# A nonlinear supercooled Stefan problem

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**Abstract.**

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## 1. Introduction

Supercooled Stefan problems describe the freezing of a liquid initially cooled below its freezing point. The practical importance of solids formed from a supercooled liquid motivates the need for the theorical understanding of the associated phase change process.

We study a one-phase supercooled Stefan problem in one space dimension for a non-linear heat conduction equation on a semi-infinite region $x > 0$ with a nonlinear thermal conductivity $k(\theta)$ given by

$$k(\theta) = \frac{\rho c}{(a + b\theta)^2} \tag{1.1}$$

where $a, b$ are positive parameters, $c, \rho$ are the specific heat and the density of the medium respectively. This kind of thermal conductivity or diffusion coefficient was considered in [2, 3, 6, 7, 20, 28, 33, 40].

In [5] one-phase Stefan problem with this non-linear thermal conductivity with a boundary Robin condition at the fixed face is considered. Sufficient conditions for data in order to have a parametric representation of the solution of similarity type for $t \geq t_0$ are obtained, where $t_0$ is a positive arbitrary time. In [31] analogous problems with temperature and flux-type conditions on the fixed face $x = 0$ were studied and parametric representations of the similarity type solutions were obtained. In such context, free boundary problems for a non linear diffusion equation and convective term with the same

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type of conductivity given by (1.1) was also considered in [4, 30, 37]. In [4] under the Bäcklund transformation a Stefan problem with a Dirichlet boundary condition at the fixed face $x = 0$ is reduced to an associated free boundary problem, the existence and uniqueness local in time of the solution is proved by using the Friedman Rubinstein integral representation and the Banach contraction theorem. Necessary and sufficient conditions for the existence of a parametric representation of the solution of the similarity type was found in [30]. On the other hand, in [37] a Neumann boundary condition at the fixed face $x = 0$ is studied. A reciprocal transformation to the Stefan problem is applied and a parametric representation of the similarity type of the solution is obtained through the unique solution of a Cauchy problem.

Several free boundary problems with constant thermal conductivity have been studied by other authors in conection with the freezing of a supercooled liquid. In [32] a supercooled one-phase Stefan problem with constant coefficients and a temperature boundary condition at the fixed face was considered. The explicit solutions are obtained and the relation between the temperature boundary data and the posibility of continuing the solution for arbitrary large time intervals was analyzed. The relationship between the time for which there exists solution to one-phase Stefan problem and the behavior of initial variable temperature was analyzed in [18]. In [10] a one-phase Stefan problem with initial temperature equal to zero and a heat flux depends on the time at the fixed face was analyzed. The behaviour of the free boundary of the solution of a Stefan problem when an integral condition is assigned, is considered in [11]. On the other hand, convexity and smoothness properties of the free boundary were showed in [16, 22, 23, 26] and a review of this subject was given in [34]. Some remarks on the regularization of supercooled one-phase Stefan problems can be seen in [17]. Other papers in the subject are [14, 15, 24, 25].

The mathematical formulation of our free boundary problem consists in determining the evolution of the moving phase separation $x = s(t)$ and the temperature distribution $\theta = \theta(x,t) \geq 0$ satisfying the conditions

$$\rho c \frac{\partial \theta}{\partial t} = \frac{\partial}{\partial x}\left(k(\theta)\frac{\partial \theta}{\partial x}\right) \ , \ 0 < x < s(t) \ , \ t > 0 \tag{1.2}$$

$$\theta(0,t) = -B < 0, \ t > 0 \tag{1.3}$$

$$k\left(\theta\left(s(t),t\right)\right)\frac{\partial \theta}{\partial x}\left(s(t),t\right) = -\rho l \dot{s}(t) \ , \ t > 0 \tag{1.4}$$

$$\theta\left(s(t),t\right) = 0 \ , \ t > 0 \tag{1.5}$$

$$\theta(x,0) = h(x) < 0 \ , \ 0 < x < 1 \ , \tag{1.6}$$

$$s(0) = 1 \tag{1.7}$$

where $l$ is the latent heat of fusion of the medium, the phase change temperature is $\theta_f = 0$ and $h(x)$ is the initial temperature of the material. We impose a temperature boundary condition $-B < h(x) < 0$ on $x = 0$ which corresponds to a supercooled liquid. The classical Stefan problem $(-B > 0, \quad h > 0)$ was well studied in the literature, as for example [8, 21, 38].

In Section 2 under reciprocal transformations the Stefan problem is reduced to an associated free boundary problem which admits a similarity type solution.

In Section 3 we give some preliminary results to prove the existence and uniqueness of a solution local in time and finite time blow-up of problem (1.2) – (1.7) through the unique solution of an integral equation with the time as a parameter.

This type of exact solution to problems with parameters is useful to test by benchmarking with numerical methods for different data values. Phase-change problems appear frequently in industrial processes and other problems of technological interest [1, 8, 9, 12, 13, 19, 27, 29]. A large bibliography on the subject is given in [39].

## 2. Application of reciprocal transformations. A similarity type solution

We consider the free boundary problem (1.2) – (1.7) where the parameters $a, b$, the coefficients $l, c$, the temperature on the fixed face $(-B)$ and the initial temperature $h$ satisfy the following conditions

$$bl - ac > 0, \qquad a - bB > 0, \qquad -B < h(x) < 0, \quad 0 \leq x \leq 1 \tag{2.1}$$

We give several transformations [35, 36] to obtain an equivalent problem to (1.2) – (1.7) which admits a similarity type solution. Firstly we define

$$\Theta = \frac{1}{a + b\theta}\ , \tag{2.2}$$

then the problem (1.2) – (1.7) becomes

$$\frac{\partial \Theta}{\partial t} = \Theta^2 \frac{\partial^2 \Theta}{\partial x^2}\ ,\ 0 < x < s(t)\ ,\ t > 0 \tag{2.3}$$

$$\Theta(0, t) = \frac{1}{a - bB}\ ,\ t > 0 \tag{2.4}$$

$$\frac{\partial \Theta}{\partial x}(s(t), t) = \frac{bl}{c}\dot{s}(t)\ ,\ t > 0 \tag{2.5}$$

$$\Theta(s(t), t) = \frac{1}{a}\ ,\ t > 0 \tag{2.6}$$

$$\Theta(x, 0) = \frac{1}{a + bh(x)},\ 0 < x < 1 \tag{2.7}$$

$$s(0) = 1. \tag{2.8}$$

Let us perform the transformation

$$\chi(x, t) = \int_0^x \frac{d\eta}{\Theta(\eta, t)}\ ,\ \Psi(\chi, t) = \Theta(x, t) \tag{2.9}$$

and

$$S(t) = \chi(s(t), t)\ \ . \tag{2.10}$$

then the problem (2.3) – (2.8) becomes

$$\frac{\partial\Psi}{\partial t} = \frac{\partial^2\Psi}{\partial\chi^2} - (a-bB)\frac{\partial\Psi}{\partial\chi}(0,t)\frac{\partial\Psi}{\partial\chi}\ ,\ 0<\chi<S(t)\ ,\ t>0 \tag{2.11}$$

$$\Psi(0,t) = \frac{1}{a-bB},\ t>0 \tag{2.12}$$

$$\frac{\partial\Psi}{\partial\chi}(S(t),t) = \frac{bl}{a\,(ac-bl)}\left[\dot{S}(t) - \frac{\partial\Psi}{\partial\chi}(0,t)(a-bB)\right]\ ,\ t>0 \tag{2.13}$$

$$\Psi(S(t),t) = \frac{1}{a}\ ,\ t>0 \tag{2.14}$$

$$\Psi(\chi,0) = H(\chi) = \frac{1}{a+bh(W(\chi))} \tag{2.15}$$

$$S(0) = A = \int_0^1 (a+bh(\eta))d\eta \tag{2.16}$$

where

$$W(\chi) = \int_0^\chi \Psi(\eta,0)d\eta \tag{2.17}$$

and

$$\dot{S}(t) = \left(a - \frac{bl}{c}\right)\dot{s}(t) + (a-bB)\frac{\partial\Psi}{\partial\chi}(0,t)\ . \tag{2.18}$$

If we introduce the similarity variable

$$\xi = \frac{\chi}{S(t)} \tag{2.19}$$

and the solution is sought of type

$$\Psi\left(\chi,t\right) = \varphi\left(\xi\right) = \varphi\left(\frac{\chi}{S(t)}\right) \tag{2.20}$$

then the free boundary $S(t)$ of the problem (2.11) – (2.16) must satisfies

$$S(t)\dot{S}(t) = \lambda\ ,\ t>0 \tag{2.21}$$

with $\lambda$ is an unknown coefficient to be determined.
The problem (2.11) – (2.16) yields

$$\varphi''\left(\xi\right) + \varphi'\left(\xi\right)\left(\xi\lambda - w\right) = 0\ ,\ 0<\xi<1 \tag{2.22}$$

$$\varphi\left(0\right) = \frac{1}{a-bB} \tag{2.23}$$

$$\varphi\left(1\right) = \frac{1}{a} \tag{2.24}$$

$$\varphi'\left(1\right) = \frac{bl}{a(ac-bl)}\left(\lambda - w\right) \tag{2.25}$$

where

$$w = \varphi'\left(0\right)\left(a-bB\right) < 0 \tag{2.26}$$

and the condition (2.15) becomes

$$\varphi\left(\frac{\chi}{A}\right) = H(\chi),\quad 0<\chi<A \tag{2.27}$$

where

$$\chi=\chi(x,0)=\int_0^x \frac{d\eta}{\Theta(\eta,0)}=\int_0^x a+bh(\eta)d\eta.$$

If we integrate (2.22) we obtain

$$\varphi\left(\xi\right)=C\int_0^{\xi} exp\left(-\frac{z^2}{2}\lambda+wz\right)dz+D,\;\;0<\xi<1 \tag{2.28}$$

from conditions (2.23) – (2.24) we have that

$$C=\frac{-bB}{a(a-bB)\int_0^1 exp\left(-\frac{z^2}{2}\lambda+wz\right)dz} \tag{2.29}$$

$$D=\frac{1}{a-bB} \tag{2.30}$$

where the unknowns $\lambda$ and $w$ will be determined from (2.25) and (2.26) which are equivalent to

$$\frac{bBexp\left(-\frac{\lambda}{2}+w\right)}{a(a-bB)\int_0^1 exp\left(-\frac{z^2}{2}\lambda+wz\right)dz}=p\left(\lambda-w\right) \tag{2.31}$$

$$w=\frac{-bB}{a\int_0^1 exp\left(-\frac{z^2}{2}\lambda+wz\right)dz} \tag{2.32}$$

with

$$p=\frac{bl}{a(bl-ac)}>0. \tag{2.33}$$

Moreover from (2.27), we have function $h$ satisfies

$$\varphi\left(\frac{\int_0^x a+bh(\eta)d\eta}{\int_0^1 a+bh(\eta)d\eta}\right)=C\int_0^{\frac{\int_0^x a+bh(\eta)d\eta}{\int_0^1 a+bh(\eta)d\eta}} exp\left(-\frac{z^2}{2}\lambda+wz\right)dz+D. \tag{2.34}$$

## 3. Preliminary results

Returning to (2.21) two possible cases for the free boundary $S(t)$ we should consider, one of this is with $\lambda<0$ and the other one is with $\lambda>0$.

Next we are going to analyze the existence the solution to system of equations (2.31) – (2.32) for the two cases.

First we consider

$$\lambda<0 \tag{3.1}$$

We can enunciate the following results:

**Lemma 3.1.** *Under the hypothesis* (2.1)*, if there exist* $\lambda<0$ *and* $w$ *solutions to* (2.31) – (2.32) *then the following statements hold:*

*a)* $\dot{S}(t)<0$ *and*

$$S(t)=\sqrt{2\lambda t+A^2},\;0\le t\le\frac{-A^2}{2\lambda}, \tag{3.2}$$

*b)* $\dot{s}(t)<0$,

*c)* $w<\lambda$,

*d) the free boundary $s(t)$ is given by*

$$s(t) = 1 + \frac{\lambda - w}{\lambda\left(a - \frac{bl}{c}\right)}\left(\sqrt{A^2 + 2\lambda t} - A\right), \; 0 \leq t < \frac{-A^2}{2\lambda} \tag{3.3}$$

*e)*

$$A = \frac{\left(a - \frac{bl}{c}\right)\lambda}{\lambda - w} \tag{3.4}$$

*Proof.* a) If we consider $\lambda < 0$ from (2.21) we have $\dot{S}(t) < 0$ and

$$\left(\frac{S^2(t)}{2}\right)' = \lambda.$$

Integrating and taking into account $S(0) = A$ we have

$$S(t) = \sqrt{2\lambda t + A^2}, \; 0 \leq t \leq \frac{-A^2}{2\lambda}. \tag{3.5}$$

b) From (2.3) – (2.8) it follows that $\Theta_x(s(t), t) < 0$ then $\dot{s}(t) < 0$.

c) By (2.9) and (2.20) we have that (2.5) is equivalent to

$$\frac{\varphi'(1)}{S(t)} \frac{1}{\Theta(S(t), t)} = \frac{bl}{c}\dot{s}(t),$$

and taking into account (2.25) we obtain

$$\dot{s}(t) = \frac{\lambda - w}{S(t)\left(a - \frac{bl}{c}\right)} \tag{3.6}$$

and because $ac - bl < 0$ we have that $w < \lambda < 0$.

d) On substituting (3.2) into (3.6) and integrating we have (3.3).

e) From (2.9), (2.10)

$$S(t) = \chi(s(t), t) = \int_0^{s(t)} \frac{d\eta}{\Theta(\eta, t)}$$

then

$$s(t) = 0 \Leftrightarrow S(t) = 0 \Leftrightarrow t = \frac{-A^2}{2\lambda}$$

thus

$$s\left(\frac{-A^2}{2\lambda}\right) = 0 \Leftrightarrow A = \frac{\left(a - \frac{bl}{c}\right)\lambda}{\lambda - w}.$$

□

**Corollary 3.2.** *For the case $\lambda < 0$ the free boundary is given by*

$$s(t) = \frac{1}{A}\sqrt{A^2 + 2\lambda t} \quad , \quad 0 \leq t < \frac{-A^2}{2\lambda} \tag{3.7}$$

*which satisfies*

$$\lim_{t \to \left(\frac{-A^2}{2\lambda}\right)^-} s(t) = 0, \qquad \lim_{t \to \left(\frac{-A^2}{2\lambda}\right)^-} \dot{s}(t) = -\infty \tag{3.8}$$

*so finite time blow-up occurs.*

To solve (2.31) – (2.32) it is convenient to define

$$\sigma = \frac{\lambda - w}{\sqrt{-2\lambda}} > 0, \qquad \mu = \sqrt{\frac{-\lambda}{2}} \tag{3.9}$$

then equations (2.31) – (2.32) are equivalent to

$$\frac{bB}{2pa(a-bB)} \frac{exp(\sigma^2)}{\sigma} = \int_{\sigma}^{\sigma+\mu} exp(z^2)dz \tag{3.10}$$

$$\frac{bB}{2a} \frac{exp((\sigma+\mu)^2)}{\sigma+\mu} = \int_{\sigma}^{\sigma+\mu} exp(z^2)dz \tag{3.11}$$

in the unknowns $\sigma$ and $\mu$.

**Lemma 3.3.** *Under the hypothesis* (2.1) *we have:*

*If*

$$\int_{\sigma_0}^{\sqrt{0.5}} exp(z^2)dz > \frac{bB\sqrt{2e}}{2a} \tag{3.12}$$

*then there exist unique solution* $w, \lambda < 0$ *to* (2.31) – (2.32) *with the coefficient* $\sigma_0 = J_1^{-1}(p(a-bB)\sqrt{2e})$ *where* $J_1^{-1}$ *is the inverse function of* $J_1 = J/(0, \sqrt{0.5})$ *the restriction of* $J(x) = \frac{exp(x^2)}{x}$ *to the interval* $\left(0, \sqrt{0.5}\right)$.

*Proof.* First, we define

$$J(x) = \frac{exp(x^2)}{x} \tag{3.13}$$

which satisfies

$$J(0) = +\infty, \qquad J(+\infty) = +\infty,$$

$$J'(x) = \begin{cases} < 0, & 0 < x < \sqrt{0.5} \\ = 0, & x = \sqrt{0.5} \\ > 0, & x > \sqrt{0.5} \end{cases}$$

Then, from (3.10) and (3.11) we have

$$\frac{J(\sigma)}{p(a-bB)} = J(\sigma+\mu) \tag{3.14}$$

In fact,

$$\mu = V_1(\sigma) - \sigma, \qquad 0 \le \sigma < \sigma_0 \tag{3.15}$$

where

$$V_1(\sigma) = J_1^{-1}\left(\frac{J_1(\sigma)}{p(a-bB)}\right), \tag{3.16}$$

$J_1^{-1}$ is the inverse function of $J_1 = J/(0, \sqrt{0.5})$ the restriction of $J$ to the interval $\left(0, \sqrt{0.5}\right)$ and $\sigma_0 = J_1^{-1}(p(a-bB)\sqrt{2e})$.

Under the hypotheses (2.1) and (2.33) we have that $p(a-bB) > 1$.

If we replace (3.15) in (3.10) we have the following equation in unknown $\sigma$

$$\frac{bB}{2pa(a-bB)} J_1(\sigma) = P(\sigma), \qquad 0 \le \sigma < \sigma_0 \tag{3.17}$$

where the function $P(\sigma) = \int_{\sigma}^{V_1(\sigma)} exp(z^2)dz$ is an increasing function, $P(0) = 0$ and $P(\sigma_0) = \int_{\sigma_0}^{\sqrt{0.5}} exp(z^2)dz$.

Since the properties of functions $J$ and $P$ it is enough to ask

$$P(\sigma_0) > \frac{bB\sqrt{2e}}{2a}$$

then there exists a unique $\sigma \in (0, \sigma_0)$ which satisfies (3.17). So there exists a unique

$$\mu = V_1(\sigma) - \sigma$$

such that $\sigma$, $\mu$ are the solutions of $(3.10) - (3.11)$. Therefore, we have that there exist unique solutions to the system $(2.31) - (2.32)$ given by

$$w = -2\mu(\sigma + \mu), \qquad \lambda = -2\mu^2.$$

$\square$

**Theorem 3.4.** *Under the hypothesis* (2.1) *and* (3.12) *the problem* $(2.22) - (2.26)$ *has a unique solution given by*

$$\varphi(\xi) = \frac{-bB \int_0^{\xi} exp\left(-\frac{z^2}{2}\lambda + wz\right) dz}{a(a - bB)\int_0^1 exp\left(-\frac{z^2}{2}\lambda + wz\right) dz} + \frac{1}{a - bB}, \quad 0 < \xi < 1 \tag{3.18}$$

*where* $w, \lambda < 0$ *is the unique solution to* $(2.31) - (2.32)$.

Now, we analize the existence of solution to problem $(2.22) - (2.26)$ for the case

$$\lambda > 0.$$

We define $\eta = -w > 0$ .

**Lemma 3.5.** *There is not solution* $\lambda > 0$, $w = -\eta$ *to* $(2.31) - (2.32)$.

*Proof.* Let $\alpha = \frac{\lambda+\eta}{\sqrt{2\lambda}}$ and $\epsilon = \frac{\eta}{\sqrt{2\lambda}}$ be. Then the conditions (2.31) and (2.32) are equivalent to

$$\frac{bB}{ap(a - bB)\sqrt{\pi}} R(\alpha) = erf\,(\alpha) - erf\,(\epsilon) \tag{3.19}$$

$$\frac{bB}{a\sqrt{\pi}} R(\epsilon) = erf\,(\alpha) - erf\,(\epsilon) \tag{3.20}$$

where $R(x) = exp(-x^2)/x$ and $p$ is given as before.

From $(3.19) - (3.20)$ we have $\alpha = W(\epsilon) = R^{-1}(p(a - bB)R(\epsilon))$ which is an increasing and convex function that satisfies $W(0) = 0$ and $W(+\infty) = +\infty$.

Then the equation (3.20) become

$$W(\epsilon) = F(\epsilon), \qquad \epsilon > Q^{-1}\left(\frac{bB}{a}\right) = \epsilon_0 \tag{3.21}$$

where

$$F(\epsilon) = erf^{-1}\left(g\left(\epsilon, \frac{bB}{a\sqrt{\pi}}\right)\right) \tag{3.22}$$

and

$$Q(x)=\sqrt{\pi}x exp(x^2)(1-erf(x)). \tag{3.23}$$

It's easy to see that $W(\epsilon)<F(\epsilon)$ for all $\epsilon>\epsilon_0$, then there is not solution whatever the initial data of the problem.

□

*Remark* 3.6. There is not solution to problem (2.22) – (2.26) with $\lambda>0$.

## 4. Existence and uniqueness of solution to the nonlinear supercooled Stefan problem

Therefore, under hypothesis (2.1) and (3.12) if we invert the transformation (2.20) we have that there exist unique solution to (2.11) – (2.17) given by

$$\Psi(\chi,t)=\frac{-bB\int_0^{\frac{\chi}{S(t)}} exp\left(-\frac{z^2}{2}\lambda+wz\right)dz}{a(a-bB)\int_0^1 exp\left(-\frac{z^2}{2}\lambda+wz\right)dz}+\frac{1}{a-bB}, 0<\chi<S(t), \tag{4.1}$$

$$S(t)=\sqrt{2\lambda t+A^2}\ ,\ 0\leq t\leq\frac{-A^2}{2\lambda} \tag{4.2}$$

where $\lambda<0$ and $w$ are the unique solutions of equations (2.31) and (2.32).

Then, by transformation (2.9) and taking into account (2.18) we have

$$\Theta(x,t)=\frac{-bB\left[U\left(\sqrt{\frac{-\lambda}{2}}\frac{\int_0^x\frac{d\eta}{\Theta(\eta,t)}}{\sqrt{A^2+2\lambda t}}+\frac{w}{\sqrt{-2\lambda}}\right)-U\left(\frac{w}{\sqrt{-2\lambda}}\right)\right]}{a(a-bB)\left[U\left(\sqrt{\frac{-\lambda}{2}}+\frac{w}{\sqrt{-2\lambda}}\right)-U\left[\frac{w}{\sqrt{-2\lambda}}\right)\right]}+\frac{1}{a-bB}, \tag{4.3}$$

for $0\leq x\leq s(t)$, the free boundary $s(t)$ is given by (3.3) and

$$U(x)=\int_0^x exp(z^2)dz.$$

An equivalent formulation of (4.3) is

$$\Theta(x,t)=\frac{-bB\left[U(\sigma+\mu)-U\left(\sigma+\mu-\frac{\mu\int_0^x\frac{d\eta}{\Theta(\eta,t)}}{\sqrt{A^2-4\mu^2t}}\right)\right]}{a(a-bB)\left[U(\sigma+\mu)-U(\sigma)\right]}+\frac{1}{a-bB}\ , \tag{4.4}$$

for $0\leq x\leq s(t),\quad 0\leq t<\frac{A^2}{4\mu^2}$, where $\mu$ and $\sigma$ are the unique solutions of (3.10) – (3.11) and the free boundary is

$$s(t)=\frac{1}{A}\sqrt{A^2-4\mu^2t},\ \ 0\leq t<\frac{A^2}{4\mu^2} \tag{4.5}$$

with

$$A=\frac{(bl-ac)\,\mu}{c\sigma}. \tag{4.6}$$

Note that we have actually proved that $\Theta=\Theta(x,t)$ is a solution, in variable $x$, of the integral equation (4.4) .

**Theorem 4.1.** *Let us assume the hypothesis* (2.1) *and* (3.12).

*(i) If* $(\Theta, s)$ *is a solution of the free boundary problem* $(2.3)-(2.8)$ *then* $\Theta = \Theta(x,t)$ *is a solution, in variable* $x$*, of the integral equation* (4.4) *and the free boundary is given by* (4.5).

*Moreover, the function* $Y(x,t)$ *defined by*

$$Y(x,t) = \sigma + \mu - \frac{\mu \int_0^x \frac{d\eta}{\Theta(\eta,t)}}{\sqrt{A^2 - 4\mu^2 t}}, \qquad 0 \le x \le s(t)\ ,\ 0 \le t < \frac{A^2}{4\mu^2} \tag{4.7}$$

*satisfies the conditions*

$$\frac{\partial Y}{\partial x}(x,t) = \frac{-\mu}{\sqrt{A^2 - 4\mu^2 t}} \frac{1}{\Theta(x,t)} \tag{4.8}$$

$$Y(0,t) = \sigma + \mu \tag{4.9}$$

$$\frac{\partial Y}{\partial t}(x,t) = \frac{-\mu^2}{A^2 - 4\mu^2 t}\left(\frac{bBexp(Y^2(x,t))}{(a-bB)a\Theta(x,t)\left[U(\sigma+\mu) - U(\sigma)\right]} - 2Y(x,t)\right) \tag{4.10}$$

$$Y\left(s(t),t\right) = \sigma \tag{4.11}$$

$$Y(x,0) = \sigma + \mu - \frac{\mu \int_0^x a + bh(z)dz}{A}. \tag{4.12}$$

*(ii) Conversely, if* $\Theta$ *is a solution of the integral equation* (4.4) *with* $s$ *given by* (4.5) *and function* $Y$ *defined by* (4.7) *satisfies the conditions* $(4.8)-(4.12)$ *where* $\sigma$ *and* $\mu$ *are the unique solutions of equations* $(3.10)-(3.11)$ *then* $(\Theta, s)$ *is a solution of the free boundary problem* $(2.3)-(2.8)$.

*Proof.* (i) From the previous computation we have $\Theta = \Theta(x,t)$ is a solution of the integral equation (4.4). It follows easily that function $Y$, defined by (4.7), satisfies the conditions (4.8), (4.9), (4.12) and

$$\frac{\partial Y}{\partial t}(x,t) = \frac{-\mu}{\sqrt{A^2 - 4\mu^2 t}}\left(\int_0^x \frac{-\frac{\partial \Theta}{\partial t} d\eta}{\Theta^2(\eta,t)} + \frac{2\mu^2}{A^2 - 4\mu^2 t}\int_0^x \frac{d\eta}{\Theta(\eta,t)}\right) =$$

$$= \frac{-\mu}{\sqrt{A^2 - 4\mu^2 t}}\left(-\Theta_x(x,t) + \Theta_x(0,t) + \frac{2\mu^2}{A^2 - 4\mu^2 t}\int_0^x \frac{d\eta}{\Theta(\eta,t)}\right) =$$

$$= \frac{-\mu^2}{A^2 - 4\mu^2 t}\left(\frac{bBexp\left(Y^2(x,t)\right)}{a(a-bB)\Theta(x,t)\left[U(\sigma+\mu) - U(\sigma)\right]}\right.$$

$$\left. - \frac{bBexp(\sigma+\mu)^2}{a\left[U(\sigma+\mu) - U(\sigma)\right]} + \frac{2\mu}{\sqrt{A^2 - 4\mu^2 t}}\int_0^x \frac{d\eta}{\Theta(\eta,t)}\right)$$

and from (3.11) we obtain (4.10).

Finally we get

$$Y\left(s(t),t\right) = \sigma + \mu - \frac{\mu \int_0^{s(t)} \frac{d\eta}{\Theta(\eta,t)}}{\sqrt{A^2 - 4\mu^2 t}} = \sigma + \mu - \mu \frac{S(t)}{\sqrt{A^2 - 4\mu^2 t}} = \sigma$$

that is (4.11).

(ii) Conversely, let $\Theta$ the solution an integral equation (4.4). In order to prove that $(\Theta, s)$ is a solution of the free boundary problem (2.3) – (2.8) we get:

a)

$$\Theta_x(x,t) = \frac{bBexp\left(Y^2(x,t)\right)}{a(a-bB)\left[U(\sigma+\mu)-U(\sigma)\right]}\frac{\partial Y}{\partial x}$$

and

$$\Theta_{xx}(x,t) = \frac{bBexp\left(Y^2(x,t)\right)}{a(a-bB)\left[U(\sigma+\mu)-U(\sigma)\right]}\left(2Y(x,t)\left(\frac{\partial Y}{\partial x}\right)^2+\frac{\partial^2 Y}{\partial x^2}\right)$$

By using (4.8) we obtain

$$\frac{\partial^2 Y}{\partial x^2}(x,t) = \frac{\mu}{\sqrt{A^2-4\mu^2 t}}\frac{1}{\Theta^2(x,t)}\frac{\partial \Theta}{\partial x}$$

and

$$\begin{aligned}\Theta_{xx}(x,t)\Theta^2(x,t) &= \frac{bBexp\left(Y^2(x,t)\right)}{a(a-bB)\left[U(\sigma+\mu)-U(\sigma)\right]}\left(\frac{2Y(x,t)\mu^2}{A^2-4\mu^2 t}+\frac{\mu\Theta_x}{\sqrt{A^2-4\mu^2 t}}\right)\\ &= \frac{bB\mu^2 exp\left(Y^2(x,t)\right)}{a(a-bB)(A^2-4\mu^2 t)\left[U(\sigma+\mu)-U(\sigma)\right]^2}\cdot\\ &\cdot\left(2Y(x,t)\left[U(\sigma+\mu)-U(\sigma)\right]-\frac{bBexp\left(Y^2(x,t)\right)}{a(a-bB)\Theta(x,t)}\right)\end{aligned}$$

b)

$$\begin{aligned}\Theta_t(x,t) &= \frac{-bBexp\left(Y^2(x,t)\right)\mu^2}{a(a-bB)\left(A^2-4\mu^2 t\right)\left[U(\sigma+\mu)-U(\sigma)\right]^2}\cdot\\ &\cdot\left(\frac{bBexp\left(Y^2(x,t)\right]}{a(a-bB)\Theta(x,t)}-2Y(x,t)\left[U(\sigma+\mu)-U(\sigma)\right]\right)\end{aligned}$$

then (2.3) holds.

c) It is easy to see

$$\Theta(0,t) = \frac{1}{a-bB}.$$

d) By (4.11) we have

$$\Theta(s(t),t) = \frac{-bB\left[U\left(\sigma+\mu\right)-U\left(Y(s(t),t)\right)\right]}{a(a-bB)\left[U(\sigma+\mu)-U(\sigma)\right]}+\frac{1}{a-bB} = \frac{1}{a}.$$

e) We have

$$\Theta_x(s(t),t) = \frac{-bBexp\left(\sigma^2\right)\frac{\partial Y}{\partial x}(s(t),t)}{a(a-bB)\left[U(\sigma+\mu)-U(\sigma)\right]},$$

from (4.8) and the above item we have

$$\frac{\partial Y}{\partial x}(s(t),t) = \frac{-\mu a}{\sqrt{A^2-4\mu^2 t}}$$

then by (4.5) and since $\sigma, \mu$ satisfy (3.10) we obtain

$$\Theta_x(s(t),t) = \frac{-2\mu\sigma bl}{(bl-ac)\sqrt{A^2-4\mu^2 t}} = \frac{bl}{c}\dot{s}(t) \ ,$$

that is (2.5).

f) Taking into account (4.8) and (4.12) we get

$$\frac{\partial Y}{\partial x}(x,0) = \frac{\mu}{A}\frac{1}{\Theta(x,0)} \ ,$$

and

$$\frac{\partial Y}{\partial x}(x,0) = -\mu\frac{a+bh(x)}{A}$$

then we deduce (2.7). □

**Theorem 4.2.** *Let us assume the hypothesis* (2.1) *and* (3.12).

*(i) The integral equation* (4.4) *has a unique solution for* $0 \le t \le t_0 < \frac{A^2}{4\mu^2}$ *where* $t_0$ *is an arbitrary positive time.*

*(ii) The free boundary problem* $(1.2)-(1.7)$ *has a unique similarity type solution* $(\theta, s)$ *for* $0 \le t \le t_0 < \frac{A^2}{4\mu^2}$ *and a finite blow-up ocurrs at* $t = \frac{A^2}{4\mu^2}$ *which is given by*

$$\theta(x,t) = \frac{1}{b}\left[\frac{1}{\Theta(x,t)} - a\right], \quad 0 < x < s(t), \tag{4.13}$$

$s(t)$ *given by* (4.5), *where* $\Theta$ *is the unique solution of the integral equation* (4.4) *and the coefficients* $\mu$ *and* $\sigma$ *are the unique solutions of equations* (3.10) *and* (3.11) *with* $A$ *given by* (4.6) .

*Proof.* (i) If we define $Y(x,t)$ by (4.7) then (4.4) is equivalent to the following Cauchy differential problem

$$\begin{cases} \frac{\partial Y}{\partial x}(x,t) = \frac{-\mu}{\sqrt{A^2-4\mu^2 t}} = \frac{1}{C_1+D_1U(Y(x,t))} \ , \quad 0 < x < s(t) \ , \\ Y(0,t) = \sigma+\mu \ , \end{cases} \tag{4.14}$$

with a parameter $0 \le t \le t_0 < \frac{A^2}{4\mu^2}$, the coefficients $C_1, D_1$ are given by

$$C_1 = \frac{1}{a-bB} - \frac{bBU(\sigma+\mu)}{a(a-bB)\left[U(\sigma+\mu)-U(\sigma)\right]}$$

and

$$D_1 = \frac{bBU(\sigma+\mu)}{a(a-bB)\left[U(\sigma+\mu)-U(\sigma)\right]}.$$

We have

$$\frac{\partial G}{\partial Y} = \frac{\mu}{\sqrt{A^2-4\mu^2 t}}\frac{D_1 exp(Y^2)}{\left[C_1+D_1U(Y)\right]^2}$$

If we define the function $p(z) = \frac{exp(z^2)}{[C_1+D_1U(z)]^2}$ it's easy to see that there exists $K > 0$ such that

$$\left|\frac{\partial G}{\partial Y}\right| \le \frac{D_1\mu K}{\sqrt{A^2-4\mu^2 t}},$$

which is bounded for all $0 \leq t \leq t_0 < \frac{A^2}{4\mu^2}$, $0 \leq x \leq s(t)$, for an arbitrary positive time $t_0$.

(ii) It follows taking into account Theorem 4.1, Corollary 3.2 and elementary computations.

□

## 5. Conclusions

A supercooled one-phase Stefan problem for a semi-infinite material with temperature-dependent thermal conductivity at the fixed face $x = 0$ was studied. In order to have existence of solution of similarity type, local in time, we obtained sufficient conditions for the data. Moreover we showed that finite time blow-up occurs. This explicit solution was obtained through the unique solution of an integral equation with the time as a parameter.

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## References

[1] V. Alexiades, A. D. Solomon, Mathematical modeling of melting and freezing processes, Washington : Hemisphere - Taylor & Francis, 1983.

[2] D. A. Barry, G. C. Sander, Exact solutions for water infiltration with an arbitrary surface flux or nonlinear solute adsorption, Water Resources Research 27, No 10 (1991), 2667-2680 .

[3] G. Bluman, S. Kumei, On the remarkable nonlinear diffusion equation, J. Math Phys. 21 (1980), 1019-1023.

[4] A. C. Briozzo, M. F. Natale, On a non-classical non-linear moving boundary problem for a diffusion convection equation, International Journal of Non-Linear Mechanics, 47 (2012), 712-718.

[5] A. C. Briozzo, M. F. Natale, One-phase Stefan problem with temperature-dependent thermal conductivity and a boundary condition of Robin type, J. Appl.Anal., 21 (2) (2015), 89-97. DOI: 10.1515/JAA-2015-0009

[6] P. Broadbridge, Non-integrability of non-linear diffusion-convection equations in two spatial dimensions, J. Phys. A: Math. Gen, 19 (1986), 1245-1257.

[7] P. Broadbridge, Integrable forms of the one-dimensional flow equation for unsaturated heterogeneous porous media, J. Math. Phys., 29 (1988), 622-627.

[8] J. R. Cannon, The one-dimensional heat equation, Addison - Wesley, Menlo Park, 1984.

[9] H. S. Carslaw, J. C. Jaeger, Conduction of heat in solids, London: Oxford University Press, 1959.

[10] E. Comparini, R. Ricci, D. A. Tarzia, Remarks on a one-dimensional Stefan problem related to the diffusion-consumption model, Z. Angew. Math. Mech., 64 (1984), 543-550.

[11] E. Comparini, D. A. Tarzia, A Stefan problem for the heat equation subject to an integral condition, Rend. Sem. Maten. Univ. Padova, 73 (1985), 119-136.

[12] J. Crank, Free and moving boundary problems, Oxford: Clarendon Press, 1984.

[13] J. I. Diaz, M. A. Herrero, A. Liñan, J. L. Vazquez (Eds.), Free boundary problems: theory and applications, Essex: Pitman Research Notes in Mathematics, Series 323, Longman, 1995.

[14] E. Di Benedetto, A. Friedman, The Ill-Posed Hele-Shaw Model and The Stefan Problem for Supercooled Water, Trans. Amer. Math. Soc., 282 n 1 (1984), 183-204.

[15] A. Fasano, M. Primicerio, General Free-Boundary Problems for the Heat Equation I, J. Math. Anal. Appl., 57 (1977), 694-723.

[16] A. Fasano, M. Primicerio, Convexity of the free boundary in some classical parabolic free boundary problems, Riv. Mat. Univ. Parma, 5 (1979), 635-645.

[17] A. Fasano, M. Primicerio, S. Howison, J. Ockendon, Some remarks on the regularization of supercooled one-phase Stefan problem in one dimension, Quart. Appl. Math., 48 (1990), 153-168.

[18] A. Fasano, M. Primicerio, New results on some classical parabolic free-boundary problems, Quart. Appl. Math., 38 (1981), 49-460.

[19] A. Fasano, M. Primicerio (Eds.), Nonlinear diffusion problems, Berlin, Lecture Notes in Math.N.1224, Springer Verlag, 1986.

[20] A. S. Fokas, Y. C. Yortsos, On the exactly solvable equation $S_t = \left[(\beta S+\gamma)^{-2} S_x\right]_x + \alpha (\beta S+\gamma)^{-2} S_x$ occurring in two-phase flow in porous media, SIAM J. Appl. Math., 42 No 2 (1982), 318-331.

[21] A. Friedman, Partial differential equations of parabolic type, Prentice-Hall, Englewood Cliff 1964.

[22] A. Friedman, Parabolic variational inequalities in one space dimension and smoothness of the free boundary, J. Funct. Anal., 18 (1975), 151-176.

[23] A. Friedman, R. Jensen, Convexity of the free boundary in the Stefan problem and in dam problem, Arch. Rat. Mech. Anal., 67 (1977), 1-24.

[24] M. A. Herrero, J. J. L. Velázquez, Singularity formation in the one dimensional supercooled Stefan problem, Eur. J. Appl., 7 (1994), 115-150.

[25] M. A. Herrero, J. J. L. Velázquez, The birth of a cusp in the two-dimensional, undercooled Stefan problem, Quart. Appl. Math., 58 n 3 (2000), 473-494.

[26] R. Jensen, Smoothness of the free boundary in the Stefan problem with supercooled water, Illinois J. Math., 22 (1978), 623-629.

[27] N. Kenmochi (Ed.), Free Boundary Problems: Theory and Applications, I,II"Tokyo: Gakuto International Series: Mathematical Sciences and Applications, Vol. 13, 14, Gakkotosho, 2000.

[28] J. H. Knight, J. R. Philip, Exact solutions in nonlinear diffusion, J. Engrg. Math., 8 (1974), 219-227 .

[29] V. J. Lunardini, Heat transfer with freezing and thawing, Amsterdam, Elsevier, 1991.

[30] M. F. Natale, D. A. Tarzia, Explicit solutions to the one-phase Stefan problem with temperature-dependent thermal conductivity and a convective term, Int. J. Engng. Sci., 41 (2003), 1685-1698.

[31] M. F. Natale, D. A. Tarzia, Explicit solutions for a one-phase Stefan problem with temperature-dependent thermal conductivity, Boll. Unione Math. Ital. (8) 9-B (2006), 79-99.

[32] A. Petrova, D. Tarzia and C. Turner, The one-phase supercooled Stefan problem with temperature-dependent thermal conductivity and a convective term, Advances in Mathematical Sciences and Applications, Vol. 4 N1 (1994), 35-50.

[33] R. Philip, General method of exact solution of the concentration-dependent diffusion equation, Australian J. Physics, 13 (1960), 1-12.

[34] M. Primicerio, Qualitative properties of some one-dimensional parabolic free boundary problem, Proc. Seminar on Free Boundary Problems, E. Magenes (Ed.), Ist. Naz. di Alta Matematica, Roma (1980) Vol. 1, 451-460.

[35] C. Rogers, Application of a reciprocal transformation to a two-phase Stefan problem, J. Phys. A: Mathematical and General, 18 (1985), L105-L109.

[36] C. Rogers, On a class of moving boundary problems in nonlinear heat conduction: application of a Bäcklund transformation, Int. J. Nonlinear Mechanics, 21 (1986), 249-256.

[37] C. Rogers, P. Broadbridge, On a nonlinear moving boundary problem with heterogeneity: application of reciprocal transformation, Journal of Applied Mathematics and Physics (ZAMP), 39 (1988), 122-129.

[38] L.I. Rubinstein, The Stefan problem, Trans. Math. Monographs, 27 , Amer. Math. Soc. , Providence, 1971.

[39] D. A. Tarzia, A bibliography on moving - free boundary problems for the heat-diffusion equation. The Stefan and related problems, MAT-Serie A, 2 (2000) (with 5869 titles on the subject, 300 pages). www.austral.edu.ar/MAT-SerieA/2(2000).

[40] P. Tritscher, P. Broadbridge, A similarity solution of a multiphase Stefan problem incorporating general non-linear heat conduction, Int. J. Heat Mass Transfer, 37 No 14 (1994), 2113-2121.

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