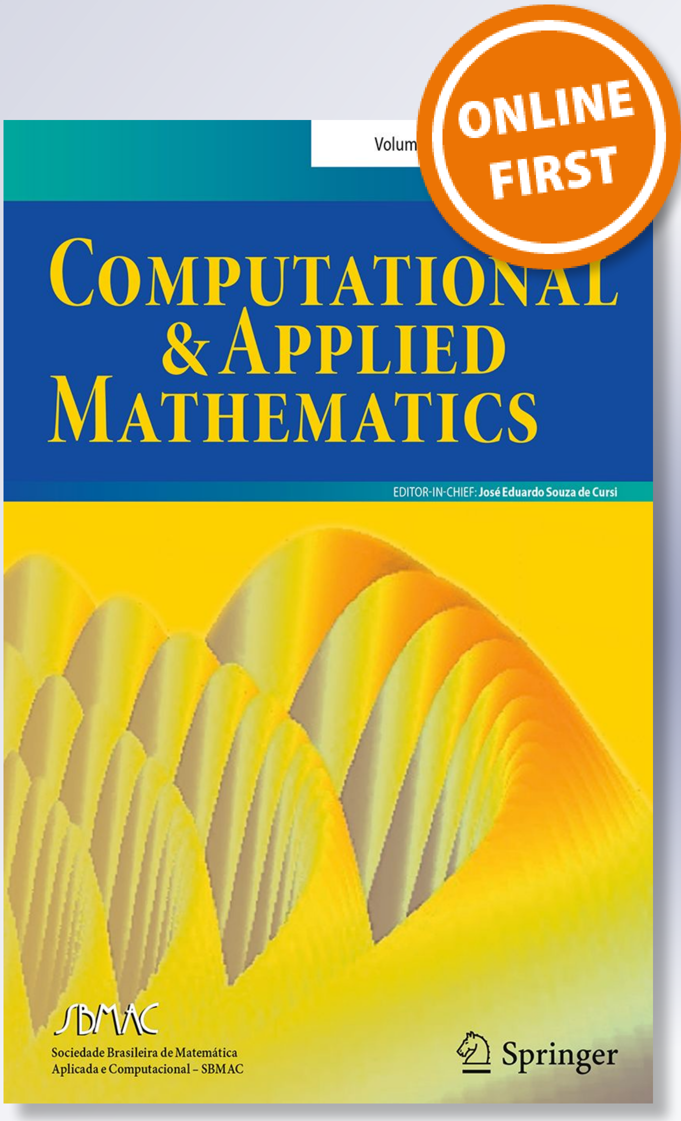
Computational and Applied Mathematics

ISSN 0101-8205

Comp. Appl. Math.
DOI 10.1007/s40314-017-0524-z



Springer

Comp. Appl. Math.
https://doi.org/10.1007/s40314-017-0524-z

Determination of unknown thermal coefficients in a Stefan problem for Storm's type materials

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Received: 18 October 2016 / Revised: 2 February 2017 / Accepted: 10 October 2017

Abstract We consider a nonlinear one-dimensional Stefan problem for a semi-infinite material $x > 0$, with phase change temperature T_f. We assume that the heat capacity and the thermal conductivity satisfy a Storm's condition. A convective boundary condition and a heat flux over-specified condition on the fixed face $x = 0$ are considered. Unknown thermal coefficients are determined for the free boundary problem and for the associate moving boundary problem and we give sufficient conditions to obtain a parametric representation of a similarity type solution. Moreover, we give formulae for the thermal coefficients in both cases.

Keywords Stefan problem · Free boundary problem · Phase-change process · Similarity solution · Unknown thermal coefficient · Over-specified boundary condition

Mathematics Subject Classification 35R35 · 80A22 · 35C05

1 Introduction

Heat transfer problem with change-phase such as melting and freezing has been studied in the last century due to their wide scientific and technological applications (Alexiades and Solomon 1993; Cannon 1984; Carslaw and Jaeger 1965; Crank 1984; Fasano 2005; Gupta 2003; Lunardini 1991; Rubinstein 1971).

We consider the following one phase nonlinear unidimensional Stefan problem for a semi-infinite material $x > 0$, with phase change temperature T_f and with a over-specified condition at the fixed face $x = 0$, (Briozzo and Natale 2014, 2016; Hill and Hart 1986; Solomon et al. 1983; Tarzia 1981)

Communicated by Antonio Leitao.

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Published online: 03 February 2018

$$\rho c(T)\frac{\partial T}{\partial t}=\frac{\partial}{\partial x}\left[k(T)\frac{\partial T}{\partial x}\right],\quad 0<x<X(t),\quad t>0, \tag{1}$$

$$k(T(0,t))\frac{\partial T}{\partial x}(0,t)=\frac{h}{\sqrt{t}}[T(0,t)-T_{\rm m}],\ h>0,\quad t>0, \tag{2}$$

$$k(T(0,t))\frac{\partial T}{\partial x}(0,t)=\frac{q_0}{\sqrt{t}},\ q_0>0,\quad t>0, \tag{3}$$

$$T(X(t),t)=T_f, \tag{4}$$

$$k(T_f)\frac{\partial T}{\partial x}(X(t),t)=\rho L\stackrel{\bullet}{X}(t),\quad t>0, \tag{5}$$

$$X(0)=0 \tag{6}$$

where L is the latent heat of fusion of the medium, ρ is the density (assumed constant), $T_{\rm m}$ is the temperature of the medium $T_{\rm m}<T(0,t)<T_f$, h is the positive heat transfer coefficient and q_0 is the coefficient which characterize the heat flux on the fixed face $x=0$.

We assume that the metal exhibits nonlinear thermal characteristics such that the heat capacity $c(T)>0$ and the thermal conductivity $k(T)>0$ satisfy a Storm's condition (Briozzo et al. 1999; Briozzo and Natale 2014, 2016; Knight and Philip 1974; Natale and Tarzia 2000; Storm 1951)

$$\frac{\frac{\rm d}{{\rm d}T}\left(\sqrt{\frac{\rho c(T)}{k(T)}}\right)}{\rho c(T)}=\lambda=\text{const.}>0, \tag{7}$$

Condition (7) was originally obtained by Storm (1951) in an investigation of heat conduction in simple monoatomic metals. There, the validity of the approximation (7) was examined for aluminum, silver, sodium, cadmium, zinc, copper and lead.

In Briozzo and Natale (2014) two nonlinear Stefan problems analogous to (1)–(6) with phase change temperature T_f and the Storm's condition (7) are considered. In one case a heat flux boundary condition of the type $q(t)=\frac{q_0}{\sqrt{t}}$ and in the other case a temperature boundary condition $T=T_s<T_f$ at the fixed face $x=0$ are assumed. Solutions of similarity type are obtained in both cases and the equivalence of the two problems is demonstrated. In Briozzo and Natale (2016) the analogous one phase nonlinear Stefan problem for a with phase change temperature T_f is considered with the assumption of a Storm's condition for the heat capacity and thermal conductivity and a convective condition at the fixed face given by (2). Existence and uniqueness of a similarity type solution are obtained. Moreover, the convergence of this problem to problem with temperature condition at the fixed face when $h\to+\infty$ was proved.

In this paper the convective over-specified boundary condition on the fixed face to the semi-infinite material allows us to consider some thermal coefficients as unknowns and to calculate them, under certain specified restrictions upon data.

Several papers on the determination of thermal coefficients in free boundary problems are found in the bibliography, see Briozzo et al. (1999), Ceretani and Tarzia (2015), Ceretani and Tarzia (2016), Tarzia (1982, 1983, 1998). Determination of one and two unknown constant thermal coefficients through an inverse one-phase Stefan problem with a temperature boundary condition and an overspecied heat flux condition at fixed face were considered in Tarzia (1982, 1983). An analogous problem for a thermal conductivity as an affine function of the temperature was given in Tarzia (1998). In Briozzo et al. (1999) unknown thermal coefficients of a semi-infinite material of Storm's type through a phase-change process, analogous to Stefan problem here considered, with temperature boundary condition and a heat flux over-specified condition on the fixed face, was determined. That is a similar Stefan problem to the one considered here, but this manuscript differs from that in the over condition imposed on the fixed face. In Ceretani and Tarzia (2015) and Ceretani and Tarzia (2016) determination of one and two constant unknown thermal coefficients through a mushy zone model with a convective over-specified boundary condition, for the free boundary problem and for the moving boundary problem, respectively, were considered. In Tarzia (2015) explicit expression for one unknown thermal coefficient of the semi-infinite material through the one-phase fractional Lamé-Clapeyron Stefan problem with an over-specified boundary condition on the fixed face $x = 0$ is obtained.

The goal of this paper is to determine the temperature $T = T(x, t)$, one or two unknown thermal coefficients chosen among ρ, L, $k(T)$ and $c(T)$ as a function of data q_0, T_f, T_0, λ depending if $X = X(t)$ is a free (unknown function) or a moving (known function) boundary, which satisfies the problem (1)–(7). In Sect. 2 we show how to find a unique solution of the similarity type for the free boundary problem and we give the corresponding restrictions of data and formulas to determine one unknown thermal coefficient and the coefficient that characterizes the free boundary.

In Sect. 3 we consider the moving boundary problem (1)–(7) where $X = X(t)$ is assumed known. We determine the temperature $T = T(x, t)$ and two unknown thermal coefficients chosen among ρ, L, $k(T)$ and $c(T)$. Since inverse Stefan problems are usually ill-posed problems, it is expected that restrictions on data have to be set to obtain solution to problem (1)–(7).

The results are summarized in Tables 1 and 2.

2 Unknown thermal coefficients through a free boundary problem

Following Briozzo and Natale (2014, 2016), Carslaw and Jaeger (1965), Hill and Hart (1986) we consider the problem (1)–(7) and we propose a similarity type solution given by Briozzo and Natale (2014, 2016), Carslaw and Jaeger (1965), Hill and Hart (1986)

$$T(x, t) = \Phi(\xi), \;\; \xi = \frac{x}{X(t)} \tag{8}$$

where

$$X(t) = \sqrt{2\gamma t}, \;\; t > 0 \tag{9}$$

is the free boundary and γ is assumed a positive constant to be determined.

Then we have that the problem (1)–(6) is equivalent to

$$k(\Phi)\Phi''(\xi) + k'(\Phi)\Phi'^2(\xi) + \gamma\rho c(\Phi)\Phi'(\xi)\xi = 0, \;\; 0 < \xi < 1, \tag{10}$$

$$k(\Phi(0))\Phi'(0) = h\sqrt{2\gamma}[\Phi(0) - T_{\mathrm{m}}], \tag{11}$$

$$k(\Phi(0))\Phi'(0) = q_0\sqrt{2\gamma}, \tag{12}$$

$$\phi(1) = T_f, \tag{13}$$

$$k(\Phi(1))\Phi'(1) = \rho L\gamma\;. \tag{14}$$

If we define

$$y(\xi) = \sqrt{\frac{k}{s}\,(\Phi(\xi))}, \tag{15}$$

then a parametrization of the Storm condition (7) is

$$c(\Phi) = -\frac{1}{\rho\lambda y^2}\frac{\mathrm{d}y}{\mathrm{d}\Phi}, \;\;\; k(\Phi) = -\frac{1}{\lambda}\frac{\mathrm{d}y}{\mathrm{d}\Phi} \tag{16}$$

Table 1 Formulae for the one unknown thermal coefficient and condition on data for the free boundary problem

Case	Formulae for the parameters $\tilde{u}_0$ and $\tilde{u}_1$	Formulae for the coefficient γ and unknown thermal coefficient	Condition for data
1	$\tilde{u}_0 = \sqrt{2}\lambda q_0$	$\tilde{\gamma} = \frac{2\lambda^2 q_0^2 F^2_{\sqrt{2}\lambda q_0}(\tilde{u}_1) Y_2^2}{\exp(-2\lambda^2 q_0^2)\tilde{\rho}}$	(44)
	$\tilde{u}_1 = \sqrt{2} erf^{-1}\left(g\left(\lambda q_0, \frac{1}{\sqrt{\pi}}\left(1-\frac{Y_1}{Y_0}\right)\right)\right)$	$\tilde{\rho} = \frac{\exp(-\tilde{u}_1^2)}{Y_1^2 \epsilon^2 F^2_{\sqrt{2}\lambda q_0}(\tilde{u}_1)}$	
2	$\tilde{u}_0 = \sqrt{2}\lambda q_0$	$\tilde{\gamma} = \frac{2\lambda^2 q_0^2 F^2_{\sqrt{2}\lambda q_0}(\tilde{u}_1) Y_2^2}{\exp(-2\lambda^2 q_0^2)\tilde{\rho}}$	(44)
	$\tilde{u}_1 = \sqrt{2}\mathrm{erf}^{-1}\left(g\left(\lambda q_0, \frac{1}{\sqrt{\pi}}\left(1-\frac{Y_1}{Y_0}\right)\right)\right)$	$\tilde{L} = \frac{\exp\left(-\frac{\tilde{u}_1^2}{2}\right)}{Y_1^2 F_{\sqrt{2}\lambda q_0}(\tilde{u}_1)\sqrt{\rho}}$	
3	$\tilde{u}_0 = \sqrt{2}\lambda q_0$	$\tilde{\gamma} = \frac{\exp(-2\tilde{\sigma}^2)}{\rho^2\epsilon^2\frac{\pi}{2}\left[g\left(\lambda q_0, \frac{1}{\sqrt{\pi}}\right)-\mathrm{erf}(\tilde{\sigma})\right]^2}$	
	$\tilde{u}_1 = \sqrt{2}\tilde{\sigma}$	$\tilde{k}(T) = \frac{c(T)}{\left(-\frac{\sqrt{\rho}\epsilon f_{\lambda q_0}(\tilde{\sigma})}{\exp(-\tilde{\sigma}^2)}+\sqrt{\rho}\lambda\int_{T_f}^{T} c(T)\mathrm{d}T\right)^2}$	
	where $\tilde{\sigma}$ satisfy (63)		
4	$\tilde{u}_0 = \sqrt{2}\lambda q_0$	$\tilde{\gamma} = \frac{\exp(-2\tilde{\sigma}^2)}{\rho^2\epsilon^2\frac{\pi}{2}\left[g\left(\lambda q_0, \frac{1}{\sqrt{\pi}}\right)-\mathrm{erf}(\tilde{\sigma})\right]^2}$	(70)
	$\tilde{u}_1 = \sqrt{2}\tilde{\sigma}$	$\tilde{c}(T) = \frac{k(T)}{\left(\frac{-\exp(-\tilde{\sigma}^2)}{\sqrt{\rho}\epsilon f_{\lambda q_0}(\tilde{\sigma}}) - \sqrt{\rho}\lambda\int_{T_f}^{T} k(T)\mathrm{d}T\right)^2}$	
	where $\tilde{\sigma}$ satisfy (75)		

and we have that the following problem is equivalent to (10)–(14)

$$\frac{\mathrm{d}^2 y}{\mathrm{d}\xi^2} + \frac{\gamma\xi}{y^2}\frac{\mathrm{d}y}{\mathrm{d}\xi} = 0, \quad 0 < \xi < 1, \tag{17}$$

$$y'(0) = -\lambda h\sqrt{2\gamma}\left[P(y^2(0)) - T_\mathrm{m}\right], \tag{18}$$

$$y'(0) = -\lambda q_0\sqrt{2\gamma}, \tag{19}$$

$$y'(1) = -\rho L\lambda\gamma, \tag{20}$$

$$y(1) = y_1 = \sqrt{\frac{k}{\rho c}(T_f)}\ . \tag{21}$$

where P is the inverse function of the decreasing function $\frac{k}{\rho c} = \frac{k}{\rho c}(T)$.

A parametric solution to the problem (17)–(21) is given by Briozzo and Natale (2014, 2016), Hill and Hart (1986)

$$\xi = \varphi_1(u) = \frac{F_{u_0}(u)}{F_{u_0}(u_1)}, \tag{22}$$

Table 2 Formulae for two unknown thermal coefficients and condition on data for the moving boundary problem

Case	Formulae for the parameters $\tilde{u}_0$ and $\tilde{u}_1$	Formulae for two coefficients	Condition
5	$\tilde{u}_0=\sqrt{2}\lambda q_0$	$\tilde{\rho}=\frac{Y_2^2\lambda^2 q_0^2 f_{\lambda q_0}^2(\frac{\tilde{u}_1}{\sqrt{2}})}{\gamma\exp(-2\lambda^2 q_0^2)}$	(44)
	$\tilde{u}_1=\sqrt{2}\operatorname{erf}^{-1}\left(g\left(\lambda q_0,\frac{1}{\sqrt{\pi}}\left(1-\frac{Y_1}{Y_2}\right)\right)\right)$	$\tilde{L}=\frac{\exp\left(\frac{-\tilde{u}_1^2}{2}\right)}{\lambda\tilde{\rho}[g(\lambda q_0,\frac{1}{\sqrt{\pi}})-\operatorname{erf}(\frac{\tilde{u}_1}{2})]}$	
6	$\tilde{u}_0=\sqrt{2}\lambda q_0$	$\tilde{\rho}=\frac{\exp(-\tilde{\sigma}^2)}{\sqrt{\gamma}\epsilon\sqrt{\frac{\pi}{2}}\left[g\left(\lambda q_0,\frac{1}{\sqrt{\pi}}\right)-\operatorname{erf}(\tilde{\sigma})\right]}$	
	$\tilde{u}_1=\sqrt{2}\tilde{\sigma}$	$\tilde{k}(T)=\frac{c(T)}{\left(-\frac{\sqrt{\tilde{\rho}}\epsilon f_{\lambda q_0}(\tilde{\sigma})}{\exp(-\tilde{\sigma}^2)}+\sqrt{\tilde{\rho}}\lambda\int_{T_f}^{T}c(T)\mathrm{d}T\right)^2}$	
	where $\tilde{\sigma}$ satisfy (98)		
7	$\tilde{u}_0=\sqrt{2}\lambda q_0$	$\tilde{\rho}=\frac{\exp(-\tilde{\sigma}^2)}{\sqrt{\gamma}\epsilon\sqrt{\frac{\pi}{2}}\left[g\left(\lambda q_0,\frac{1}{\sqrt{\pi}}\right)-\operatorname{erf}(\tilde{\sigma})\right]}$	(104)
	$\tilde{u}_1=\sqrt{2}\tilde{\sigma}$	$\tilde{c}(T)=\frac{k(T)}{\left(\frac{-\exp(-\tilde{\sigma}^2)}{\sqrt{\rho}\epsilon f_{\lambda q_0}(\tilde{\sigma})}-\sqrt{\rho}\lambda\int_{T_f}^{T}k(T)\mathrm{d}T\right)^2}$	
	where $\tilde{\sigma}$ satisfy (109)		
8	$\tilde{u}_0=\sqrt{2}\lambda q_0$	$\tilde{L}=\frac{\exp(-\tilde{\sigma}^2)}{\lambda\sqrt{\gamma}\rho\sqrt{\frac{\pi}{2}}\left[g\left(\lambda q_0,\frac{1}{\sqrt{\pi}}\right)-\operatorname{erf}(\tilde{\sigma})\right]}$	
	$\tilde{u}_1=\sqrt{2}\tilde{\sigma}$	$\tilde{k}(T)=\frac{c(T)}{\left(-\frac{\sqrt{\tilde{\rho}}\lambda\tilde{L}f_{\lambda q_0}(\tilde{\sigma})}{\exp(-\tilde{\sigma}^2)}+\sqrt{\rho}\lambda\int_{T_f}^{T}c(T)\mathrm{d}T\right)^2}$	
	where $\tilde{\sigma}$ satisfy (120)		
9	$\tilde{u}_0=\sqrt{2}\lambda q_0$	$\tilde{L}=\frac{\exp(-\tilde{\sigma}^2)}{\lambda\sqrt{\gamma}\rho\sqrt{\frac{\pi}{2}}\left[g\left(\lambda q_0,\frac{1}{\sqrt{\pi}}\right)-\operatorname{erf}(\tilde{\sigma})\right]}$	(104)
	$\tilde{u}_1=\sqrt{2}\tilde{\sigma}$	$\tilde{c}(T)=\frac{k(T)}{\left(\frac{-\exp(-\tilde{\sigma}^2)}{\sqrt{\rho}\lambda\tilde{L}f_{\lambda q_0}(\tilde{\sigma})}-\sqrt{\rho}\lambda\int_{T_f}^{T}k(T)\mathrm{d}T\right)^2}$	
	where $\tilde{\sigma}$ satisfy (127)		

$$y=\varphi_2(u)=\frac{\sqrt{\gamma}\sqrt{\frac{\pi}{2}}\left[\operatorname{erf}\left(\frac{u}{\sqrt{2}}\right)-g\left(\frac{u_0}{\sqrt{2}},\frac{1}{\sqrt{\pi}}\right)\right]}{F_{u_0}(u_1)}, \tag{23}$$

for

$$u_0\le u\le u_1$$

the function $F_{u_0}=F_{u_0}(u)$ was defined in Briozzo and Natale (2014) as follow

$$F_{u_0}(u)=\exp(-\frac{u^2}{2})+u\left(\int_{u_0}^{u}\exp(-\frac{z^2}{2})dz-\frac{\exp(-\frac{u_0^2}{2})}{u_0}\right) \tag{24}$$

$$= \sqrt{\frac{\pi}{2}} u \left[g\left(\frac{u}{\sqrt{2}}, \frac{1}{\sqrt{\pi}}\right) - g\left(\frac{u_0}{\sqrt{2}}, \frac{1}{\sqrt{\pi}}\right) \right], \ u \geq u_0 \tag{25}$$

with u_0, u_1 are the parameter values which verify that $\xi = \varphi_1(u_0) = 0$ and $\xi = \varphi_1(u_1) = 1$,

$$g(x, p) = \operatorname{erf}(x) + pR(x), \ \ p > 0, \ \ x > 0 \tag{26}$$

where

$$R(x) = \frac{\exp(-x^2)}{x}$$

and

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-z^2) dz, \ \ x > 0.$$

The unknowns $\gamma, \ u_0, \ u_1$ and an unknown coefficient (chosen among ρ, L, $k(T)$ and $c(T)$) must verify the following system of equations

$$u_0 = \sqrt{2} \lambda q_0 \tag{27}$$

$$\sqrt{\frac{k}{\rho c}(T_f)} = \frac{-\exp(-\frac{u_1^2}{2})}{\rho L \lambda F_{u_0}(u_1)} \tag{28}$$

$$\sqrt{\gamma} = \frac{\exp(-\frac{u_1^2}{2})}{\rho L \lambda \sqrt{\frac{\pi}{2}} \left[g\left(\frac{u_0}{\sqrt{2}}, \frac{1}{\sqrt{\pi}}\right) - \operatorname{erf}\left(\frac{u_1}{\sqrt{2}}\right) \right]} \tag{29}$$

$$\frac{k}{\rho c} \left(\frac{q_0}{h} + T_{\mathrm{m}} \right) = \frac{\gamma \exp(-u_0^2)}{\left[u_0 F_{u_0}(u_1) \right]^2} \tag{30}$$

Remark 1 From conditions (2) and (3) we have that the temperature at fixed face $x = 0$ is constant and it satisfies

$$T_0 := T(0, t) = \frac{q_0}{h} + T_{\mathrm{m}} \tag{31}$$

Before to solve the system of Eqs. (27)–(30) we enunciate the following results.

Lemma 1 *The function* g *given by* (26) *satisfies the following properties* (Briozzo et al. 1999):

$$g(0^+, p) = +\infty, \ \forall p > 0, \ g\left(Q^{-1}(p\sqrt{\pi}), p\right) = 1 \ \ \textit{for} \ \ 0 < p < \frac{1}{\sqrt{\pi}},$$

$$g(+\infty, p) = \begin{cases} 1^+, & p \geq \frac{1}{\sqrt{\pi}}, \\ 1^-, & 0 < p < \frac{1}{\sqrt{\pi}}, \end{cases}$$

$$\frac{\partial g}{\partial x}(x,p) = \begin{cases} < 0, & x > 0, \quad p \geq \frac{1}{\sqrt{\pi}}, \\ < 0, & 0 < x < \sqrt{\frac{p}{2(\frac{1}{\sqrt{\pi}}-p)}}, \quad 0 < p < \frac{1}{\sqrt{\pi}}, \\ = 0, & x = \sqrt{\frac{p}{2(\frac{1}{\sqrt{\pi}}-p)}}, \quad 0 < p < \frac{1}{\sqrt{\pi}}, \\ > 0, & x > \sqrt{\frac{p}{2(\frac{1}{\sqrt{\pi}}-p)}}, \quad 0 < p < \frac{1}{\sqrt{\pi}}. \end{cases}$$

Lemma 2 *The function* F_{u_0} *defined by* (25) *satisfies the following properties* (Briozzo et al. 1999; Briozzo and Natale 2014, 2016):

$$F_{u_0}(u_0) = 0, \;\; F(+\infty) = -\infty \tag{32}$$

$$F'_{u_0}(x) = \sqrt{\frac{\pi}{2}}\left\{\operatorname{erf}\left(\frac{x}{\sqrt{2}}\right) - g\left(\frac{u_0}{\sqrt{2}}, \frac{1}{\sqrt{\pi}}\right)\right\} < 0. \tag{33}$$

To simplify the expressions we define the new variables and the parameter

$$\eta = \frac{u_0}{\sqrt{2}}, \quad \sigma = \frac{u_1}{\sqrt{2}}, \quad \epsilon = \lambda L \tag{34}$$

and we denote

$$f_\eta(\sigma) = \sqrt{\pi}\sigma\left[g\left(\sigma, \frac{1}{\sqrt{\pi}}\right) - g\left(\eta, \frac{1}{\sqrt{\pi}}\right)\right] = F_{u_0}(u_1), \;\; \sigma \geq \eta \tag{35}$$

then the system (27)–(30) can be expressed as follow

$$\eta = \lambda q_0 \tag{36}$$

$$\sqrt{\frac{k}{c}(T_f)} = \frac{-\exp(-\sigma^2)}{\sqrt{\rho}\epsilon f_\eta(\sigma)} \tag{37}$$

$$\sqrt{\gamma} = \frac{\exp(-\sigma^2)}{\rho\epsilon\sqrt{\frac{\pi}{2}}\left[g\left(\eta, \frac{1}{\sqrt{\pi}}\right) - \operatorname{erf}(\sigma)\right]} \tag{38}$$

$$\sqrt{\frac{k}{c}(T_0)} = -\frac{\sqrt{\rho}\sqrt{\gamma}\exp(-\eta^2)}{\sqrt{2}\eta f_\eta(\sigma)} \tag{39}$$

in the unknowns η, σ, γ and an unknown coefficient chosen among ρ, $k(T)$, $c(T)$ and L (or ϵ).

We shall give necessary and sufficient conditions to obtain η, σ (this is u_0 and u_1) and we also give formulae for the coefficients γ and an unknown thermal coefficient in the following four cases:

Case 1: Determination of ρ
Case 2: Determination of L (or ϵ)
Case 3: Determination of $k(T)$
Case 4: Determination of $c(T)$

which are the solutions to system (36)–(39).

To solve Case 3 and Case 4 we will add (7) which is equivalent to the relations established in the following lemma

Lemma 3 *The condition (7) is equivalent to each of the following relationships*

$$\sqrt{\frac{c}{k}(T)} = \sqrt{\frac{c}{k}(T_{\mathrm{m}})} + \sqrt{\rho}\lambda \int\limits_{T_{\mathrm{m}}}^{T} c(T)\mathrm{d}T \tag{40}$$

or

$$k(T) = \frac{c(T)}{\left(\sqrt{\frac{c}{k}(T_{\mathrm{m}})} + \sqrt{\rho}\lambda \int\limits_{T_{\mathrm{m}}}^{T} c(T)\mathrm{d}T\right)^2} \tag{41}$$

$$\sqrt{\frac{k}{c}(T)} = \sqrt{\frac{k}{c}(T_{\mathrm{m}})} - \sqrt{\rho}\lambda \int\limits_{T_{\mathrm{m}}}^{T} k(T)\mathrm{d}T \tag{42}$$

$$c(T) = \frac{k(T)}{\left(\sqrt{\frac{k}{c}(T_{\mathrm{m}})} - \sqrt{\rho}\lambda \int\limits_{T_{\mathrm{m}}}^{T} k(T)\mathrm{d}T\right)^2} \tag{43}$$

Proof We solve the ordinary differential Eq. (7) in unknown $k(T)$ or $c(T)$. □

Hereinafter we solve all cases of determination of coefficients whose results are given in the following lemmas. At the end of the paper, we give a summary table (Table 1) with the formulae and the corresponding restrictions.

Lemma 4 (Case 1) *Let γ and ρ be unknowns.*
If

$$\lambda > \frac{1}{q_0} Q^{-1}\left(1 - \frac{\sqrt{\frac{k}{c}(T_f)}}{\sqrt{\frac{k}{c}(T_0)}}\right) \tag{44}$$

then there exists unique solution to (27)–(30) which is given by

$$\tilde{u}_0 = \sqrt{2}\lambda q_0 \tag{45}$$

$$\tilde{u}_1 = \sqrt{2}\mathrm{erf}^{-1}\left(g\left(\lambda q_0, \frac{1}{\sqrt{\pi}}\left(1 - \frac{\sqrt{\frac{k}{c}(T_f)}}{\sqrt{\frac{k}{c}(T_0)}}\right)\right)\right) \tag{46}$$

$$\tilde{\gamma} = \frac{2\lambda^2 q_0^2 F^2_{\sqrt{2}\lambda q_0}(\tilde{u}_1) Y_2^2}{\exp(-2\lambda^2 q_0^2)\tilde{\rho}} \tag{47}$$

and

$$\tilde{\rho} = \frac{\exp(-\tilde{u}_1^2)}{Y_1^2 \epsilon^2 F^2_{\sqrt{2}\lambda q_0}(\tilde{u}_1)} \tag{48}$$

Proof From (36) we have $\tilde{\eta} = \lambda q_0$.
The Eqs. (37) and (39) are equivalent to

$$\sqrt{\rho} = \frac{-\exp(-\sigma^2)}{Y_1 \epsilon f_\eta(\sigma)} \tag{49}$$

and

$$\sqrt{\gamma} = -\frac{\sqrt{2}\eta f_\eta(\sigma) Y_2}{\exp(-\eta^2)\sqrt{\rho}} \tag{50}$$

respectively, where $Y_1 = \sqrt{\frac{k}{c}(T_f)}$ and $Y_2 = \sqrt{\frac{k}{c}(T_0)}$. Then, from (35), (38), (49) and (50) we have

$$g\left(\lambda q_0, \frac{1}{\sqrt{\pi}}\left(1 - \frac{Y_1}{Y_2}\right)\right) = \operatorname{erf}(\sigma). \tag{51}$$

By $T_0 < T_f$ we have $\frac{Y_1}{Y_2} < 1$. If $\tilde{\eta} > Q^{-1}\left(1 - \frac{Y_1}{Y_2}\right)$, this is (44), it follows that

$$\tilde{\sigma} = \operatorname{erf}^{-1}\left(g\left(\tilde{\eta}, \frac{1}{\sqrt{\pi}}\left(1 - \frac{Y_1}{Y_2}\right)\right)\right). \tag{52}$$

The solutions $\tilde{\gamma}$ and $\tilde{\rho}$ are given by

$$\sqrt{\tilde{\gamma}} = -\frac{\sqrt{2}\lambda q_0 f_{\lambda q_0}(\tilde{\sigma}) Y_2}{\exp(-\lambda^2 q_0^2)\sqrt{\tilde{\rho}}} \tag{53}$$

and

$$\sqrt{\tilde{\rho}} = \frac{-\exp(-\tilde{\sigma}^2)}{Y_1 \epsilon f_{\lambda q_0}(\tilde{\sigma})} \tag{54}$$

respectively. With (34) the proof is completed. □

Lemma 5 (Case 2) *If the coefficients γ and L (i.e ϵ) are unknowns and the coefficients satisfy* (44) *then there exists a unique solution to* (27)–(30) *given by*

$$\tilde{u}_0 = \sqrt{2}\lambda q_0 \tag{55}$$

$$\tilde{u}_1 = \sqrt{2}\operatorname{erf}^{-1}\left(g\left(\lambda q_0, \frac{1}{\sqrt{\pi}}\left(1 - \frac{\sqrt{\frac{k}{c}(T_f)}}{\sqrt{\frac{k}{c}(T_0)}}\right)\right)\right) \tag{56}$$

$$\tilde{\gamma} = \frac{2\lambda^2 q_0^2 F^2_{\sqrt{2}\lambda q_0}(\tilde{u}_1) Y_2^2}{\exp(-2\lambda^2 q_0^2)\tilde{\rho}} \tag{57}$$

and

$$L = \frac{\exp\left(-\frac{\tilde{u}_1^2}{2}\right)}{Y_1^2 F_{\sqrt{2}\lambda q_0}(\tilde{u}_1)\sqrt{\rho}} \tag{58}$$

Proof It follows with analogous reasoning of Lemma 2.

Lemma 6 (Case 3) *If the coefficients γ and $k(T)$ are unknowns then there exists a unique solution to* (27)–(30) *given by*

$$\tilde{u}_0 = \sqrt{2}\lambda q_0 \tag{59}$$

$$\tilde{u}_1 = \sqrt{2}\tilde{\sigma} \tag{60}$$

$$\tilde{\gamma} = \frac{\exp(-2\tilde{\sigma}^2)}{\rho^2 \epsilon^2 \frac{\pi}{2}\left[g\left(\lambda q_0, \frac{1}{\sqrt{\pi}}\right) - \operatorname{erf}(\tilde{\sigma})\right]^2} \tag{61}$$

and

$$\tilde{k}(T) = \frac{c(T)}{\left(-\frac{\sqrt{\rho}\epsilon f_{\lambda q_0}(\tilde{\sigma})}{\exp(-\tilde{\sigma}^2)} + \sqrt{\rho}\lambda \int\limits_{T_f}^{T} c(T)\mathrm{d}T\right)^2} \tag{62}$$

where $\tilde{\sigma}$ *is the unique solution of equation*

$$\frac{\epsilon f_{\lambda q_0}(\sigma)}{\exp(-\sigma^2)}\left\{\lambda q_0 \exp(\lambda^2 q_0^2)\sqrt{\pi}\left[g\left(\lambda q_0, \frac{1}{\sqrt{\pi}}\right) - \mathrm{erf}(\sigma)\right] - 1\right\} = \lambda \int\limits_{T_f}^{T_0} c(T)\mathrm{d}T \tag{63}$$

Proof Taking into account (40) we can rewrite (37) and (39) as follow

$$\frac{-\sqrt{\rho}\epsilon f_\eta(\sigma)}{\exp(-\sigma^2)} = \sqrt{\frac{c}{k}(T_\mathrm{m})} + \sqrt{\rho}\lambda \int\limits_{T_\mathrm{m}}^{T_f} c(T)\mathrm{d}T \tag{64}$$

$$\sqrt{\frac{c}{k}(T_\mathrm{m})} + \sqrt{\rho}\lambda \int\limits_{T_\mathrm{m}}^{T_f} c(T)\mathrm{d}T + \sqrt{\rho}\lambda \int\limits_{T_f}^{T_0} c(T)\mathrm{d}T = -\frac{\sqrt{2}\eta f_\eta(\sigma)}{\sqrt{\rho}\sqrt{\gamma}\exp(-\eta^2)} \tag{65}$$

Now we solve the system (36), (64), (38) and (65) in the unknowns η, σ, γ and $\sqrt{\frac{c}{k}(T_\mathrm{m})}$. From (36) we determine $\tilde{\eta} = \lambda q_0$. By (64) and (65) we have

$$\frac{\sqrt{2}\tilde{\eta} f_{\tilde{\eta}}(\sigma)}{\sqrt{\rho}\sqrt{\gamma}\exp(-\tilde{\eta}^2)} - \sqrt{\rho}\lambda \int\limits_{T_0}^{T_f} c(T)\mathrm{d}T = \frac{\sqrt{\rho}\epsilon f_{\tilde{\eta}}(\sigma)}{\exp(-\sigma^2)} \tag{66}$$

Taking into account (38) the Eq. (66) is equivalent to

$$\frac{\epsilon f_{\tilde{\eta}}(\sigma)}{\exp(-\sigma^2)}\left\{\tilde{\eta}\exp(\tilde{\eta}^2)\sqrt{\pi}\left[g\left(\tilde{\eta}, \frac{1}{\sqrt{\pi}}\right) - \mathrm{erf}(\sigma)\right] - 1\right\} = \lambda \int\limits_{T_0}^{T_f} c(T)\mathrm{d}T \tag{67}$$

in unknown σ.

The function

$$Z(\sigma) = \frac{\epsilon f_{\tilde{\eta}}(\sigma)}{\exp(-\sigma^2)}\left\{\tilde{\eta}\exp(\tilde{\eta}^2)\sqrt{\pi}\left[g\left(\tilde{\eta}, \frac{1}{\sqrt{\pi}}\right) - \mathrm{erf}(\sigma)\right] - 1\right\}$$

satisfies

$$Z(\tilde{\eta}) = 0, \qquad Z(+\infty) = +\infty \qquad Z'(\sigma) > 0$$

therefore there exist a unique solution $\tilde{\sigma}$ to the Eq. (67).

From (38) we have

$$\tilde{\gamma} = \frac{\exp(-2\tilde{\sigma}^2)}{\rho^2\epsilon^2\frac{\pi}{2}\left[g\left(\tilde{\eta}, \frac{1}{\sqrt{\pi}}\right) - \mathrm{erf}(\tilde{\sigma})\right]^2} \tag{68}$$

and from (64) we obtain

$$\sqrt{\frac{c}{k}(T_\mathrm{m})} = -\frac{\sqrt{\rho}\epsilon f_{\tilde{\eta}}(\tilde{\sigma})}{\exp(-\tilde{\sigma}^2)} - \sqrt{\rho}\lambda \int\limits_{T_\mathrm{m}}^{T_f} c(T)\mathrm{d}T. \tag{69}$$

Then by (41) we determine $\tilde{k}(T)$ which is given by (62). □

Lemma 7 (Case 4) *If the coefficients γ and $c(T)$ are unknowns and the data satisfy*

$$\int_{T_0}^{T_f} k(T)\mathrm{d}T > \frac{4q_0^2}{\rho L} \tag{70}$$

then there exists a unique solution to (27)–(30) given by

$$\tilde{u}_0 = \sqrt{2}\lambda q_0 \tag{71}$$

$$\tilde{u}_1 = \sqrt{2}\tilde{\sigma} \tag{72}$$

$$\tilde{\gamma} = \frac{\exp(-2\tilde{\sigma}^2)}{\rho^2\epsilon^2\frac{\pi}{2}\left[g\left(\lambda q_0, \frac{1}{\sqrt{\pi}}\right) - \operatorname{erf}(\tilde{\sigma})\right]^2} \tag{73}$$

and

$$\tilde{c}(T) = \frac{k(T)}{\left(\frac{-\exp(-\tilde{\sigma}^2)}{\sqrt{\rho}\epsilon f_{\lambda q_0}(\tilde{\sigma}}) - \sqrt{\rho}\lambda\int_{T_f}^{T} k(T)\mathrm{d}T\right)^2} \tag{74}$$

where $\tilde{\sigma}$ is the unique solution of equation

$$\frac{\exp(-\sigma^2)}{\sqrt{\rho}\epsilon f_{\lambda q_0}(\sigma)}\left\{1 - \frac{\exp(-\lambda^2 q_0^2)}{\sqrt{\pi}\lambda q_0\left[g\left(\lambda q_0, \frac{1}{\sqrt{\pi}}\right) - \operatorname{erf}(\sigma)\right]}\right\} = \sqrt{\rho}\lambda\int_{T_0}^{T_f} k(T)\mathrm{d}T \tag{75}$$

Proof By (43) we will to find $\sqrt{\frac{k}{c}(T_m)}$ to obtain $c(T)$. Taking into account (42) we can rewrite (37) and (39) as follow

$$\frac{-\exp(-\sigma^2)}{\sqrt{\rho}\epsilon f_\eta(\sigma)} = \sqrt{\frac{k}{c}(T_m)} - \sqrt{\rho}\lambda\int_{T_m}^{T_f} k(T)\mathrm{d}T \tag{76}$$

$$\frac{-\exp(-\eta^2)\sqrt{\rho}\sqrt{\gamma}}{\sqrt{2}\eta f_\eta(\sigma)} = \sqrt{\frac{k}{c}(T_m)} - \sqrt{\rho}\lambda\int_{T_m}^{T_0} k(T)\mathrm{d}T \tag{77}$$

Now we solve the system given by (36), (76), (38) and (77) in the unknowns η, σ, γ and $\sqrt{\frac{k}{c}(T_m)}$. From (36) we determine $\tilde{\eta} = \lambda q_0$.

By (76) and (77) we have

$$\frac{-\exp(-\eta^2)\sqrt{\rho}\sqrt{\gamma}}{\sqrt{2}\eta f_\eta(\sigma)} = \frac{-\exp(-\sigma^2)}{\sqrt{\rho}\epsilon f_\eta(\sigma)} + \sqrt{\rho}\lambda\int_{T_0}^{T_f} k(T)\mathrm{d}T \tag{78}$$

Taking into account (38) the Eq. (78) is equivalent to equation

$$\frac{\exp(-\sigma^2)}{\sqrt{\rho}\epsilon f_{\tilde{\eta}}(\sigma)}\left\{1 - \frac{\exp(-\tilde{\eta}^2)}{\tilde{\eta}\sqrt{\pi}\left[g\left(\tilde{\eta}, \frac{1}{\sqrt{\pi}}\right) - \operatorname{erf}(\sigma)\right]}\right\} = \sqrt{\rho}\lambda\int_{\frac{q_0}{h}+T_m}^{T_f} k(T)\mathrm{d}T \tag{79}$$

in unknown σ.

The function

$$Y(\sigma) = \frac{\exp(-\sigma^2)}{\sqrt{\rho}\epsilon f_{\tilde{\eta}}(\sigma)} \left\{ 1 - \frac{\exp(-\tilde{\eta}^2)}{\tilde{\eta}\sqrt{\pi}\left[g\left(\tilde{\eta}, \frac{1}{\sqrt{\pi}}\right) - \operatorname{erf}(\sigma)\right]} \right\}$$

$$= \frac{\exp(-\sigma^2)}{\sqrt{\rho}\epsilon f_{\tilde{\eta}}(\sigma)} \left[\frac{\operatorname{erf}(\tilde{\eta}) - \operatorname{erf}(\sigma)}{g\left(\tilde{\eta}, \frac{1}{\sqrt{\pi}}\right) - \operatorname{erf}(\sigma)} \right]$$

satisfies

$$Y(\tilde{\eta}) = \frac{4\tilde{\eta}^2}{\epsilon\sqrt{\rho}}, \qquad Y(+\infty) = +\infty, \qquad Y'(\sigma) > 0$$

therefore, if

$$\sqrt{\rho}\lambda \int_{T_0}^{T_f} k(T)\mathrm{d}T > \frac{4\tilde{\eta}^2}{\epsilon\sqrt{\rho}}$$

(this is (70)), there exist a unique solution $\tilde{\sigma}$ to Eq. (79) .

From (38) we have

$$\tilde{\gamma} = \frac{\exp(-2\tilde{\sigma}^2)}{\rho^2\epsilon^2 \frac{\pi}{2} \left[g\left(\tilde{\eta}, \frac{1}{\sqrt{\pi}}\right) - \operatorname{erf}(\tilde{\sigma})\right]^2} \tag{80}$$

and from (76) we obtain

$$\sqrt{\frac{k}{c}(T_{\mathrm{m}})} = \frac{-\exp(-\sigma^2)}{\sqrt{\rho}\epsilon f_{\eta}(\sigma)} + \sqrt{\rho}\lambda \int_{T_{\mathrm{m}}}^{T_f} k(T)\mathrm{d}T \tag{81}$$

then by (43) we determine

$$\tilde{c}(T) = \frac{k(T)}{\left(\frac{-\exp(-\tilde{\sigma}^2)}{\sqrt{\rho}\epsilon f_{\tilde{\eta}}(\tilde{\sigma}}) - \sqrt{\rho}\lambda \int_{T_f}^{T} k(T)\mathrm{d}T\right)^2}.$$

□

Now, we enunciate the following Theorem

Theorem 1 *Under the corresponding hypothesis established in the previous Lemmas, the free boundary problem* (1)–(7) *has a unique similarity type solution given by*

$$T(x,t) = P\left(\left(\varphi_2\left(\varphi_1^{-1}(x/X(t))\right)\right)^2\right), \qquad 0 < x < X(t) \tag{82}$$

where

$$X(t) = \sqrt{2\tilde{\gamma}t}, \quad t > 0 \tag{83}$$

is the free boundary,

$$\varphi_1(u) = \frac{F_{\tilde{u}_0}(u)}{F_{\tilde{u}_0}(\tilde{u}_1)}, \tag{84}$$

$$\varphi_2(u) = \frac{\sqrt{\tilde{\gamma}}\sqrt{\frac{\pi}{2}}\left[\operatorname{erf}\left(\frac{u}{\sqrt{2}}\right) - g\left(\frac{\tilde{u}_0}{\sqrt{2}}, \frac{1}{\sqrt{\pi}}\right)\right]}{F_{\tilde{u}_0}(\tilde{u}_1)}, \tag{85}$$

$\tilde{u}_0$, $\tilde{u}_1$, $\tilde{\gamma}$ and a thermal coefficient chosen among ρ, L, $k(T)$ and $c(T)$ is the unique solution of (27)–(30) *and* $P = \left(\frac{k}{\rho c}\right)^{-1}$ *is the inverse function of the function* $\frac{k}{\rho c}$.

Proof Fixed the data of the problem (1)–(7), under the corresponding restrictions on them (see previous lemmas) we obtain the solutions of the Eqs. (27)–(30) given by $\tilde{u}_0$, $\tilde{u}_1$, $\tilde{\gamma}$ and the corresponding thermal coefficient (see Table 1).

Next, we obtain φ_1 and φ_2 given by (84), (85) respectively and the free boundary is $X(t) = \sqrt{2\tilde{\gamma}t}$. Taking into account that φ_1 is an increasing function we determine $\varphi_1^{-1}\left(\frac{x}{X(t)}\right)$. Finally, we invert the relation (15) and from (8) we obtain (82). □

3 Determination of two thermal coefficients through a moving boundary problem

To determine two unknown thermal coefficients we consider the inverse Stefan problem (1)–(6). We assume that $X = X(t)$ is known and it is defined by $X(t) = \sqrt{2\gamma t}$ for a given $\gamma > 0$, and the thermal coefficients of material verify condition (7). The temperature T of this problem is given by (82) where ϕ_1 and ϕ_2 are given by (84) and (85) respectively. Then the two unknown coefficients can be chosen among ρ, L (or ϵ), $k(T)$ and $c(T)$, which must verify Eqs. (36)–(39) and the condition (7) when $k(T)$ or $c(T)$ is one of the thermal coefficient to determinate.

We shall give necessary and sufficient conditions to obtain η, σ (this is u_0 and u_1) and we also give formulae for the two unknown thermal coefficients which are the solutions to system (36)–(39) in the following five cases:

Case 5: Determination of ρ and L
Case 6: Determination of ρ and $k(T)$
Case 7: Determination of ρ and $c(T)$
Case 8: Determination of L and $k(T)$
Case 9: Determination of L and $c(T)$

The results are summarize in Table 2.

Lemma 8 (Case 5) *Let L and ρ be unknowns. If* (44) *its satisfied then there exists unique solution to* (27)–(30) *which is given by*

$$\tilde{u}_0 = \sqrt{2}\lambda q_0 \tag{86}$$

$$\tilde{u}_1 = \sqrt{2}\operatorname{erf}^{-1}\left(g\left(\lambda q_0, \frac{1}{\sqrt{\pi}}\left(1 - \frac{\sqrt{\frac{k}{c}(T_f)}}{\sqrt{\frac{k}{c}(T_0)}}\right)\right)\right) \tag{87}$$

$$\tilde{\rho} = \frac{Y_2^2\lambda^2 q_0^2 f_{\lambda q_0}^2(\frac{\tilde{u}_1}{\sqrt{2}})}{\gamma\exp(-2\lambda^2 q_0^2)} \tag{88}$$

$$\tilde{L} = \frac{\exp\left(\frac{-\tilde{u}_1^2}{2}\right)}{\lambda\tilde{\rho}[g(\lambda q_0, \frac{1}{\sqrt{\pi}}) - \operatorname{erf}(\frac{\tilde{u}_1}{2})]} \tag{89}$$

Proof From (36) we have $\tilde{\eta} = \lambda q_0$. From (37), (38) and (39) we obtain

$$g\left(\lambda q_0, \frac{1}{\sqrt{\pi}}\left(1 - \frac{Y_1}{Y_2}\right)\right) = \operatorname{erf}(\sigma). \tag{90}$$

where $Y_1 = \sqrt{\frac{k}{c}(T_f)}$ and $Y_2 = \sqrt{\frac{k}{c}(T_0)}$.

If $\tilde{\eta} > Q^{-1}\left(1 - \frac{Y_1}{Y_2}\right)$, this is (44), it follows that

$$\tilde{\sigma} = \operatorname{erf}^{-1}\left(g\left(\lambda q_0, \frac{1}{\sqrt{\pi}}\left(1 - \frac{Y_1}{Y_2}\right)\right)\right). \tag{91}$$

The solutions $\tilde{\rho}$ and $\tilde{L}$ are given by

$$\tilde{\rho} = \frac{Y_2^2 \tilde{\eta}^2 f_{\tilde{\eta}}^2(\tilde{\sigma})}{\gamma \exp(-2\tilde{\eta}^2)} \tag{92}$$

$$\tilde{L} = \frac{\exp\left(-\tilde{\sigma}^2\right)}{\lambda \tilde{\rho}[g(\lambda q_0, \frac{1}{\sqrt{\pi}}) - \operatorname{erf}(\tilde{\sigma})]} \tag{93}$$

With (34) the proof is completed. □

Lemma 9 (Case 6) *If the coefficients ρ and $k(T)$ are unknowns then there exists a unique solution to* (27)–(30) *given by*

$$\tilde{u}_0 = \sqrt{2}\lambda q_0 \tag{94}$$

$$\tilde{u}_1 = \sqrt{2}\tilde{\sigma} \tag{95}$$

$$\tilde{\rho} = \frac{\exp(-\tilde{\sigma}^2)}{\sqrt{\gamma}\epsilon\sqrt{\frac{\pi}{2}}\left[g\left(\lambda q_0, \frac{1}{\sqrt{\pi}}\right) - \operatorname{erf}(\tilde{\sigma})\right]} \tag{96}$$

and

$$\tilde{k}(T) = \frac{c(T)}{\left(-\frac{\sqrt{\tilde{\rho}}\epsilon f_{\lambda q_0}(\tilde{\sigma})}{\exp(-\tilde{\sigma}^2)} + \sqrt{\tilde{\rho}}\lambda \int_{T_f}^{T} c(T)\mathrm{d}T\right)^2} \tag{97}$$

where $\tilde{\sigma}$ is the unique solution of equation

$$\frac{\epsilon f_{\lambda q_0}(\sigma)}{\exp(-\sigma^2)}\left\{\lambda q_0 \exp(\lambda^2 q_0^2)\sqrt{\pi}\left[g\left(\lambda q_0, \frac{1}{\sqrt{\pi}}\right) - \operatorname{erf}(\sigma)\right] - 1\right\} = \lambda \int_{T_f}^{T_0} c(T)\mathrm{d}T \tag{98}$$

Proof As in Case 3, taking into account (40) we can rewrite (37) and (39) as follow

$$\frac{-\sqrt{\rho}\epsilon f_\eta(\sigma)}{\exp(-\sigma^2)} = \sqrt{\frac{c}{k}(T_\mathrm{m})} + \sqrt{\rho}\lambda \int_{T_\mathrm{m}}^{T_f} c(T)\mathrm{d}T \tag{99}$$

$$\sqrt{\frac{c}{k}(T_\mathrm{m})} + \sqrt{\rho}\lambda \int_{T_\mathrm{m}}^{T_f} c(T)\mathrm{d}T + \sqrt{\rho}\lambda \int_{T_f}^{T_0} c(T)\mathrm{d}T = -\frac{\sqrt{2}\eta f_\eta(\sigma)}{\sqrt{\rho}\sqrt{\gamma}\exp(-\eta^2)} \tag{100}$$

Now we solve the system (36), (99), (38) and (100) in the unknowns η, σ, ρ and $\sqrt{\frac{c}{k}(T_\mathrm{m})}$.

From (36) we determine $\tilde{\eta} = \lambda q_0$. By (99) and (100) we have

$$\frac{\sqrt{2}\tilde{\eta} f_{\tilde{\eta}}(\sigma)}{\sqrt{\rho}\sqrt{\gamma}\exp(-\tilde{\eta}^2)} - \sqrt{\rho}\lambda \int_{T_0}^{T_f} c(T)\mathrm{d}T = \frac{\sqrt{\rho}\epsilon f_{\tilde{\eta}}(\sigma)}{\exp(-\sigma^2)} \tag{101}$$

Taking into account (38) the Eq. (101) is equivalent to

$$\frac{\epsilon f_{\tilde{\eta}}(\sigma)}{\exp(-\sigma^2)} \left\{ \tilde{\eta}\exp(\tilde{\eta}^2)\sqrt{\pi} \left[g\left(\tilde{\eta}, \frac{1}{\sqrt{\pi}}\right) - \mathrm{erf}(\sigma) \right] - 1 \right\} = \lambda \int_{T_0}^{T_f} c(T)\mathrm{d}T \tag{102}$$

which has a unique solution $\tilde{\sigma}$.

From (38) we have (96) and from (99) we obtain

$$\sqrt{\frac{c}{k}(T_\mathrm{m})} = -\frac{\sqrt{\rho}\epsilon f_{\tilde{\eta}}(\tilde{\sigma})}{\exp(-\tilde{\sigma}^2)} - \sqrt{\rho}\lambda \int_{T_\mathrm{m}}^{T_f} c(T)\mathrm{d}T. \tag{103}$$

Then by (41) we determine $\tilde{k}(T)$ which is given by (97). □

Lemma 10 (Case 7) *If the coefficients ρ and $c(T)$ are unknowns and the data satisfy*

$$\int_{T_0}^{T_f} k(T)\mathrm{d}T \le \frac{2q_0\sqrt{\gamma}}{\sqrt{2}} \tag{104}$$

then there exists a unique solution to (27)–(30) given by

$$\tilde{u}_0 = \sqrt{2}\lambda q_0 \tag{105}$$

$$\tilde{u}_1 = \sqrt{2}\tilde{\sigma} \tag{106}$$

$$\tilde{\rho} = \frac{\exp(-\tilde{\sigma}^2)}{\sqrt{\gamma}\epsilon\sqrt{\frac{\pi}{2}}\left[g\left(\lambda q_0, \frac{1}{\sqrt{\pi}}\right) - \mathrm{erf}(\tilde{\sigma})\right]} \tag{107}$$

and

$$\tilde{c}(T) = \frac{k(T)}{\left(\frac{-\exp(-\tilde{\sigma}^2)}{\sqrt{\rho}\epsilon f_{\lambda q_0}(\tilde{\sigma})} - \sqrt{\rho}\lambda \int_{T_f}^{T} k(T)\mathrm{d}T \right)^2} \tag{108}$$

where $\tilde{\sigma}$ is solution of equation

$$\frac{\sqrt{\gamma\pi}}{\sqrt{2} f_{\tilde{\eta}}(\sigma)} \left[\mathrm{erf}(\tilde{\eta}) - \mathrm{erf}(\sigma) \right] = \lambda \int_{T_0}^{T_f} k(T)\mathrm{d}T \tag{109}$$

Proof By (43) we will to find $\sqrt{\frac{k}{c}(T_\mathrm{m})}$ to obtain $c(T)$. Taking into account (42) we can rewrite (37) and (39) as follow

$$\frac{-\exp(-\sigma^2)}{\sqrt{\rho}\epsilon f_{\eta}(\sigma)} = \sqrt{\frac{k}{c}(T_\mathrm{m})} - \sqrt{\rho}\lambda \int_{T_\mathrm{m}}^{T_f} k(T)\mathrm{d}T \tag{110}$$

$$\frac{-\exp(-\eta^2)\sqrt{\rho}\sqrt{\gamma}}{\sqrt{2}\eta f_\eta(\sigma)} = \sqrt{\frac{k}{c}(T_\mathrm{m})} - \sqrt{\rho}\lambda \int_{T_\mathrm{m}}^{T_0} k(T)\mathrm{d}T \tag{111}$$

Now we solve the system given by (36), (110), (38) and (111) in the unknowns η, σ, ρ and $\sqrt{\frac{k}{c}(T_\mathrm{m})}$. From (36) we determine $\tilde{\eta} = \lambda q_0$.

By (110) and (111) we have

$$\frac{-\exp(-\eta^2)\sqrt{\rho}\sqrt{\gamma}}{\sqrt{2}\eta f_\eta(\sigma)} = \frac{-\exp(-\sigma^2)}{\sqrt{\rho}\epsilon f_\eta(\sigma)} + \sqrt{\rho}\lambda \int_{T_0}^{T_f} k(T)\mathrm{d}T \tag{112}$$

Taking into account (38) the Eq. (112) is equivalent to equation

$$\frac{\sqrt{\gamma\pi}}{\sqrt{2}f_{\tilde{\eta}}(\sigma)}\left[\mathrm{erf}(\tilde{\eta}) - \mathrm{erf}(\sigma)\right] = \lambda \int_{T_0}^{T_f} k(T)\mathrm{d}T \tag{113}$$

in unknown σ.

The function

$$U(\sigma) = \frac{\sqrt{\gamma\pi}}{\sqrt{2}f_{\tilde{\eta}}(\sigma)}\left[\mathrm{erf}(\tilde{\eta}) - \mathrm{erf}(\sigma)\right]$$

satisfies

$$U(\tilde{\eta}) = \frac{2\tilde{\eta}\sqrt{\gamma}}{\sqrt{2}}, \qquad U(+\infty) = 0$$

therefore, if

$$\lambda \int_{T_0}^{T_f} k(T)\mathrm{d}T \le \frac{2\tilde{\eta}\sqrt{\gamma}}{\sqrt{2}}$$

(this is (104)), there exist at least a solution $\tilde{\sigma}$ to Eq. (113) .

From (38) we have

$$\tilde{\rho} = \frac{\exp(-\tilde{\sigma}^2)}{\sqrt{\gamma}\epsilon\sqrt{\frac{\pi}{2}}\left[g\left(\lambda q_0, \frac{1}{\sqrt{\pi}}\right) - \mathrm{erf}(\tilde{\sigma})\right]} \tag{114}$$

and from (110) we obtain

$$\sqrt{\frac{k}{c}(T_\mathrm{m})} = \frac{-\exp(-\sigma^2)}{\sqrt{\tilde{\rho}}\epsilon f_{\tilde{\eta}}(\tilde{\sigma})} + \sqrt{\tilde{\rho}}\lambda \int_{T_\mathrm{m}}^{T_f} k(T)\mathrm{d}T \tag{115}$$

then by (43) we determine

$$\tilde{c}(T) = \frac{k(T)}{\left(\frac{-\exp(-\tilde{\sigma}^2)}{\sqrt{\tilde{\rho}}\epsilon f_{\tilde{\eta}}(\tilde{\sigma})} - \sqrt{\tilde{\rho}}\lambda \int_{T_f}^{T} k(T)\mathrm{d}T\right)^2}$$

□

Lemma 11 (Case 8) *If the coefficients* L *and* $k(T)$ *are unknowns then there exists a unique solution to* (27)–(30) *given by*

$$\tilde{u}_0 = \sqrt{2}\lambda q_0 \tag{116}$$

$$\tilde{u}_1 = \sqrt{2}\tilde{\sigma} \tag{117}$$

$$\tilde{L} = \frac{\exp(-\tilde{\sigma}^2)}{\lambda\sqrt{\gamma}\rho\sqrt{\frac{\pi}{2}}\left[g\left(\lambda q_0, \frac{1}{\sqrt{\pi}}\right) - \operatorname{erf}(\tilde{\sigma})\right]} \tag{118}$$

and

$$\tilde{k}(T) = \frac{c(T)}{\left(-\frac{\sqrt{\rho}\lambda\tilde{L} f_{\lambda q_0}(\tilde{\sigma})}{\exp(-\tilde{\sigma}^2)} + \sqrt{\rho}\lambda\int\limits_{T_f}^{T} c(T)\mathrm{d}T\right)^2} \tag{119}$$

where $\tilde{\sigma}$ *is the unique solution of equation*

$$\frac{\lambda q_0 \exp(\lambda^2 q_0^2)\sqrt{2} f_{\lambda q_0}(\sigma)\left[\operatorname{erf}(\sigma) - \operatorname{erf}(\lambda q_0)\right]}{\rho\sqrt{\gamma}\left[g\left(\lambda q_0, \frac{1}{\sqrt{\pi}}\right) - \operatorname{erf}(\sigma)\right]} = -\lambda\int\limits_{T_0}^{T_f} c(T)\mathrm{d}T \tag{120}$$

Proof From (36) we determine $\tilde{\eta} = \lambda q_0$. By (38), (99) and (66) we have

$$\frac{\tilde{\eta}\exp(\tilde{\eta}^2)\sqrt{2} f_{\tilde{\eta}}(\sigma)\left[\operatorname{erf}(\sigma) - \operatorname{erf}(\tilde{\eta})\right]}{\rho\sqrt{\gamma}\left[g\left(\tilde{\eta}, \frac{1}{\sqrt{\pi}}\right) - \operatorname{erf}(\sigma)\right]} = -\lambda\int\limits_{T_0}^{T_f} c(T)\mathrm{d}T \tag{121}$$

The function $J = J(\sigma)$ given by

$$J(\sigma) = \frac{\tilde{\eta}\exp(\tilde{\eta}^2)\sqrt{2} f_{\tilde{\eta}}(\sigma)\left[\operatorname{erf}(\sigma) - \operatorname{erf}(\tilde{\eta})\right]}{\rho\sqrt{\gamma}\left[g\left(\tilde{\eta}, \frac{1}{\sqrt{\pi}}\right) - \operatorname{erf}(\sigma)\right]} \tag{122}$$

satisfies $J(\tilde{\eta}) = 0$, $J(+\infty) = -\infty$ and $J'(\sigma) < 0$.

Then there exists unique $\tilde{\sigma}$ which solve (121).

From (38) we have (118). From (64), (118) and (41) we determine $\tilde{k}(T)$ which is given by (119). □

Lemma 12 (Case 9) *If the coefficients* L *and* $c(T)$ *are unknowns and the data satisfy* (104) *then there exists a unique solution to* (27)–(30) *given by*

$$\tilde{u}_0 = \sqrt{2}\lambda q_0 \tag{123}$$

$$\tilde{u}_1 = \sqrt{2}\tilde{\sigma} \tag{124}$$

$$\tilde{L} = \frac{\exp(-\tilde{\sigma}^2)}{\lambda\sqrt{\gamma}\rho\sqrt{\frac{\pi}{2}}\left[g\left(\lambda q_0, \frac{1}{\sqrt{\pi}}\right) - \operatorname{erf}(\tilde{\sigma})\right]} \tag{125}$$

and

$$\tilde{c}(T) = \frac{k(T)}{\left(\frac{-\exp(-\tilde{\sigma}^2)}{\sqrt{\rho}\lambda\tilde{L} f_{\lambda q_0}(\tilde{\sigma})} - \sqrt{\rho}\lambda\int\limits_{T_f}^{T} k(T)\mathrm{d}T\right)^2} \tag{126}$$

where $\tilde{\sigma}$ is solution of equation

$$\frac{\sqrt{\gamma\pi}}{\sqrt{2}f_{\tilde{\eta}}(\sigma)}\left[\operatorname{erf}(\tilde{\eta})-\operatorname{erf}(\sigma)\right]=\lambda\int_{T_0}^{T_f}k(T)\mathrm{d}T \tag{127}$$

Proof The proof is similar to that given in Case 7. □

4 Conclusions

A nonlinear one-dimensional Stefan problem for a semi-infinite material $x > 0$, with phase change temperature T_f was considered. The heat capacity and the thermal conductivity was assumed to satisfy a Storm's condition. A convective boundary condition and a heat flux over-specified condition at the fixed face $x = 0$ were considered. Under certain restrictions on data, a similarity type solution, the free boundary and one unknown thermal coefficient were determined. Then, the associate moving boundary problem with the same boundary conditions and assumption on the thermal conductivity and heat capacity was considered. In this problem, five different cases were considered and two unknowns thermal coefficients were determined in each one. From four cases of single parameter identification studied in the free boundary problem (1)–(7) we conclude that case 3 and case 4 are the most difficult to solve since the determined thermal coefficients depend on a parameter that is a unique solution of a given transcendental equation. For the same reason, we remark that cases 6–9 are the most difficult to solve in the simultaneous identification of two parameters in the moving boundary problem (1)–(7) where the phase change position is known. The results are summarized in Tables 1 and 2.

Acknowledgements This paper has been partially sponsored by the Proyect PIP No. 112-200801-00534 "Inecuaciones Variacionales, Control Optimo y Problemas de Frontera Libre: teoría, análisis numérico y aplicaciones", from CONICET-UA, Rosario-Argentina and Proyect: "Problemas de frontera libre unidimensionales", Universidad Austral, Rosario-Argentina.

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