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## One-Dimensional Nonlinear Stefan Problems in Storm's Materials

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**Abstract:** We consider two one-phase nonlinear one-dimensional Stefan problems for a semi-infinite material  $x > 0$ , with phase change temperature  $T_f$ . We assume that the heat capacity and the thermal conductivity satisfy a Storm's condition. In the first case, we assume a heat flux boundary condition of the type  $q(t) = \frac{q_0}{\sqrt{t}}$ , and in the second case, we assume a temperature boundary condition  $T = T_s < T_f$  at the fixed face. Solutions of similarity type are obtained in both cases, and the equivalence of the two problems is demonstrated. We also give procedures in order to compute the explicit solution.

**Keywords:** Stefan problem; free boundary problem; phase-change process; similarity solution

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### 1. Introduction

As in [1–3] we consider the following one-phase nonlinear unidimensional Stefan problem for a semi-infinite material  $x > 0$ , with phase change temperature  $T_f$

$$s(T) \frac{\partial T}{\partial t} = \frac{\partial}{\partial x} \left[ k(T) \frac{\partial T}{\partial x} \right], \quad 0 < x < X(t), \quad t > 0 \quad (1)$$

$$k(T(0, t)) \frac{\partial T}{\partial x}(0, t) = \frac{q_0}{\sqrt{t}}, \quad q_0 > 0, \quad t > 0 \quad (2)$$

$$T(X(t), t) = T_f \tag{3}$$

$$k(T_f) \frac{\partial T}{\partial x}(X(t), t) = \alpha \dot{X}(t), \quad t > 0 \tag{4}$$

$$X(0) = 0 \tag{5}$$

where the positive constant,  $\alpha$ , is  $\rho L$ ,  $L$  is the latent heat of fusion of the medium and  $\rho$  is the density (assumed constant). The partial differential equation of heat conduction is a nonlinear equation when the temperature dependence of the thermal parameters is taken into account. We assume that the metal exhibits nonlinear thermal characteristics, such that the heat capacity,  $c_p(T)$ , and the thermal conductivity,  $k(T)$ , satisfy a Storm’s condition [4–8]:

$$\frac{\frac{d}{dT} \left( \sqrt{\frac{s(T)}{k(T)}} \right)}{s(T)} = \lambda = const. > 0 \tag{6}$$

where  $s(T) = \rho c_p(T)$ . Condition (6) was originally obtained by [8] in an investigation of heat conduction in simple monoatomic metals. In that paper, it was shown that if this condition is satisfied, then the partial differential equation of the heat conduction can be transformed to the linear form. There, the validity of the approximation (6) was examined for aluminum, silver, sodium, cadmium, zinc, copper and lead.

In [7], the free boundary problems (1)–(5) (fusion case) for the particular case  $k(T) = \rho c / (a + bT)^2$  and  $s(T) = \rho c = \text{constant}$  was studied. The explicit solution of this problem was obtained through the unique solution of an integral equation with time as a parameter. A similar case with the constant temperature at the fixed face  $x = 0$  was also studied.

The goal of this paper is to determine the temperature  $T = T(x, t)$  and the position of the phase change boundary at time  $t$ ,  $X = X(t)$ , which satisfy the problems (1)–(5). In the section after, we show how to find a parametric solution for this problem.

In Section 3, we consider the free boundary problems (1), (3)–(5) and a temperature boundary condition  $T = T_s < T_f$  at the fixed face  $x = 0$  instead of the heat flux condition (2). We improve [1], obtaining the explicit solution and showing the existence and uniqueness of this type of solution in both cases in which only numerical results for the case with a temperature boundary condition at the fixed face  $x = 0$  were presented in that work.

We also give procedures in order to compute the explicit solution in both cases.

In Section 4, we prove the equivalence of the two free boundary problems: the first with the Neumann boundary condition (2) is considered in Section 2, and the second one with the Dirichlet constant boundary condition (44) is considered in Section 3.

## 2. Solution to the Stefan Problem with the Heat Flux Condition on the Fixed Face

We consider the problems (1)–(5), and we propose a similarity type solution given by [1,9]:

$$T(x, t) = \Phi(\xi), \quad \xi = \frac{x}{X(t)} \tag{7}$$

where:

$$X(t) = \sqrt{2\gamma t}, \quad t > 0 \tag{8}$$

is the free boundary and  $\gamma$  is assumed a positive constant to be determined.

Then, we have that the problems (1)–(5) are equivalent to:

$$k(\Phi)\Phi''(\xi) + k'(\Phi)\Phi'^2(\xi) + \gamma s(\Phi)\Phi'(\xi)\xi = 0 \quad , \quad 0 < \xi < 1 \tag{9}$$

$$k(\Phi(0))\Phi'(0) = \sqrt{2\gamma}q_0 \tag{10}$$

$$\Phi(1) = T_f \tag{11}$$

$$k(\Phi(1))\Phi'(1) = \alpha\gamma \tag{12}$$

If we define:

$$y(\xi) = \sqrt{\frac{k(\Phi(\xi))}{s(\Phi(\xi))}} \tag{13}$$

then a parametrization of the Storm condition is:

$$s(\Phi) = -\frac{1}{\lambda y^2} \frac{dy}{d\Phi} \quad , \quad k(\Phi) = -\frac{1}{\lambda} \frac{dy}{d\Phi} \tag{14}$$

Then, we have that the following problem is equivalent to Equations (9)–(12):

$$\frac{d^2y}{d\xi^2} + \frac{\gamma\xi}{y^2} \frac{dy}{d\xi} = 0 \quad , \quad 0 < \xi < 1 \tag{15}$$

$$y'(0) = -\sqrt{2\gamma}\lambda q_0 \tag{16}$$

$$y'(1) = -\alpha\lambda\gamma \tag{17}$$

$$y(1) = y_1 = \sqrt{\frac{k(T_f)}{s(T_f)}} \tag{18}$$

**Lemma 1.** A parametric solution to the problems (15)–(18) is given by:

$$\xi = \varphi_1(u) = \frac{\exp(-\frac{u^2}{2}) + u \left( \int_{u_0}^u \exp(-\frac{x^2}{2}) dx - \frac{\exp(-\frac{u_0^2}{2})}{u_0} \right)}{\exp(-\frac{u_1^2}{2}) + u_1 \left( -\frac{\exp(-\frac{u_0^2}{2})}{u_0} + \int_{u_0}^{u_1} \exp(-\frac{x^2}{2}) dx \right)} \tag{19}$$

$$y = \varphi_2(u) = \frac{\sqrt{\gamma} \left\{ -\frac{\exp(-\frac{u_0^2}{2})}{u_0} + \int_{u_0}^u \exp(-\frac{x^2}{2}) dx \right\}}{\exp(-\frac{u_1^2}{2}) + u_1 \left( -\frac{\exp(-\frac{u_0^2}{2})}{u_0} + \int_{u_0}^{u_1} \exp(-\frac{x^2}{2}) dx \right)} \tag{20}$$

for:

$$u_0 < u < u_1$$

where  $u_0$  and  $u_1$  are the parameter values, which verify that  $\xi = \varphi_1(u_0) = 0$  and  $\xi = \varphi_1(u_1) = 1$ . The unknowns,  $\gamma$ ,  $u_0$  and  $u_1$ , must verify the following system of equations:

$$u_0 = \sqrt{2\gamma}\lambda q_0 \tag{21}$$

$$\sqrt{\gamma} = \frac{\exp(-\frac{u_1^2}{2})}{\alpha\lambda \left( \frac{\exp(-\frac{u_0^2}{2})}{u_0} - \int_{u_0}^{u_1} \exp(-\frac{x^2}{2}) dx \right)} \tag{22}$$

$$y_1 = \frac{-\exp(-\frac{u_1^2}{2})}{\alpha\lambda \left\{ \exp(-\frac{u_1^2}{2}) + u_1 \left( -\frac{\exp(-\frac{u_0^2}{2})}{u_0} + \int_{u_0}^{u_1} \exp(-\frac{x^2}{2}) dx \right) \right\}} \tag{23}$$

**Proof.** A parametric solution of Equation (15) was deduced in [1], and it is given by:

$$\xi = \varphi_1(u) = C_2 \left( \exp(-\frac{u^2}{2}) + u \left( \int_0^u \exp(-\frac{x^2}{2}) dx + C_1 \right) \right) \tag{24}$$

$$y = \varphi_2(u) = \sqrt{\gamma} C_2 \left( \int_0^u \exp(-\frac{x^2}{2}) dx + C_1 \right), \quad u > 0 \tag{25}$$

where  $C_1$  and  $C_2$  are integration constants to be determined.

We choose  $u_0$  and  $u_1$  to be such that  $\varphi_1(u_0) = 0$  and  $\varphi_1(u_1) = 1$ ; we obtain that:

$$C_1 = -\frac{\exp(-\frac{u_0^2}{2})}{u_0} - \int_0^{u_0} \exp(-\frac{x^2}{2}) dx \tag{26}$$

$$C_2 = \left\{ \exp(-\frac{u_1^2}{2}) + u_1 \left( -\frac{\exp(-\frac{u_0^2}{2})}{u_0} + \int_{u_0}^{u_1} \exp(-\frac{x^2}{2}) dx \right) \right\}^{-1} \tag{27}$$

Then, we have that:

$$\xi = \varphi_1(u) = \frac{\exp(-\frac{u^2}{2}) + u \left( \int_{u_0}^u \exp(-\frac{x^2}{2}) dx - \frac{\exp(-\frac{u_0^2}{2})}{u_0} \right)}{\exp(-\frac{u_1^2}{2}) + u_1 \left( -\frac{\exp(-\frac{u_0^2}{2})}{u_0} + \int_{u_0}^{u_1} \exp(-\frac{x^2}{2}) dx \right)}, \quad u_0 < u < u_1 \tag{28}$$

and:

$$y = \varphi_2(u) = \frac{\sqrt{\gamma} \left\{ -\frac{\exp(-\frac{u_0^2}{2})}{u_0} + \int_{u_0}^u \exp(-\frac{x^2}{2}) dx \right\}}{\exp(-\frac{u_1^2}{2}) + u_1 \left( -\frac{\exp(-\frac{u_0^2}{2})}{u_0} + \int_{u_0}^{u_1} \exp(-\frac{x^2}{2}) dx \right)}, \quad u_0 < u < u_1 \tag{29}$$

is a parametric solution to Equations (15)–(18).

Next, we prove that the unknowns,  $u_0$ ,  $u_1$  and  $\gamma$ , must satisfy Equations (21)–(23). From Equations (28) and (29), we have:

$$y'(\xi) = \frac{\varphi_2'(u)}{\varphi_1'(u)} = \frac{\sqrt{\gamma} \exp(-\frac{u^2}{2})}{\int_{u_0}^u \exp(-\frac{x^2}{2}) dx - \frac{\exp(-\frac{u_0^2}{2})}{u_0}} \tag{30}$$

Then:

$$y'(0) = -\sqrt{\gamma}u_0 \tag{31}$$

and from Equation (16), we have Equation (21).

Analogously, we have:

$$y'(1) = \frac{\varphi'_{2}(u_1)}{\varphi'_{1}(u_1)} = \frac{\sqrt{\gamma} \exp(-\frac{u_1^2}{2})}{\int_{u_0}^{u_1} \exp(-\frac{x^2}{2})dx - \frac{\exp(-\frac{u_0^2}{2})}{u_0}} \tag{32}$$

and by Equation (17), we have:

$$\frac{\sqrt{\gamma} \exp(-\frac{u_1^2}{2})}{\int_{u_0}^{u_1} \exp(-\frac{x^2}{2})dx - \frac{\exp(-\frac{u_0^2}{2})}{u_0}} = -\alpha\lambda\gamma \tag{33}$$

that is, Equation (22).

Last, we have:

$$y(1) = \varphi_2(u_1) = \frac{\sqrt{\gamma} \left\{ -\frac{\exp(-\frac{u_0^2}{2})}{u_0} + \int_{u_0}^{u_1} \exp(-\frac{x^2}{2})dx \right\}}{\exp(-\frac{u_1^2}{2}) + u_1 \left( -\frac{\exp(-\frac{u_0^2}{2})}{u_0} + \int_{u_0}^{u_1} \exp(-\frac{x^2}{2})dx \right)} \tag{34}$$

and taking into account Equation (18), we obtain Equation (23). The Lemma 1 is proven.

Next, we want to find  $u_0$ ,  $u_1$  and  $\gamma$ , the solutions to Equations (21)–(23). Obviously, the solution  $u_0 > 0$  is determined by Equation (21). To obtain  $u_1$  and  $\gamma$ , previously, we define the family of functions  $F_m = F_m(x)$  for  $x \geq m$ , with  $m > 0$  given by:

$$F_m(x) = \exp(-\frac{x^2}{2}) + x \left( \int_m^x \exp(-\frac{z^2}{2})dz - \frac{\exp(-\frac{m^2}{2})}{m} \right) \tag{35}$$

$$= \sqrt{\frac{\pi}{2}}x \left[ g\left(\frac{x}{\sqrt{2}}, \frac{1}{\sqrt{\pi}}\right) - g\left(\frac{m}{\sqrt{2}}, \frac{1}{\sqrt{\pi}}\right) \right], \quad x \geq m \tag{36}$$

where [4]:

$$g(x, p) = erf(x) + p \frac{\exp(-x^2)}{x}, \quad p > 0, \quad x > 0 \tag{37}$$

and:

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-z^2)dz, \quad x > 0$$

Now, the system of Equations (22)–(23) can be expressed as follows:

$$y_1 = \frac{-\exp(-\frac{u_1^2}{2})}{\alpha\lambda F_{\sqrt{2}\lambda q_0}(u_1)} \tag{38}$$

$$\sqrt{\gamma} = \frac{\exp(-\frac{u_1^2}{2})}{\alpha\lambda \left( \frac{\exp(-\frac{u_0^2}{2})}{u_0} - \int_{u_0}^{u_1} \exp(-\frac{x^2}{2}) dx \right)} \tag{39}$$

**Lemma 2.** For each  $m > 0$ , the function  $F_m = F_m(x)$  satisfies the following properties:

$$F_m(m) = 0, F(+\infty) = -\infty \tag{40}$$

$$F'_m(x) = \frac{\sqrt{\pi}}{2} \left\{ \operatorname{erf} \left( \frac{x}{\sqrt{2}} \right) - g \left( \frac{m}{\sqrt{2}}, \frac{1}{\sqrt{\pi}} \right) \right\} < 0 \tag{41}$$

**Lemma 3.** The system of Equations (22)–(23) has a unique solution  $u_1, \gamma$ , where  $u_1$  satisfies:

$$F_{\sqrt{2}\lambda q_0}(u_1) = \frac{-\exp(-\frac{u_1^2}{2})}{\alpha\lambda y_1} \tag{42}$$

and  $\gamma$  is given by:

$$\sqrt{\gamma} = \frac{\exp(-\frac{u_1^2}{2})}{\alpha\lambda \left( \frac{\exp(-\frac{\lambda^2 q_0^2}{2})}{\sqrt{2}\lambda q_0} - \int_{\sqrt{2}\lambda q_0}^{u_1} \exp(-\frac{x^2}{2}) dx \right)} \tag{43}$$

Proof: From the properties of function  $F_{\sqrt{2}\lambda q_0}$ , it is easy to see that Equation (42) has a unique solution,  $u_1$ . The unknown  $\gamma$  is determined by Equation (43). Lemma 3 is proven.

Summarizing, we can enunciate the following theorem.

**Theorem 4.** The problems (1)–(5) have a unique solution of a similarity type.

Now, we give a procedure in order to compute the explicit solution. Fixing the data,  $\alpha, \lambda, q_0, T_f, k(T)$  and  $s(T)$ , of the problems (1)–(5), to obtain the free boundary,  $X(t)$ , and the temperature,  $T(x, t)$ , for  $0 < x < X(t), t > 0$ , we follow the following process:

(i) We obtain the unique solutions,  $u_0, u_1$  and  $\gamma$ , of Equations (21)–(23).

(ii) For  $t > 0$ , we compute:

$$X(t) = \sqrt{2\gamma t}$$

and for each  $0 < x < X(t)$ , we obtain:

$$\frac{x}{X(t)}$$

(iii) Taking into account that  $\varphi_1(u)$  is an increasing function, we determine:

$$u = \varphi_1^{-1} \left( \frac{x}{X(t)} \right) \text{ and } \varphi_2(u)$$

where  $\varphi_1$  and  $\varphi_2$  are given by Equations (28) and (29).

(iv) We have:

$$P((\varphi_2(u))^2) = \left(\frac{k}{s}\right)^{-1} ((\varphi_2(u))^2)$$

where  $P = \left(\frac{k}{s}\right)^{-1}$  is the inverse function of the function,  $\frac{k}{s}$ , which is an increasing function by the condition (6).

(v) We obtain the temperature:

$$T(x, t) = P\left((\varphi_2(\varphi_1^{-1}(x/X(t))))^2\right)$$

### 3. Solution to the Stefan Problem with a Temperature Boundary Condition on the Fixed Face

In this section, we will prove the existence and uniqueness of the solution to the problems (1), (3)–(6) and the temperature boundary condition at the fixed face  $x = 0$  given by:

$$T(0, t) = T_s \quad , \quad t > 0 \tag{44}$$

We define the same transformations, (7), (8), (13) and (14), as was done for the problem in the previous section. We obtain an equivalent problem given by Equations (15), (17), (18) and:

$$y(0) = y_0 = \sqrt{\frac{k(T_s)}{s(T_s)}} \tag{45}$$

**Remark 1.** Assumption (6) enables one to deduce that  $y_1 < y_0$ .

**Lemma 5.** A parametric solution to the problems (15), (17), (18) and (45) is given by Equations (19) and (20), where the unknown  $u_0$ ,  $u_1$  and  $\gamma$  must satisfy the following system of equations:

$$y_1 = \sqrt{\gamma} \frac{F_{u_0}(u_1) - \exp(-\frac{u_1^2}{2})}{u_1 F_{u_0}(u_1)} \tag{46}$$

$$\sqrt{\gamma} = \frac{u_1 y_1}{1 + \lambda \alpha y_1} \tag{47}$$

$$y_0 = -\sqrt{\gamma} \frac{\exp(-\frac{u_0^2}{2})}{u_0 F_{u_0}(u_1)} \tag{48}$$

where  $F_{u_0}$  was defined in Equation (35).

Proof: Proceeding as in the previous section, we determine a parametric solution of Equation (15) given by Equations (28) and (29).

Next, we prove that the unknowns,  $u_0$ ,  $u_1$  and  $\gamma$ , must satisfy Equations (46)–(48). From Equations (19), (20) and (30), we have:

$$\sqrt{\gamma} \frac{\xi}{y} - \frac{1}{\sqrt{\gamma}} y'(\xi) = \sqrt{\gamma} \frac{\varphi_1(u)}{\varphi_2(u)} - \frac{1}{\sqrt{\gamma}} \frac{\varphi_2(u)}{\varphi_1(u)} = u \tag{49}$$

Then, for  $\xi = 1$ , we have:

$$\sqrt{\gamma} \frac{1}{y_1} - \frac{1}{\sqrt{\gamma}} y'(1) = u_1 \tag{50}$$

and taking into account Equation (17), we obtain:

$$\sqrt{\gamma} \frac{1}{y_1} - \frac{1}{\sqrt{\gamma}}(-\alpha\lambda\gamma) = u_1 \tag{51}$$

that is, Equation (47).

From Equations (18) and (29), it is easy to see that Equation (46) is obtained.

Finally, from Equations (29) and (45), we have Equation (48).

Lemma 5 is proven.

**Lemma 6.** *The system of Equations (46)–(48) has a unique solution,  $u_0$ ,  $u_1(u_0)$  and  $\gamma = \gamma(u_1(u_0))$ .*

**Proof.** In order to solve the system of Equations (46)–(48), first, we replace the expression of  $\sqrt{\gamma}$  given by Equation (47) in Equations (46) and (48); we get:

$$\frac{y_1}{y_0(1 + \lambda\alpha y_1)} \frac{\exp(-\frac{u_0^2}{2})}{u_0} = \frac{-F_{u_0}(u_1)}{u_1} \tag{52}$$

$$-\lambda\alpha y_1 F_{u_0}(u_1) = \exp(-\frac{u_1^2}{2}) \tag{53}$$

By using Lemma 2, for each  $u_0 > 0$ , we have that there exists a unique  $u_1 = u_1(u_0)$  solution to Equation (53).

If we define:

$$V(x) = g\left(\frac{x}{\sqrt{2}}, \beta\right), \quad W(x) = g\left(\frac{u_1(x)}{\sqrt{2}}, \frac{1}{\sqrt{\pi}}\right), \quad x > 0$$

where  $\beta = \frac{1}{\sqrt{\pi}} \left(1 - \frac{y_1}{y_0(1 + \lambda\alpha y_1)}\right) \in \left(0, \frac{1}{\sqrt{\pi}}\right)$  (see Remark 1), then the Equation (52) can be rewritten as

$$V(u_0) = W(u_0), \quad u_0 > 0 \tag{54}$$

Taking into account that this functions satisfy the following properties [4]:

$$V(0^+) = +\infty, \quad V(+\infty) = 1^-$$

$$V'(x) = \begin{cases} < 0 & \text{if } 0 < x < \sqrt{\frac{\beta}{2\left(\frac{1}{\sqrt{\pi}} - \beta\right)}} \\ = 0 & \text{if } x = \sqrt{\frac{\beta}{2\left(\frac{1}{\sqrt{\pi}} - \beta\right)}} \\ > 0 & \text{if } x > \sqrt{\frac{\beta}{2\left(\frac{1}{\sqrt{\pi}} - \beta\right)}} \end{cases}$$

$$W(0) = g\left(\frac{u_1(0)}{\sqrt{2}}, \frac{1}{\sqrt{\pi}}\right) > 1, \quad W(+\infty) = 1^+, \quad W'(x) < 0, \quad x > 0$$

Equation (54) admits a unique solution  $u_0 > 0$ .

Therefore, the system of Equations (46)–(48) has a unique solution  $u_0$ ,  $u_1(u_0)$  and  $\gamma = \gamma(u_1(u_0))$ .

Lemma 6 is proven.

**Remark 2.** *The solution of Equation (54) verifies  $u_0 < Q^{-1}(\beta\sqrt{\pi})$ , where:*

$$Q(x) = \sqrt{\pi}x \exp(x^2) (1 - erf(x))$$



Finally, we have the following theorem as before:

**Theorem 7.** *The problems (1), (3)–(5) and (44) have a unique solution of a similarity type .*

Next, we give a procedure similar to that given in the previous section to compute the solution of the problems (1), (3)–(5) and (44).

Fixing the data  $\alpha, \lambda, T_0, T_f, k(T)$  and  $s(T)$ , in order to obtain the free boundary,  $X(t)$ , and the temperature,  $T(x, t)$ , for  $0 < x < X(t), t > 0$ , we can follow the following process:

(i) We obtain the unique solutions,  $u_0, u_1$  and  $\gamma$ , of Equations (46)–(48).

(ii) For  $t > 0$  we compute:

$$X(t) = \sqrt{2\gamma t}$$

and for each  $0 < x < X(t)$ , we obtain:

$$\frac{x}{X(t)}$$

(iii) We determine:

$$u = \varphi_1^{-1} \left( \frac{x}{X(t)} \right) \text{ and } \varphi_2(u)$$

where  $\varphi_1$  and  $\varphi_2$  are given by Equations (28)–(29).

(iv) We have

$$P \left( (\varphi_2(u))^2 \right) = \left( \frac{k}{s} \right)^{-1} \left( (\varphi_2(u))^2 \right)$$

where  $P = \left( \frac{k}{s} \right)^{-1}$  is the inverse function of the function,  $\frac{k}{s}$ .

(v) We obtain the temperature:

$$T(x, t) = P \left( \left( \varphi_2 \left( \varphi_1^{-1} \left( \frac{x}{X(t)} \right) \right) \right)^2 \right) \tag{55}$$

#### 4. Equivalence of the Two Free Boundary Problems

We consider the solution,  $T(x, t), X(t)$ , of the problems (1), (3)–(5) and (44), given by Equation (55);  $X(t) = \sqrt{2\gamma t}$  is the free boundary,  $\varphi_1(u)$  and  $\varphi_2(u)$  are given by Equations (28) and (29) with  $u_0, u_1(u_0)$  and  $\gamma = \gamma(u_1(u_0))$  the unique solutions to Equations (46)–(48). We compute:

$$T_x(x, t) = P' \left( y^2(\xi) \right) 2y(\xi) y'(\xi) \xi_x \tag{56}$$

and we have:

$$T_x(0, t) = P' \left( y_0^2 \right) 2y_0 \left( -u_0 \sqrt{\gamma} \right) \frac{1}{\sqrt{2\gamma t}} \tag{57}$$

from Equation (6), and the definition of function  $P$  results:

$$P' \left( y_0^2 \right) = \frac{-\sqrt{s(T_s)}}{2\lambda k^{3/2}(T_s)} \tag{58}$$

Then, we have:

$$T_x(0, t) = \frac{u_0}{\lambda k(T_s) \sqrt{2t}} \tag{59}$$

This is:

$$k(T(0, t)) T_x(0, t) = \frac{u_0}{\lambda\sqrt{2t}} \tag{60}$$

meaning that the heat flux at the fixed face  $x = 0$  is of the type  $\tilde{q}_0/\sqrt{t}$ . If we replace  $q_0$  by  $\tilde{q}_0 = u_0/\sqrt{2\lambda}$  in the condition (2) and we solve the problems (1)–(5), we obtain the solution:

$$\tilde{T}(x, t) = P((\tilde{\varphi}_2(u))^2), \quad x = \sqrt{2\tilde{\gamma}t} \tilde{\varphi}_1(u), \quad \tilde{X}(t) = \sqrt{2\tilde{\gamma}t}, \quad t > 0 \tag{61}$$

where  $\tilde{\varphi}_1$  and  $\tilde{\varphi}_2$  are given by:

$$\tilde{\varphi}_1(u) = \frac{F_{\tilde{u}_0}(u)}{F_{\tilde{u}_0}(\tilde{u}_1)}, \quad \tilde{u}_0 < u < \tilde{u}_1 \tag{62}$$

$$\tilde{\varphi}_2(u) = \frac{\sqrt{\tilde{\gamma}} \left\{ -\frac{\exp(-\tilde{u}_0^2)}{\tilde{u}_0} + \int_{\tilde{u}_0}^u \exp(-\frac{x^2}{2}) dx \right\}}{F_{\tilde{u}_0}(\tilde{u}_1)}, \quad \tilde{u}_0 < u < \tilde{u}_1 \tag{63}$$

and  $\tilde{u}_0, \tilde{u}_1, \tilde{\gamma}$  are the solutions of the following system:

$$\tilde{u}_0 = \sqrt{2\lambda\tilde{q}_0} \tag{64}$$

$$F_{\sqrt{2\lambda\tilde{q}_0}}(\tilde{u}_1) = \frac{-\exp(-\frac{\tilde{u}_1^2}{2})}{\alpha\lambda y_1} \tag{65}$$

$$\sqrt{\tilde{\gamma}} = \frac{\exp(-\frac{\tilde{u}_1^2}{2})}{\alpha\lambda \left( \frac{\exp(-\lambda^2\tilde{q}_0^2)}{\sqrt{2\lambda\tilde{q}_0}} - \int_{\tilde{u}_0}^{\tilde{u}_1} \exp(-\frac{x^2}{2}) dx \right)} \tag{66}$$

Next, we prove that  $\tilde{u}_0 = u_0, \tilde{u}_1 = u_1$  and  $\tilde{\gamma} = \gamma$ , which are the solutions of Equations (46)–(48).

From Equation (64) and the definition of  $\tilde{q}_0$ , we obtain  $\tilde{u}_0 = u_0$ . By Equations (47)–(48), we have that  $u_0$  and  $u_1$  satisfy:

$$y_0 = -\frac{u_1 y_1 \exp(-\frac{u_0^2}{2})}{(1 + \lambda\alpha y_1) u_0 F_{u_0}(u_1)} \tag{67}$$

which is equivalent to:

$$F_{u_0}(u_1) = \frac{-\exp(-\frac{u_1^2}{2})}{\alpha\lambda y_1} \tag{68}$$

Therefore, we have that  $u_1$  is the solution of Equation (65); then  $\tilde{u}_1 = u_1$ .

Finally, we have that Equation (65) is equivalent to:

$$\sqrt{\tilde{\gamma}} = \frac{-\exp(-\frac{u_1^2}{2})\sqrt{\tilde{\gamma}}}{\alpha\lambda y_1 F_{u_0}(u_1)} \tag{69}$$

and by using Equation (68), we obtain  $\tilde{\gamma} = \gamma$ .

Then,  $\tilde{T}(x, t) = T(x, t)$  and  $\tilde{X}(t) = X(t)$  for all  $t > 0$  and  $0 < x < X(t)$ .

## 5. Conclusions

Two one-phase nonlinear, one-dimensional Stefan problems for a semi-infinite material  $x > 0$ , with phase change temperature  $T_f$  have been considered with the assumption of a Storm's condition for the heat capacity and thermal conductivity. In one of them, a heat flux boundary condition of the type  $q(t) = \frac{q_0}{\sqrt{t}}$  has been considered, and in the other problem, a temperature boundary condition  $T = T_s < T_f$  at the fixed face has been studied. The existence and uniqueness of solutions of a similarity type has been obtained in both cases. Furthermore, the procedures to compute the solutions are given. Finally, the equivalence of two problems is proven.

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## Conflicts of Interest

The authors declare no conflict of interest.

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