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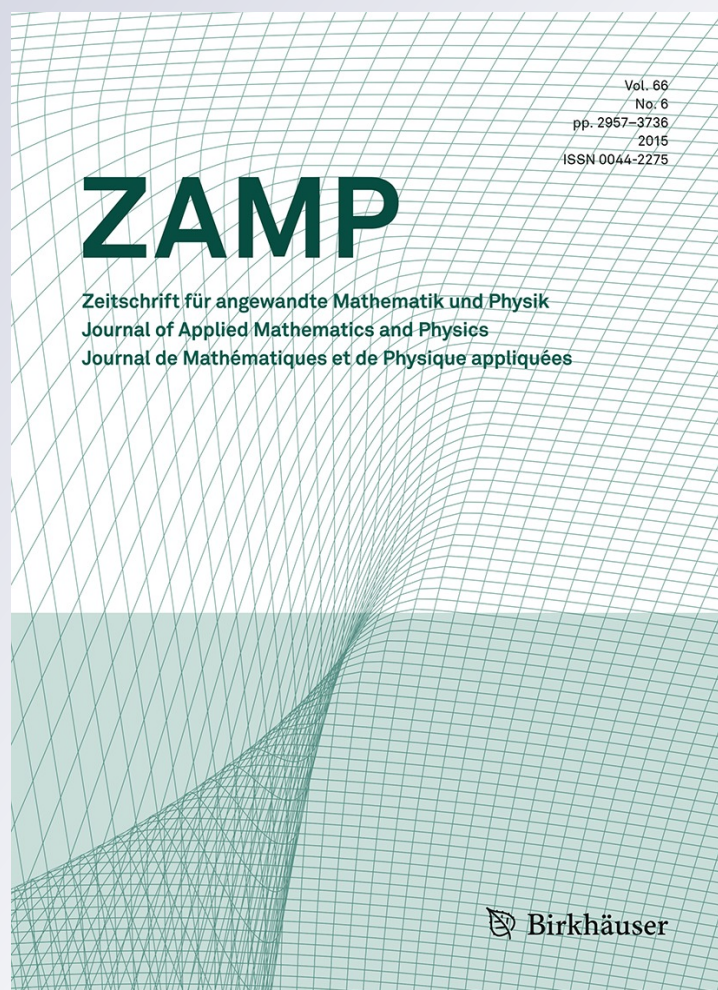
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Nonlinear Stefan problem with convective boundary condition in Storm's materials

Adriana C. Briozzo and Maria F. Natale

Abstract. We consider a nonlinear one-dimensional Stefan problem for a semi-infinite material $x > 0$, with phase change temperature T_f . We assume that the heat capacity and the thermal conductivity satisfy a Storm's condition, and we assume a convective boundary condition at the fixed face $x = 0$. A unique explicit solution of similarity type is obtained. Moreover, asymptotic behavior of the solution when $h \rightarrow +\infty$ is studied.

Mathematics Subject Classification. 35R35 · 80A22 · 35C05.

Keywords. Stefan problem · Free boundary problem · Phase change process · Similarity solution.

1. Introduction

As in [4, 11, 13], we consider the following one-phase nonlinear unidimensional Stefan problem for a semi-infinite material $x > 0$, with phase change temperature T_f

$$s(T) \frac{\partial T}{\partial t} = \frac{\partial}{\partial x} \left[k(T) \frac{\partial T}{\partial x} \right], \quad 0 < x < X(t), \quad t > 0, \tag{1.1}$$

$$k(T(0, t)) \frac{\partial T}{\partial x}(0, t) = \frac{h}{\sqrt{t}} [T(0, t) - T_m], \quad h > 0, \quad t > 0, \tag{1.2}$$

$$T(X(t), t) = T_f, \tag{1.3}$$

$$k(T_f) \frac{\partial T}{\partial x}(X(t), t) = \alpha \dot{X}(t), \quad t > 0, \tag{1.4}$$

$$X(0) = 0 \tag{1.5}$$

where the positive constant α is ρL , L is the latent heat of fusion of the medium, ρ is the density (assumed constant), T_m is the temperature of the medium $T_m < T(0, t) < T_f$ and h_0 is the positive heat transfer coefficient.

We assume that the metal exhibits nonlinear thermal characteristics such that the heat capacity $c_p(T) > 0$ and the thermal conductivity $k(T) > 0$ satisfy a Storm's condition [1, 2, 5–7, 12]

$$\frac{\frac{d}{dT} \left(\sqrt{\frac{s(T)}{k(T)}} \right)}{s(T)} = \lambda = \text{const.} > 0, \tag{1.6}$$

where $s(T) = \rho c_p(T)$.

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Condition (1.6) was originally obtained by [12] in an investigation of heat conduction in simple monoatomic metals. There, the validity of the approximation (1.6) was examined for aluminum, silver, sodium, cadmium, zinc, copper and lead.

This class of moving boundary problems for Storm materials was originally solved in [9, 10] by using a reciprocal transformation. Reciprocal Bäcklund transformations were used in [8] to obtain solutions both a one-phase and a two-phase Stefan problems in nonlinear heat conduction.

In [7], the free boundary problem (1.1)–(1.6) (fusion case) for the particular case $k(T) = \rho c / (a + bT)^2$ and $s(T) = \rho c = \text{constant}$ was studied. The explicit solution of this problem was obtained through the unique solution of an integral equation with time as a parameter. A similar case with the constant temperature at the fixed face $x = 0$ was also studied.

In [2], two nonlinear Stefan problems analogous to (1.1)–(1.5) with phase change temperature T_f and the Storm's condition (1.6) were considered. In one case, a heat flux boundary condition of the type $q(t) = \frac{q_0}{\sqrt{t}}$ and in the other case a temperature boundary condition $T = T_s < T_f$ at the fixed face $x = 0$ were assumed. Solutions of similarity type were obtained in both cases, and the equivalence of the two problems was demonstrated.

The goal of this paper is to determine the temperature $T = T(x, t)$ and the position of the phase change boundary at time t , $X = X(t)$, which satisfy the problem (1.1)–(1.6). In Sect. 2, we show how to find a unique solution of the similarity type for this problem. In Sect. 3, we study the asymptotic behavior when $h \rightarrow +\infty$. We prove that the solutions $T = T_h(x, t)$, $X = X_h(t)$ of (1.1)–(1.6) converge to the solution $T = T_\infty(x, t)$, $X = X_\infty(t)$ of an analogous Stefan problem with temperature condition $T(0, t) = T_m$ when $h \rightarrow +\infty$.

2. Existence and uniqueness of the solution to the Stefan problem with convective boundary condition on the fixed face

We consider the problem (1.1)–(1.6), and we propose a similarity-type solution given by [2–4]

$$T(x, t) = \Phi(\xi), \quad \xi = \frac{x}{X(t)} \tag{2.1}$$

where

$$X(t) = \sqrt{2\gamma t}, \quad t > 0 \tag{2.2}$$

is the free boundary and γ is assumed a positive constant to be determined. Then, we have that the problem (1.1)–(1.5) is equivalent to

$$k(\Phi)\Phi''(\xi) + k'(\Phi)\Phi'^2(\xi) + \gamma s(\Phi)\Phi'(\xi)\xi = 0, \quad 0 < \xi < 1, \tag{2.3}$$

$$k(\Phi(0))\Phi'(0) = h\sqrt{2\gamma}[\Phi(0) - T_m], \tag{2.4}$$

$$\phi(1) = T_f, \tag{2.5}$$

$$k(\Phi(1))\Phi'(1) = \alpha\gamma. \tag{2.6}$$

If we define

$$y(\xi) = \sqrt{\frac{k}{s}}(\Phi(\xi)), \tag{2.7}$$

then a parameterization of the Storm condition (1.6) is

$$s(\Phi) = -\frac{1}{\lambda y^2} \frac{dy}{d\Phi}, \quad k(\Phi) = -\frac{1}{\lambda} \frac{dy}{d\Phi} \tag{2.8}$$

and then we have that the following problem is equivalent to (2.3)–(2.6)

$$\frac{d^2y}{d\xi^2} + \frac{\gamma\xi}{y^2} \frac{dy}{d\xi} = 0, \quad 0 < \xi < 1, \tag{2.9}$$

$$y'(0) = -\lambda h \sqrt{2\gamma} [P(y^2(0)) - T_m], \tag{2.10}$$

$$y'(1) = -\alpha \lambda \gamma, \tag{2.11}$$

$$y(1) = y_1 = \sqrt{\frac{k(T_f)}{s(T_f)}} \tag{2.12}$$

where P is the inverse function of the decreasing function $\frac{k}{s}$.

Lemma 2.1. *A parametric solution to the problem (2.9)–(2.12) is given by*

$$\xi = \varphi_1(u) = \frac{F_{u_0}(u)}{F_{u_0}(u_1)}, \tag{2.13}$$

$$y = \varphi_2(u) = \frac{\sqrt{\gamma} \sqrt{\frac{\pi}{2}} \left[\operatorname{erf} \left(\frac{u}{\sqrt{2}} \right) - g \left(\frac{u_0}{\sqrt{2}}, \frac{1}{\sqrt{\pi}} \right) \right]}{F_{u_0}(u_1)}, \tag{2.14}$$

for

$$u_0 \leq u \leq u_1$$

where the function $F_{u_0} = F_{u_0}(u)$ was defined in [2] as follows

$$\begin{aligned} F_{u_0}(u) &= \exp\left(-\frac{u^2}{2}\right) + u \left(\int_{u_0}^u \exp\left(-\frac{z^2}{2}\right) dz - \frac{\exp\left(-\frac{u_0^2}{2}\right)}{u_0} \right) \\ &= \sqrt{\frac{\pi}{2}} u \left[g\left(\frac{u}{\sqrt{2}}, \frac{1}{\sqrt{\pi}}\right) - g\left(\frac{u_0}{\sqrt{2}}, \frac{1}{\sqrt{\pi}}\right) \right], \quad u \geq u_0 \end{aligned}$$

with u_0, u_1 are the parameter values which verify that $\xi = \varphi_1(u_0) = 0$ and $\xi = \varphi_1(u_1) = 1$,

$$g(x, p) = \operatorname{erf}(x) + p \frac{\exp(-x^2)}{x}, \quad p > 0, x > 0 \tag{2.15}$$

and

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-z^2) dz, \quad x > 0.$$

The unknowns γ, u_0 and u_1 must verify the following system of equations

$$u_0 = \sqrt{2} \lambda h \left[P \left(\frac{\gamma \exp(-u_0^2)}{[u_0 F_{u_0}(u_1)]^2} \right) - T_m \right], \tag{2.16}$$

$$\sqrt{\gamma} = \frac{\exp\left(-\frac{u_1^2}{2}\right)}{\sqrt{\frac{\pi}{2}} \alpha \lambda \left[g\left(\frac{u_0}{\sqrt{2}}, \frac{1}{\sqrt{\pi}}\right) - \operatorname{erf}\left(\frac{u_1}{\sqrt{2}}\right) \right]} \tag{2.17}$$

$$y_1 = \frac{-\exp\left(-\frac{u_1^2}{2}\right)}{\alpha \lambda F_{u_0}(u_1)} \tag{2.18}$$

Proof. A parametric solution of (2.9) was deduced in [4], and it is given by

$$\xi = \varphi_1(u) = C_2 \left(\exp\left(-\frac{u^2}{2}\right) + u \left(\int_0^u \exp\left(-\frac{x^2}{2}\right) dx + C_1 \right) \right) \tag{2.19}$$

$$y = \varphi_2(u) = \sqrt{\gamma} C_2 \left(\int_0^u \exp\left(-\frac{x^2}{2}\right) dx + C_1 \right) \quad u > 0 \tag{2.20}$$

where C_1 and C_2 are integration constants to be determined.

We choose u_0 and u_1 be such that $\varphi_1(u_0) = 0$ and $\varphi_1(u_1) = 1$; we obtain that

$$C_1 = -\frac{\exp\left(-\frac{u_0^2}{2}\right)}{u_0} - \int_0^{u_0} \exp\left(-\frac{x^2}{2}\right) dx, \tag{2.21}$$

$$C_2 = \left\{ \exp\left(-\frac{u_1^2}{2}\right) + u_1 \left(-\frac{\exp\left(-\frac{u_0^2}{2}\right)}{u_0} + \int_{u_0}^{u_1} \exp\left(-\frac{x^2}{2}\right) dx \right) \right\}^{-1}. \tag{2.22}$$

Then, we have

$$\xi = \varphi_1(u) = \frac{\exp\left(-\frac{u^2}{2}\right) + u \left(\int_{u_0}^u \exp\left(-\frac{x^2}{2}\right) dx - \frac{\exp\left(-\frac{u_0^2}{2}\right)}{u_0} \right)}{\exp\left(-\frac{u_1^2}{2}\right) + u_1 \left(-\frac{\exp\left(-\frac{u_0^2}{2}\right)}{u_0} + \int_{u_0}^{u_1} \exp\left(-\frac{x^2}{2}\right) dx \right)}, \quad u_0 \leq u \leq u_1 \tag{2.23}$$

and

$$y = \varphi_2(u) = \frac{\sqrt{\gamma} \left\{ -\frac{\exp\left(-\frac{u_0^2}{2}\right)}{u_0} + \int_{u_0}^u \exp\left(-\frac{x^2}{2}\right) dx \right\}}{\exp\left(-\frac{u_1^2}{2}\right) + u_1 \left(-\frac{\exp\left(-\frac{u_0^2}{2}\right)}{u_0} + \int_{u_0}^{u_1} \exp\left(-\frac{x^2}{2}\right) dx \right)}, \quad u_0 \leq u \leq u_1 \tag{2.24}$$

that is (2.13)–(2.14).

Next we prove that the unknowns u_0 , u_1 and γ must satisfy (2.16)–(2.18). From (2.23) and (2.24), we have

$$y'(\xi) = \frac{\varphi_2'(u)}{\varphi_1'(u)} = \frac{\sqrt{\gamma} \exp\left(-\frac{u^2}{2}\right)}{\int_{u_0}^u \exp\left(-\frac{x^2}{2}\right) dx - \frac{\exp\left(-\frac{u_0^2}{2}\right)}{u_0}} \tag{2.25}$$

then

$$y'(0) = -\sqrt{\gamma}u_0 \tag{2.26}$$

and taking into account that

$$y(0) = \frac{-\sqrt{\gamma} \exp\left(-\frac{u_0^2}{2}\right)}{u_0 F_{u_0}(u_1)}$$

and from (2.10), we have (2.16).

Analogously we have

$$y'(1) = \frac{\varphi_2'(u_1)}{\varphi_1'(u_1)} = \frac{\sqrt{\gamma} \exp\left(-\frac{u_1^2}{2}\right)}{\int_{u_0}^{u_1} \exp\left(-\frac{x^2}{2}\right) dx - \frac{\exp\left(-\frac{u_0^2}{2}\right)}{u_0}} \tag{2.27}$$

and by (2.11) we have

$$\frac{\sqrt{\gamma} \exp\left(-\frac{u_1^2}{2}\right)}{\int_{u_0}^{u_1} \exp\left(-\frac{x^2}{2}\right) dx - \frac{\exp\left(-\frac{u_0^2}{2}\right)}{u_0}} = -\alpha\lambda\gamma \tag{2.28}$$

that is (2.17).

Last, we have

$$y(1) = \varphi_2(u_1) = \frac{\sqrt{\gamma} \left\{ -\frac{\exp\left(-\frac{u_0^2}{2}\right)}{u_0} + \int_{u_0}^{u_1} \exp\left(-\frac{x^2}{2}\right) dx \right\}}{\exp\left(-\frac{u_1^2}{2}\right) + u_1 \left(-\frac{\exp\left(-\frac{u_0^2}{2}\right)}{u_0} + \int_{u_0}^{u_1} \exp\left(-\frac{x^2}{2}\right) dx \right)} \tag{2.29}$$

and taking into account (2.12) and (2.17) we obtain (2.18). □

Next we want to find u_0 , u_1 and γ the solutions to the equations (2.16)–(2.18). We can rewrite the system (2.16)–(2.18) as follows

$$P^{-1} \left(\frac{u_0}{\sqrt{2}h\lambda} + T_m \right) = \frac{\gamma \exp(-u_0^2)}{[u_0 F_{u_0}(u_1)]^2} \tag{2.30}$$

$$\sqrt{\gamma} = \frac{\exp\left(-\frac{u_1^2}{2}\right)}{\alpha\lambda\sqrt{\frac{\pi}{2}} \left[g\left(\frac{u_0}{\sqrt{2}}, \frac{1}{\sqrt{\pi}}\right) - \operatorname{erf}\left(\frac{u_1}{\sqrt{2}}\right) \right]} \tag{2.31}$$

$$M(u_1) = g\left(\frac{u_0}{\sqrt{2}}, \frac{1}{\sqrt{\pi}}\right) \tag{2.32}$$

where

$$M(x) = g\left(\frac{x}{\sqrt{2}}, \frac{1}{\sqrt{\pi}} \left(\frac{1}{\alpha\lambda y_1} + 1 \right) \right). \tag{2.33}$$

Lemma 2.2. *The real function F_{u_0} and M satisfy the following properties:*

$$F_{u_0}(u_0) = 0, F(+\infty) = -\infty \tag{2.34}$$

$$F'_{u_0}(x) = \frac{\sqrt{\pi}}{2} \left\{ \operatorname{erf}\left(\frac{x}{\sqrt{2}}\right) - g\left(\frac{u_0}{\sqrt{2}}, \frac{1}{\sqrt{\pi}}\right) \right\} < 0 \tag{2.35}$$

$$M(0) = +\infty, M(+\infty) = 1 \text{ and } M'(x) < 0. \tag{2.36}$$

Proof. See [1] and [2]. □

Lemma 2.3. (Existence of the solution)

There exists a solution of the system (2.30)–(2.32) given by

$$\tilde{u}_1 = M^{-1} \left(g\left(\frac{\tilde{u}_0}{\sqrt{2}}, \frac{1}{\sqrt{\pi}}\right) \right) \tag{2.37}$$

$$\tilde{\gamma} = \frac{\exp(-\tilde{u}_1^2)}{\alpha^2\lambda^2 \left(\frac{\exp\left(-\frac{\tilde{u}_0^2}{2}\right)}{\tilde{u}_0} - \int_{\tilde{u}_0}^{\tilde{u}_1} \exp\left(-\frac{x^2}{2}\right) dx \right)^2} \tag{2.38}$$

where \tilde{u}_0 is a solution of

$$P^{-1} \left(\frac{u_0}{\sqrt{2}h\lambda} + T_m \right) = \frac{\gamma \exp(-u_0^2)}{\left[u_0 F_{u_0} \left(M^{-1} \left(g \left(\frac{u_0}{\sqrt{2}}, \frac{1}{\sqrt{\pi}} \right) \right) \right) \right]^2}. \tag{2.39}$$

Proof. Because M is a decreasing function, there exists the inverse function M^{-1} , and from (2.32) for each u_0 there exists a unique u_1 given by

$$u_1(u_0) = M^{-1} \left(g \left(\frac{u_0}{\sqrt{2}}, \frac{1}{\sqrt{\pi}} \right) \right). \tag{2.40}$$

If we replace (2.40) in (2.31) and (2.30), we have

$$\gamma(u_0) = \frac{\exp(-u_1^2(u_0))}{\alpha^2 \lambda^2 \left(\frac{\exp\left(-\frac{u_0^2}{2}\right)}{u_0} - \int_{u_0}^{u_1(u_0)} \exp\left(-\frac{x^2}{2}\right) dx \right)^2} \tag{2.41}$$

and

$$P^{-1} \left(\frac{u_0}{\sqrt{2}h\lambda} + T_m \right) = \frac{\gamma(u_0) \exp(-u_0^2)}{\left[u_0 F_{u_0}(u_1(u_0)) \right]^2}. \tag{2.42}$$

We define the function

$$G(u_0) := P^{-1} \left(\frac{u_0}{\sqrt{2}h\lambda} + T_m \right)$$

which satisfies $G(0) = \frac{k}{s}(T_m)$ and $G'(u_0) < 0$, and let

$$H(u_0) := \frac{\gamma(u_0) \exp(-u_0^2)}{\left[u_0 F_{u_0}(u_1(u_0)) \right]^2}.$$

From (2.18), (2.40) and (2.41), it follows that

$$H(u_0) = \frac{2y_1^2 \exp(-u_0^2)}{u_0^2 \pi \left[\operatorname{erf} \left(\frac{M^{-1} \left(g \left(\frac{u_0}{\sqrt{2}}, \frac{1}{\sqrt{\pi}} \right) \right)}{\sqrt{2}} \right) - g \left(\frac{u_0}{\sqrt{2}}, \frac{1}{\sqrt{\pi}} \right) \right]^2},$$

$H(0) = y_1^2$, $H(+\infty) = +\infty$ and $H(u_0) \geq y_1^2, \forall u_0 \geq 0$.

Since $T_m < T_f$, we conclude $G(0) = \frac{k}{s}(T_m) > \frac{k}{s}(T_f) = y_1^2 = H(0)$.

Taking into account the properties of G and H , there exists $\tilde{u}_0 < u_0^* = (T_f - T_m) \sqrt{2}h\lambda$ which satisfies (2.42).

Then by (2.40) and (2.41), we complete the solution $\tilde{u}_1 = u_1(\tilde{u}_0)$ and $\tilde{\gamma} = \gamma(\tilde{u}_0)$ to the system (2.30)–(2.32). □

Lemma 2.4. (Uniqueness of the solution)

The solution $(\tilde{u}_0, \tilde{u}_1, \tilde{\gamma})$ to the system (2.16)–(2.18) is unique.

Proof. Suppose the assertion of the lemma is false. That is there exist two solutions $(\tilde{u}_0, \tilde{u}_1, \tilde{\gamma})$ and (u_0^*, u_1^*, γ^*) to (2.16)–(2.18).

We assume that $\tilde{u}_0 < u_0^*$; then, by (2.13) we have

$$\xi = \frac{F_{u_0^*}(u)}{F_{u_0^*}(u_1^*)} = \frac{F_{\tilde{u}_0}(u)}{F_{\tilde{u}_0}(\tilde{u}_1)}, \quad \text{for } u_0^* \leq u \leq \min(\tilde{u}_1, u_1^*). \tag{2.43}$$

For $u = u_0^*$, we have

$$0 = \frac{F_{u_0^*}(u_0^*)}{F_{u_0^*}(u_1^*)} = \frac{F_{\tilde{u}_0}(u_0^*)}{F_{\tilde{u}_0}(\tilde{u}_1)} \tag{2.44}$$

then $F_{\tilde{u}_0}(u_0^*) = 0$. This is a contradiction because $F_{\tilde{u}_0}(u_0^*) = 0$ if and only if $u = \tilde{u}_0$. \square

Theorem 2.5. *The problem (1.1)–(1.6) has a unique similarity-type solution given by*

$$T(x, t) = P\left(\left(\varphi_2\left(\varphi_1^{-1}\left(x/X(t)\right)\right)\right)^2\right), \quad 0 < x < X(t) \tag{2.45}$$

where

$$X(t) = \sqrt{2\tilde{\gamma}t}, \quad t > 0 \tag{2.46}$$

is the free boundary,

$$\varphi_1(u) = \frac{F_{\tilde{u}_0}(u)}{F_{\tilde{u}_0}(\tilde{u}_1)}, \tag{2.47}$$

$$\varphi_2(u) = \frac{\sqrt{\tilde{\gamma}}\sqrt{\frac{\pi}{2}}\left[\operatorname{erf}\left(\frac{u}{\sqrt{2}}\right) - g\left(\frac{\tilde{u}_0}{\sqrt{2}}, \frac{1}{\sqrt{\pi}}\right)\right]}{F_{\tilde{u}_0}(\tilde{u}_1)}, \tag{2.48}$$

$(\tilde{u}_0, \tilde{u}_1, \tilde{\gamma})$ is the unique solution of (2.16)–(2.18) and $P = \left(\frac{k}{s}\right)^{-1}$ is the inverse function of the function $\frac{k}{s}$.

Proof. Fixed the data: α, λ, h, T_f of the problem (1.1)–(1.6), we obtain the solutions of the equations (2.16)–(2.18) given by (2.37), (2.38) and \tilde{u}_0 is the solution of (2.39).

Next, we obtain φ_1 and φ_2 given by (2.47), (2.48), respectively, and the free boundary is $X(t) = \sqrt{2\tilde{\gamma}t}$. Taking into account that φ_1 is an increasing function, we determine $\varphi_1^{-1}\left(\frac{x}{X(t)}\right)$. Finally, we invert the relation (2.7), and from (2.1) we obtain (2.45). \square

Remark 2.6. If $T(0, t) = T_s$ is constant, the convective condition (1.2) at the fixed face $x = 0$ of the problem (1.1)–(1.6) becomes a Neumann boundary condition given by

$$k(T(0, t))\frac{\partial T}{\partial x}(0, t) = \frac{q_0}{\sqrt{t}} \tag{2.49}$$

with

$$q_0 = h[T_s - T_m].$$

The Stefan problem (1.1)–(1.6) with the condition (2.49) instead (1.2) was studied in [2].

3. Asymptotic behavior of the solution when $h \rightarrow +\infty$

Let $h > 0$ and $T = T_h(x, t), X = X_h(t)$ denote the solution to the problem (1.1)–(1.6) given by (2.45)–(2.48). We will study the behavior of this solution when the transfer coefficient $h \rightarrow +\infty$. We will prove that T_h, X_h converges to the solution T_∞, X_∞ of the following parabolic free boundary problem:

$$s(T)\frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left[k(T)\frac{\partial T}{\partial x}\right], \quad 0 < x < X(t), \quad t > 0, \tag{3.1}$$

$$T(0, t) = T_m, \quad t > 0, \tag{3.2}$$

$$T(X(t), t) = T_f, \quad t > 0, \tag{3.3}$$

$$k(T_f)\frac{\partial T}{\partial x}(X(t), t) = \alpha \dot{X}(t), \quad t > 0, \tag{3.4}$$

$$X(0) = 0 \tag{3.5}$$

with the Storm's condition

$$\frac{d}{dT} \left(\sqrt{\frac{s(T)}{k(T)}} \right) = \lambda. \tag{3.6}$$

The problem (3.1)–(3.6) was studied in [2]. The solution is given by

$$T_\infty(x, t) = P \left(\left(\varphi_{2\infty} \left(\varphi_{1\infty}^{-1} \left(x / X_\infty(t) \right) \right) \right)^2 \right) \tag{3.7}$$

$$X_\infty(t) = \sqrt{2\gamma_\infty t} \tag{3.8}$$

where

$$\varphi_{1\infty}(u) = \frac{F_{v_0}(u)}{F_{v_0}(v_1)}, \tag{3.9}$$

$$\varphi_{2\infty}(u) = \frac{\sqrt{\gamma_\infty} \sqrt{\frac{\pi}{2}} \left[\operatorname{erf} \left(\frac{u}{\sqrt{2}} \right) - g \left(\frac{v_0}{\sqrt{2}}, \frac{1}{\sqrt{\pi}} \right) \right]}{F_{v_0}(v_1)} \tag{3.10}$$

with $v_0 \leq u \leq v_1$. The parameters v_0, v_1 and γ_∞ satisfy the following equations

$$y_1 = \sqrt{\gamma_\infty} \frac{F_{v_0}(v_1) - \exp \left(-\frac{v_1^2}{2} \right)}{v_1 F_{v_0}(v_1)} \tag{3.11}$$

$$\sqrt{\gamma_\infty} = \frac{v_1 y_1}{1 + \alpha \lambda y_1} \tag{3.12}$$

$$\frac{k}{s}(T_m) = y_0 = -\sqrt{\gamma_\infty} \frac{\exp \left(-\frac{v_0^2}{2} \right)}{v_0 F_{v_0}(v_1)} \tag{3.13}$$

which are equivalent to

$$\frac{k}{s}(T_m) = H(v_0) = \frac{2y_1^2 \exp \left(-v_0^2 \right)}{v_0^2 \pi \left[\operatorname{erf} \left(\frac{M^{-1} \left(g \left(\frac{v_0}{\sqrt{2}}, \frac{1}{\sqrt{\pi}} \right) \right)}{\sqrt{2}} \right) - g \left(\frac{v_0}{\sqrt{2}}, \frac{1}{\sqrt{\pi}} \right) \right]^2} \tag{3.14}$$

$$\sqrt{\gamma_\infty} = \frac{\exp \left(-\frac{v_1^2}{2} \right)}{\alpha \lambda \sqrt{\frac{\pi}{2}} \left[g \left(\frac{v_0}{\sqrt{2}}, \frac{1}{\sqrt{\pi}} \right) - \operatorname{erf} \left(\frac{v_1}{\sqrt{2}} \right) \right]} \tag{3.15}$$

$$v_1 = M^{-1} \left(g \left(\frac{v_0}{\sqrt{2}}, \frac{1}{\sqrt{\pi}} \right) \right). \tag{3.16}$$

For simplicity of notation, we write $(u_{0h}, u_{1h}, \gamma_h)$ instead of $(\tilde{u}_{0h}, \tilde{u}_{1h}, \tilde{\gamma}_h)$, which is the solution of (2.30)–(2.32). Firstly, we will prove that $(u_{0h}, u_{1h}, \gamma_h)$ converges to $(v_0, v_1, \gamma_\infty)$ when $h \rightarrow +\infty$. The proof of this statement is based on the following lemma:

Lemma 3.1. *The sequences $\{u_{0h}\}$, $\{u_{1h}\}$ and $\{\gamma_h\}$ are increasing and bounded. Moreover,*

$$\lim_{h \rightarrow +\infty} u_{0h} = v_0, \quad \lim_{h \rightarrow +\infty} u_{1h} = v_1, \quad \text{and} \quad \lim_{h \rightarrow +\infty} \gamma_h = \gamma_\infty.$$

Proof. From properties of function $G = G_h(x) = P^{-1} \left(\frac{x}{\sqrt{2h\lambda}} + T_m \right)$, we have

- a) $h_1 \leq h_2 \Rightarrow G_{h_1}(x) \leq G_{h_2}(x), \quad \forall x \geq 0$
- b) $G_h(x) \leq \frac{k}{s}(T_m), \quad \forall x \geq 0, \quad h > 0.$

We consider $h_1 < h_2$, if u_{0h_1} and u_{0h_2} are the solutions of $G_{h_1}(x) = H(x)$ and $G_{h_2}(x) = H(x)$, respectively, by a) and properties of function H we have that $u_{0h_1} < u_{0h_2}$. Moreover, from b) results $u_{0h} \leq v_0$ for all $h > 0$. Then, $\{u_{0h}\}$ is an increasing bounded sequence, and there exists \tilde{u}_0 such that

$$\lim_{h \rightarrow +\infty} u_{0h} = \tilde{u}_0.$$

Letting $h \rightarrow +\infty$ on $G_h(u_{0h}) = H(u_{0h})$ yields $\frac{k}{s}(T_m) = H(\tilde{u}_0)$. By uniqueness of the solution of (3.14) results $\tilde{u}_0 = v_0$.

From (2.32), we have

$$u_{1h} = M^{-1} \left(g \left(\frac{u_{0h}}{\sqrt{2}}, \frac{1}{\sqrt{\pi}} \right) \right) \tag{3.17}$$

Because $\{u_{0h}\}$ is increasing, M and g are decreasing functions we have that the sequence $\{u_{1h}\}$ is increasing. Moreover, taking into account $u_{0h} \leq v_0$ and (3.16) follows

$$u_{1h} = M^{-1} \left(g \left(\frac{u_{0h}}{\sqrt{2}}, \frac{1}{\sqrt{\pi}} \right) \right) \leq M^{-1} \left(g \left(\frac{v_0}{\sqrt{2}}, \frac{1}{\sqrt{\pi}} \right) \right) = v_1$$

for all $h > 0$.

By (3.17), we obtain

$$\lim_{h \rightarrow +\infty} u_{1h} = \lim_{h \rightarrow +\infty} M^{-1} \left(g \left(\frac{u_{0h}}{\sqrt{2}}, \frac{1}{\sqrt{\pi}} \right) \right) = M^{-1} \left(g \left(\frac{v_0}{\sqrt{2}}, \frac{1}{\sqrt{\pi}} \right) \right) = v_1.$$

Finally, letting $h \rightarrow +\infty$ in (2.31) we have

$$\lim_{h \rightarrow +\infty} \gamma_h = \gamma_\infty.$$

It follows easily of (2.31) and (2.32) that $\sqrt{\gamma_h} = \frac{u_{1h}y_1}{1 + \alpha\lambda y_1}$. Taking into account $u_{1h} \leq v_1$, we have

$$\sqrt{\gamma_h} = \frac{u_{1h}y_1}{1 + \alpha\lambda y_1} \leq \frac{v_1y_1}{1 + \alpha\lambda y_1} = \sqrt{\gamma_\infty} \quad \forall h > 0.$$

□

Corollary 3.2. *For each $t > 0$, the sequence $\{X_h(t)\}$ is monotonically increasing and $\lim_{h \rightarrow +\infty} X_h(t) = X_\infty(t)$.*

We can now define an extension $\tilde{T}_h = \tilde{T}_h(x, t) \in C^1 [0, X_\infty(t)]$ of $T_h(x, t)$ as follows

$$\tilde{T}_h(x, t) = \begin{cases} T_h(x, t) & \text{if } 0 \leq x < X_h(t) \\ \frac{\alpha\sqrt{2\gamma_h}}{2k(T_f)\sqrt{t}}(x - X_h(t)) + T_f & \text{if } X_h(t) \leq x \leq X_\infty(t) \end{cases} \tag{3.18}$$

Lemma 3.3. *The functions $\tilde{T}_h \in C^1 [0, X_\infty(t)]$ satisfy $\left| \frac{\partial \tilde{T}_h}{\partial x} \right| \leq M$ on $[0, X_\infty(t)]$ for all $h > 0, t > 0$.*

Proof. Let $t > 0$ and $x \in [0, X_\infty(t)]$.

If $x \in [X_h(t), X_\infty(t)]$, then

$$\left| \frac{\partial \tilde{T}_h(x, t)}{\partial x} \right| = \frac{\alpha\sqrt{2\gamma_\infty}}{2k(T_f)\sqrt{t}}.$$

For otherwise, this is $x \in [0, X_h(t))$ according to (2.1) and (2.7) we have

$$\frac{\partial \tilde{T}_h}{\partial x}(x, t) = P' \left(y_h^2 \left(\frac{x}{X_h(t)} \right) \right) 2y_h \left(\frac{x}{X_h(t)} \right) y'_h \left(\frac{x}{X_h(t)} \right) \frac{1}{X_h(t)}$$

Since $\frac{k}{s}$ is decreasing and $T_m \leq T_h(x, t) \leq T_f$, from (2.7) we have $y_1 \leq y_h \left(\frac{x}{X_h(t)} \right) \leq y_0$, for all $h > 0$. From (1.6), it follows that

$$\left| P' \left(y_h^2 \left(\frac{x}{X_h(t)} \right) \right) \right| \leq \frac{1}{2\lambda y_1 k_m}$$

where $k_m = \min \{k(T), T_m \leq T \leq T_f\}$. Taking into account (2.23), (2.24), (2.47) and Lemma 3.1, we have

$$\left| y'_h \left(\frac{x}{X_h(t)} \right) \frac{1}{X_h(t)} \right| \leq \frac{1}{\sqrt{\pi t} \left[1 - \operatorname{erf} \left(\frac{v_1}{\sqrt{2}} \right) \right]}.$$

Then for $x \in [0, X_h(t))$ results

$$\left| \frac{\partial \tilde{T}_h(x, t)}{\partial x} \right| \leq \frac{y_0}{\lambda y_1 k_m \sqrt{\pi t} \left[1 - \operatorname{erf} \left(\frac{v_1}{\sqrt{2}} \right) \right]}.$$

Summarizing, for all $h > 0$ and $x \in [0, X_\infty(t)]$ we obtain

$$\left| \frac{\partial \tilde{T}_h(x, t)}{\partial x} \right| \leq M = \max \left\{ \frac{\alpha \sqrt{2\gamma_\infty}}{2k(T_f)\sqrt{t}}, \frac{y_0}{\lambda y_1 k_m \sqrt{\pi t} \left[1 - \operatorname{erf} \left(\frac{v_1}{\sqrt{2}} \right) \right]} \right\}$$

and this precisely the assertion of the lemma. □

Lemma 3.4. We have $\lim_{h \rightarrow +\infty} \tilde{T}_h(x, t) = T_\infty(x, t)$ for each $t > 0$ and $x \in [0, X_\infty(t)]$.

Proof. Let $t > 0$ and $x \in [0, X_\infty(t))$. By Corollary 3.2, there exists $h_0 = h_0(x) > 0$ such that $x \in [0, X_h(t)]$ for all $h \geq h_0$. We consider $\tilde{T}_h(x, t)$ for $h \geq h_0$ we have

$$\tilde{T}_h(x, t) = T_h(x, t) = P \left((\varphi_{2h}(\varphi_{1h}^{-1}(x/X_h(t))))^2 \right). \tag{3.19}$$

Taking into account Lemma 3.1, Corollary 3.2, (2.47) and (2.48) we obtain that the sequence $\{T_h(x, t)\}$ converges to $T_\infty(x, t)$. If $x = X_\infty(t)$, then $\tilde{T}_h(X_\infty(t), t) = T_f = T_\infty(X_\infty(t), t)$.

Hence, the sequence $\{\tilde{T}_h(x, t)\}$ converges to $T_\infty(x, t)$ pointwise on $[0, X_\infty(t)]$ for each $t > 0$. □

Theorem 3.5. For each $t > 0$, we have the family of functions $\{\tilde{T}_h\}$ converges uniformly to T_∞ for $h \rightarrow +\infty$ on $[0, X_\infty(t)]$.

Proof. By Lemma 3.3, for any $t > 0$ the functions $\tilde{T}_h(x, t)$ are equicontinuous on $[0, X_\infty(t)]$ and from Lemma 3.4 converges pointwise to $T_\infty(x, t)$ for $h \rightarrow +\infty$. Then, by Ascoli Arzela lemma we obtain their uniform convergence on $[0, X_\infty(t)]$. □

4. Conclusions

One-phase nonlinear, one-dimensional Stefan problem for a semi-infinite material $x > 0$, with phase change temperature T_f , has been considered with the assumption of a Storm's condition for the heat capacity and thermal conductivity and a convective condition at the fixed face. Existence and uniqueness of a similarity-type solution have been obtained. Moreover, the convergence of this problem to problem with temperature condition at the fixed face when $h \rightarrow +\infty$ has been proved.

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