Differential and Integral Equations Volume 27, Numbers 11-12 (2014) , 1187–1202

TWO STEFAN PROBLEMS FOR A NON-CLASSICAL HEAT EQUATION WITH NONLINEAR THERMAL COEFFICIENTS

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(Submitted by: Reza Aftabizadeh)

Abstract. The mathematical analysis of two one-phase unidimensional and non-classical Stefan problems with nonlinear thermal coefficients is obtained. Two related cases are considered, one of them has a temperature condition on the fixed face $x = 0$ and the other one has a flux condition of the type $-q_0/\sqrt{t}$ $(q_0 > 0)$. In the first case, the source function depends on the heat flux at the fixed face $x = 0$, and in the other case it depends on the temperature at the fixed face $x = 0$. In both cases, we obtain sufficient conditions in order to have the existence of an explicit solution of a similarity type, which is given by using a double fixed point.

1. Introduction

The one-phase Stefan problem for a semi-infinite material is a free boundary problem for the classical heat equation, which requires the determination of the temperature distribution T of the liquid phase (melting problem) or the solid phase (solidification problem) and the evolution of the free boundary $x = s(t)$. Phase change problems appear frequently in industrial processes and other problems of technological interest [1, 11, 12, 17].

The Lamé-Clapeyron-Stefan problem is nonlinear even in its simplest form due to the free boundary conditions. If the thermal coefficients of the material are temperature-dependent, we have a doubly nonlinear free boundary problem.

The present study provides the existence of solutions of the similarity type to two nonlinear one-phase melting problems for non-classical heat equations. First, we consider the following non-classical free boundary problem for a semi-infinite material [4, 7, 8, 11]:

$$\rho(T)c(T)T_t = (k(T)T_x)_x - F(W(t),t) \ , \qquad 0 < x < s(t),\ t > 0 \qquad (1.1)$$

AMS Subject Classifications: 35R35, 80A22, 45G10.
Accepted for publication: July 2014.

$$T(0,t) = T_b \tag{1.2}$$

$$T(s(t),t) = T_m \tag{1.3}$$

$$k\left(T(s(t),t)\right) T_x(s(t),t) = -\rho_0\, l\, \dot{s}\,(t) \tag{1.4}$$

$$s(0) = 0, \tag{1.5}$$

where $T = T(x,t)$ is the temperature of the liquid phase; $\rho(T), c(T)$ and $k(T)$ are the body's density, its specific heat, and its thermal conductivity, respectively; T_m is the phase-change temperature, $T_b > T_m$ is the temperature on the fixed face $x = 0$; $\rho_0 > 0$ is its constant density of mass at the melting temperature; $l > 0$ is the latent heat of fusion by unity of mass, and $s(t)$ is the position of phase change location. We assume that $\rho(T), c(T)$ and $k(T)$ are continuous functions of the temperature and $k(T) \geq k^* > 0$. The control function F depends on the evolution of the heat flux at the boundary $x = 0$ as follows

$$W(t) = T_x(0,t)\ ,\quad F(W(t),t) = F(T_x(0,t),t) = \tfrac{\lambda_0}{\sqrt{t}} T_x(0,t) \tag{1.6}$$

where λ_0 is a given positive constant.

Then, we consider an analogous problem (1.1), (1.3)-(1.5) and the temperature condition (1.2) will be replaced by the following flux condition

$$k(T(0,t))T_x(0,t) = -q_0/\sqrt{t} \tag{1.7}$$

at the fixed face $x = 0$ where q_0 is a positive constant. In this case, the control function F depends on the evolution of the temperature at the boundary $x = 0$ as follows

$$W(t) = T(0,t)\ ,\quad F(W(t),t) = F(T(0,t),t) = \tfrac{\lambda_0}{t} T(0,t),\ (\lambda_0 > 0)\,. \tag{1.8}$$

Here, $-q_0/\sqrt{t}$ denotes the prescribed flux on the boundary $x = 0$, which is of the type imposed in [19]. Furthermore, this kind of heat flux on the fixed boundary was also considered in several applied problems, e.g. [2, 10, 18].

The non-classical heat conduction problem for a semi-infinite material was studied in [3, 9, 13, 16, 20, 22]. A problem of this type is the following:

$$\begin{aligned} T_t - T_{xx} &= -F(W(t),t)\ , \qquad x > 0,\ t > 0 \\ T(0,t) &= f(t),\ t > 0 \quad T(x,0) = h(x),\ x > 0 \end{aligned} \tag{1.9}$$

where functions $f = f(t)$ and $h = h(x)$ are continuous real functions, and F is a given function of two variables. A particular and interesting case is the following:

$$F(W(t),t) = \tfrac{\lambda_0}{\sqrt{t}} W(t),\ (\lambda_0 > 0), \tag{1.10}$$

where $W(t)$ represents the heat flux on the boundary $x = 0$, that is, $W(t) = T_x(0,t)$. Problems of the type (1.9) and (1.10) can be thought of by modelling of a system of temperature regulation in isotropic mediums [20, 22] with nonuniform source term, which provides a cooling or heating effect depending upon the properties of F related to the course of the heat flux (or the temperature in other cases) at the boundary $x = 0$ [20].

In the particular case of a bounded domain, a class of problems, when the heat source is uniform and belongs to a given multivalued function from $\mathbb{R}$ into itself was studied in [16] regarding existence, uniqueness and asymptotic behavior. Moreover, in [3], conditions are given on the nonlinearity of the source term F so as to accelerate the convergence of the solution to the steady-state solution. Other references on the subject are in [13, 14, 15].

Non-classical free boundary problems of the Stefan type were studied in [5, 6] from a theoretical point of view by using an equivalent formulation through a system of second kind Volterra integral equations. In [7], the one-phase unidimensional Stefan problems for non-classical heat equations with constants thermal coefficients and a source function F given by (1.6) or (1.8) were considered. Exact solutions of a similarity type were obtained in all cases.

The problem (1.1)-(1.5) with null source term was firstly considered in [21] where an equivalent integral equation was obtained, however, any mathematical result is given in [21]. In [4], the existence of an explicit solution of a similarity type by using a double fixed point was given.

The plan of the paper is the following: In Section II, we prove the existence of at least one explicit solution of a similarity type for the problem (1.1)-(1.5) and the control function given by (1.6) by using a double fixed point for the integral equation (2.15) and the transcendental equation (2.20) under certain hypothesis for data.

In Section III, we consider the analogous problem (1.1), (1.3)–(1.5), (1.7) and a control function given by (1.8). We prove the existence of at least one explicit solution of a similarly type by using a double fixed point for the integral equation (2.58) and the transcendental equation (2.62) under certain hypothesis for data.

2. The one-phase non-classical Stefan problem with nonlinear thermal coefficients with temperature boundary condition on the fixed face

If we define the following transformation [4, 21]

$$\theta(x,t) = \tfrac{T(x,t)-T_m}{T_b-T_m}, \tag{2.1}$$

then the problem (1.1)-(1.5) becomes

$$N(\theta)\theta_t = \alpha_0(L(\theta)\theta_x)_x - \tfrac{F((T_b-T_m)\theta_x(0,t),t)}{c_0\rho_0(T_b-T_m)},\ 0<x<s(t),\ t>0 \tag{2.2}$$

$$\theta(0,t)=1,\ t>0 \tag{2.3}$$

$$\theta(s(t),t)=0,\ t>0 \tag{2.4}$$

$$k(T_m)\theta_x(s(t),t) = \frac{-\rho_0\, l}{T_b-T_m}\overset{\bullet}{s}(t),\ t>0 \tag{2.5}$$

$$s(0)=0, \tag{2.6}$$

where

$$N(T)=\tfrac{\rho(T)c(T)}{\rho_0 c_0},\qquad L(T)=\tfrac{k(T)}{k_0} \tag{2.7}$$

and k_0,ρ_0,c_0 and $\alpha_0=\frac{k_0}{\rho_0 c_0}$ are the reference thermal conductivity, density of mass, specific heat and thermal diffusive, respectively.

Now we assume a similarity solution of the type

$$\theta(x,t)=f(\eta),\qquad \eta=\tfrac{x}{2\sqrt{\alpha_0 t}}\ . \tag{2.8}$$

The free boundary conditions implies that the free boundary $s(t)$ must be of the type

$$s(t)=2\eta_0\sqrt{\alpha_0 t} \tag{2.9}$$

where η_0 is a positive parameter to be determined later.

Therefore, the conditions (2.2)-(2.5) is reduced to the following problem:

$$\big[L(f)f'(\eta)\big]' + 2\eta N(f)f'(\eta) = Af'(0),\ 0<\eta<\eta_0 \tag{2.10}$$

$$f(0)=1 \tag{2.11}$$

$$f(\eta_0)=0 \tag{2.12}$$

$$f'(\eta_0)=-B\eta_0, \tag{2.13}$$

where

$$A=\tfrac{2\lambda_0}{c_0\rho_0\sqrt{\alpha_0}}\ ,\qquad B=\tfrac{2\alpha_0\rho_0 l}{k(T_m)(T_b-T_m)}. \tag{2.14}$$

We have that the problem 2.10)-(2.12) is equivalent to the following nonlinear integral equation of Volterra type:

$$f(\eta)=1-\frac{\Phi\left[\eta,L(f),N(f)\right]}{\Phi\left[\eta_0,L(f),N(f)\right]}, \tag{2.15}$$

where Φ is given by

$$\Phi\left[\eta,L(f),N(f)\right] := \int_0^\eta \frac{1}{G(f)(t)}\,dt + A\int_0^\eta \frac{w(f)(t)}{G(f)(t)}\,dt, \tag{2.16}$$

and

$$G(f)(x) := \frac{L(f(x))}{L(f(0))} I(f)(x) \quad , \quad I(f)(x) := \exp\left(2\int_0^x s\frac{N(f(s))}{L(f(s))}\,ds\right), \tag{2.17}$$

$$w(f)(x) := \int_0^x \frac{G(f)(t)}{L(f)(t)} dt = \frac{1}{L(f(0))}\int_0^x I(f)(t)dt \tag{2.18}$$

with

$$L(f(0)) = L(T_m(f(0)+1)) = \frac{k\left(T_m(f(0)+1)\right)}{k_0}. \tag{2.19}$$

The condition (2.13) becomes

$$A\int_0^\eta \frac{G(f)(t)}{L(f)(t)}\,dt + 1 = B\eta_0 G(f)(\eta_0)\Phi\left[\eta_0, L(f), N(f)\right]. \tag{2.20}$$

First, in order to prove the existence of the solution of the system (2.15) and (2.20), we will obtain some preliminary results following [4, 21]. Then, we shall prove that the integral equation (2.15) has a unique solution for any given $\eta_0 > 0$ by using a fixed point theorem. Secondly, we shall consider (2.20).

For convenience of notation, we will note $\Phi[\eta, f] \equiv \Phi[\eta, L(f), N(f)]$. We suppose that there exist N_m, N_M, L_m, L_M positive constants such as

$$L_m \le L(T) \le L_M \qquad , \qquad N_m \le N(T) \le N_M. \tag{2.21}$$

Furthermore, we assume that the dimensionless thermal conductivity and specific heat are Lipschitz functions, i.e., there exist positive constants $\widetilde{L}$ and $\widetilde{N}$ such that

$$|L(g) - L(h)| \le \widetilde{L}\,\|g-h\| \quad , \quad \forall g,h \in C^0\left(R_0^+\right)\cap L^\infty\left(R_0^+\right) \tag{2.22}$$

$$|N(g) - N(h)| \le \widetilde{N}\,\|g-h\| \quad , \quad \forall g,h \in C^0\left(R_0^+\right)\cap L^\infty\left(R_0^+\right). \tag{2.23}$$

Then, we get:

Lemma 2.1. *For $0 < \eta < \eta_0$, we have*

$$\exp\left(\frac{N_m\eta^2}{L_M}\right) \le I(f)(\eta) \le \exp\left(\frac{N_M\eta^2}{L_m}\right) \tag{2.24}$$

$$\frac{L_m}{L_M} \le G(f)(\eta) \le \frac{L_M}{L_m}\exp\left(\frac{N_M}{L_m}\eta_0^2\right) \tag{2.25}$$

$$\eta_0\frac{L_m}{L_M}\exp\left(-\frac{N_M}{L_m}\eta_0^2\right) \le \int_0^\eta \frac{1}{G(f)(t)}\,dt \le \frac{L_M}{L_m}\eta_0 \tag{2.26}$$

$$\frac{\eta_0}{L_M} \le w(f)(\eta) \le \frac{\eta_0^2}{2L_m}\exp\left(\frac{N_M}{L_m}\eta_0^2\right) \tag{2.27}$$

$$\int_0^\eta \frac{w(f)(t)}{G(f)(t)}\,dt \le \frac{\eta_0^3 L_M}{2L_m^2}\exp\Big(\frac{N_M}{L_m}\eta_0^2\Big) \tag{2.28}$$

$$\frac{L_m}{L_M}\eta_0\exp\Big(-\frac{N_M}{L_m}\eta_0^2\Big) \le \Phi[\eta,f] \le \frac{L_M}{L_m}\eta_0 + A\frac{\eta_0^3 L_M}{2L_m^2}\exp\Big(\frac{N_M}{L_m}\eta_0^2\Big). \tag{2.29}$$

We consider $C^0[0,\eta_0]$, the space of continuous real functions defined on $[0,\eta_0]$ with its norm $\|f\| = \max_{\eta\in[0,\eta_0]}|f(\eta)|$.

Lemma 2.2. *Let η_0 be a given positive real number. For all $f,\ f^*\in C^0[0,\eta_0]$, $\forall\eta\in(0,\eta_0)$, we have*

$$|I(f)(\eta) - I(f^*)(\eta)| \le C_1\|f^*-f\| \tag{2.30}$$

$$|L(f(\eta))L(f^*(0)) - L(f^*(\eta))L(f(0))| \le C_2\|f^*-f\| \tag{2.31}$$

$$|G(f)(\eta) - G(f^*)(\eta)| \le C_3\|f^*-f\| \tag{2.32}$$

$$\Big|\int_0^\eta \Big(\frac{1}{G(f)(t)} - \frac{1}{G(f^*)(t)}\Big)\,dt\Big| \le \eta_0 C_4\|f^*-f\| \tag{2.33}$$

$$|w(f)(\eta) - w(f^*)(\eta)| \le C_5\|f^*-f\| \tag{2.34}$$

$$\Big|\int_0^\eta \Big(\frac{w\big(f\big)(t)}{G(f)(t)} - \frac{w(f^*)(t)}{G(f^*)(t)}\Big)\,dt\Big| \le \eta_0 C_6\|f^*-f\| \tag{2.35}$$

$$|\Phi[\eta,f] - \Phi[\eta,f^*]| \le \eta_0 C_7\|f^*-f\|, \tag{2.36}$$

where

$$C_1 = \exp\Big(\tfrac{N_M}{L_m}\eta_0^2\Big)\frac{\eta_0^2}{L_m^2}\Big(\widetilde{N}L_M + N_M\widetilde{L}\Big),\quad C_2 = 2L_M\widetilde{L}$$

$$C_3 = \frac{L_M^2 C_1 + C_2\exp(\frac{N_M\eta_0^2}{L_m})}{L_m^2},\ C_4 = \frac{L_M^2}{L_m^2}C_3$$

$$C_5 = \eta_0\frac{L_M}{L_m^2}\Big[\frac{\widetilde{L}}{L_m}\exp\big(\tfrac{N_M}{L_m}\eta_0^2\big) + C_3\Big]$$

$$C_6 = \exp\Big(\tfrac{N_M}{L_m}\eta_0^2\Big)\tfrac{L_M^2}{L_m^3}\Big[\frac{\eta_0^2}{2}C_3 + L_M C_5\Big],\ C_7 = C_4 + AC_6.$$

Proof. By using the mean value theorem and (2.21), (2.22), (2.23), we have

$$|I(f)(\eta) - I(f^*)(\eta)| = \Big|\exp\Big(2\int_0^\eta u\frac{N(f(u))}{L(f(u))}du\Big) - \exp\Big(2\int_0^\eta u\frac{N(f^*(u))}{L(f^*(u))}du\Big)\Big|$$

$$\le \exp(\tfrac{N_M}{L_m}\eta_0^2)\int_0^\eta 2u\Big|\frac{N(f(u))}{L(f(u))} - \frac{N(f^*(u))}{L(f^*(u))}\Big|\,du$$

$$\le \exp(\tfrac{N_M}{L_m}\eta_0^2)\int_0^\eta \frac{2u}{L(f(u))\,L(f^*(u)}|L(f^*(u))||N(f(u)) - N(f^*(u))|$$

$$+ |N(f^*(u))||L(f(u)) - L(f^*(u))|\,du$$

$$\leq \exp\left(\tfrac{N_M}{L_m}\eta_0^2\right)\frac{\eta_0^2}{L_m^2}\left(\widetilde{N}L_M + N_M\widetilde{L}\right)\|f^* - f\| = C_1\|f^* - f\|.$$

Taking into account (2.21), (2.22), (2.23), it is easy to see (2.31).

From (2.17), we have

$$\begin{aligned}
&|G(f)(\eta) - G(f^*)(\eta)| \\
&= \frac{|L(f(u))L(f^*(0))I(f)(\eta) - L(f^*(u))L(f(0))I(f^*)(\eta)|}{L(f^*(0))L(f(0))} \\
&\leq \frac{1}{L(f^*(0))L(f(0))}\Big[L(f(u))L(f^*(0))\,|I(f)(\eta) - I(f^*)(\eta)| \\
&\qquad + |L(f(\eta))L(f^*(0)) - L(f^*(\eta))L(f(0))|\,I(f^*)(\eta) \\
&\leq \frac{L_M^2 C_1\|f^* - f\| + C_2\|f^* - f\|\exp\left(\frac{N_M x^2}{L_m}\right)}{L_m^2} \\
&\qquad \times \frac{L_M^2 C_1 + C_2\exp(\frac{N_M x^2}{L_m})}{L_m^2}\|f^* - f\| = C_3\|f^* - f\|.
\end{aligned}$$

From the above inequality and Lemma 1, we have

$$\begin{aligned}
\Big|\int_0^\eta \Big(\frac{1}{G(f)(t)} - \frac{1}{G(f^*)(t)}\Big)\,dt\Big| &\leq \int_0^\eta \frac{|G(f)(t) - G(f^*)(t)|}{G(f)(t)\;G(f^*)(t)}\,dt \\
&\leq \frac{L_M^2}{L_m^2}\eta_0 C_3\|f^* - f\|.
\end{aligned}$$

To prove (2.34), we write

$$\begin{aligned}
&|w(f)(\eta) - w(f^*)(\eta)| \leq \int_0^\eta \Big|\frac{G(f)(t)}{L(f(t))} - \frac{G(f^*)(t)}{L(f^*(t))}\Big|\,dt \\
&\leq \int_0^\eta \frac{G(f)(t)\,|L(f(\eta)) - L(f^*(\eta))| + L(f(t))\,|G(f)(t) - G(f^*)(t)|}{L(f(t))\;L(f^*(t))}\,dt \\
&\leq \frac{\eta_0}{L_m^2}\Big[\widetilde{L}\|f - f^*\|\frac{L_M}{L_m}\exp\left(t\frac{N_M}{L_m}\eta_0^2\right) + L_M C_3\|f^* - f\|\Big] = C_5\|f^* - f\|.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
&\Big|\int_0^\eta \Big(\frac{w(f)(t)}{G(f)(t)} - \frac{w(f^*)(t)}{G(f^*)(t)}\Big)\,dt\Big| \\
&\leq \int_0^\eta \frac{w(f)(t)\,|G(f)(t) - G(f^*)(t)| + G(f(t))\,|w(f)(t) - w(f^*)(t)|}{G(f(t))\;G(f^*(t))}dt
\end{aligned}$$

$$\leq \frac{\eta_0 L_M^2}{L_m^2}\left[\frac{\eta_0^2}{2L_m}\exp\left(\tfrac{N_M}{L_m}\eta_0^2\right)C_3\left\|f^*-f\right\| + \tfrac{L_M}{L_m}\exp\left(\tfrac{N_M}{L_m}\eta_0^2\right)C_5\left\|f^*-f\right\|\right]$$

$$= \frac{\eta_0 L_M^2}{L_m^3}\exp\left(\tfrac{N_M}{L_m}\eta_0^2\right)\left[\tfrac{\eta_0^2}{2}C_3 + L_M C_5\right]\left\|f^*-f\right\| = \eta_0 C_6\left\|f^*-f\right\|.$$

Finally, taking into account (2.16), (2.33) and (2.35), it is easy to see that (2.36) holds. □

Theorem 2.1. *Let η_0 be a given positive real number. We suppose that* (2.21), (2.22), *and* (2.23) *hold. If η_0 satisfies the inequality*

$$\beta(\eta_0) := \tfrac{2L_M^3}{L_m^3}\exp^2\left(\tfrac{N_M}{L_m}\eta_0^2\right)\left[1 + A\tfrac{\eta_0^2}{2L_m}\exp\left(\tfrac{N_M}{L_m}\eta_0^2\right)\right]C_7 < 1, \qquad (2.37)$$

then there exists a unique solution $f \in C^0[0,\eta_0]$ of the integral equation (2.15).

Proof. Let $W : C^0[0,\eta_0] \longrightarrow C^0[0,\eta_0]$ be the operator defined by

$$W(f)_{(\eta)} = 1 - \frac{\Phi\left[\eta, f\right]}{\Phi\left[\eta_0, f\right]}\ , \qquad f \in C^0[0,\eta_0]. \qquad (2.38)$$

The solution of the equation (2.15) is the fixed point of the operator W, that is,

$$W(f(\eta)) = f(\eta) \quad , \quad 0 < \eta < \eta_0. \qquad (2.39)$$

We note that the nonlinear operator W is, in fact, self mapping on $C^0\left[0,\eta_0\right]$ by the assumptions on the thermal coefficients.

Let $f, f^* \in C^0[0,\eta_0]$, then we obtain

$$\left\|W(f) - W(f^*)\right\| = \underset{\eta\in[0,\eta_0]}{Max}\left|W(f(\eta)) - W(f^*(\eta))\right|$$

$$\underset{\eta\in[0,\eta_0]}{Max} \leq \left|\frac{\Phi\left[\eta, f^*\right]\Phi\left[\eta_0, f\right] - \Phi\left[\eta_0, f^*\right]\Phi\left[\eta, f\right]}{\Phi\left[\eta_0, f\right]\Phi\left[\eta_0, f^*\right]}\right|$$

$$\leq R\underset{\eta\in[0,\eta_0]}{Max}\left|\Phi\left[\eta, f^*\right]\Phi\left[\eta_0, f\right] - \Phi\left[\eta_0, f^*\right]\Phi\left[\eta, f\right]\right|$$

$$\leq R\underset{\eta\in[0,\eta_0]}{Max}\left(|\Phi[\eta, f^*]||\Phi[\eta_0, f] - \Phi[\eta_0, f^*]| + |\Phi[\eta_0, f^*]||\Phi[\eta, f^*] - \Phi[\eta, f]|\right),$$

where

$$R = \tfrac{L_M^2}{L_m^2\eta_0^2}\exp^2\left(\tfrac{N_M}{L_m}\eta_0^2\right) > 0. \qquad (2.40)$$

Finally, from Lemmas 1 and 2 and taking into account that $0 < \eta < \eta_0$, we have

$$\left\|W(f) - W(f^*)\right\| \leq \beta\left(\eta_0\right)\left\|f^* - f\right\|.$$

Then, W is a contraction operator; therefore, there exists a unique solution of (2.15) if the condition (2.37) is satisfied. □

Remark 2.1. The solution f of the integral equation (2.15), given by the Theorem 3, depends on the real number $\eta_0 > 0$. For convenience in the notation from now on, we take

$$f(\eta) = f_{\eta_0}(\eta) = f(\eta_0, \eta) \ , \qquad 0 < \eta < \eta_0 \ , \qquad \eta_0 > 0. \tag{2.41}$$

Let Ω be the set defined by

$$\Omega = \{\eta_0 \in R^+ / \beta(\eta_0) < 1\} = \{\eta_0 \in R^+ / \text{there exists a solution of (2.15)}\}.$$

Lemma 2.3. *If*

$$4L_M^5 \widetilde{L} / L_m^7 < 1, \tag{2.42}$$

then there exists a positive number η_0^* such that

$$\beta(\eta_0) < 1 \ \text{ if } \ 0 < \eta_0 < \eta_0^* \quad , \ \beta(\eta_0) \geq 1 \ \text{ if } \ \eta_0 \geq \eta_0^*.$$

Proof. We have $\beta(0) = 4L_M^5 \widetilde{L} / L_m^7$, $\beta(+\infty) = +\infty$ and $\beta'(\eta_0) > 0 \ \forall \eta_0 > 0$. Then, $\Omega = (0, \eta_0^*)$ where $\beta(\eta_0^*) = 1$. □

To prove the existence of the solution of (2.20), we rewrite it as follows

$$\frac{1}{G(f)(x)} - Bx \int_0^x \frac{1}{G(f)(t)} \, dt = ABx \int_0^x \frac{w(f)(t)}{G(f)(t)} dt - \frac{A}{G(f)(x)} \int_0^x \frac{G(f)(t)}{L(f)(t)} \, dt \tag{2.43}$$

where f is the solution of (2.15) given by Theorem 3.

We define the functions

$$W_1(x) := \frac{1}{G(f)(x)} - Bx \int_0^x \frac{1}{G(f)(t)} \, dt, \tag{2.44}$$

$$W_2(x) := ABx \int_0^x \frac{w\,(f)\,(t)}{G(f)(t)} dt - \frac{A}{G(f)(x)} \int_0^x \frac{G(f)(t)}{L(f)(t)} \, dt. \tag{2.45}$$

Thus, the equation (2.43) is equivalent to

$$W_1(x) = W_2(x) \tag{2.46}$$

Lemma 2.4. *The functions W_1 and W_2 satisfy the following properties*
(i) $W_1(0) = 1$, *(ii)* $W_1(+\infty) = -\infty$
(iii) $W_2(0) = 0$, *(iv)* $W_2(+\infty) = +\infty$.

Lemma 2.5. *If* (2.21) *holds, then*

$$W_1(x) \leq W_3(x) \tag{2.47}$$

where

$$W_3(x) := \frac{1}{I(f)(x)} \left(\tfrac{L_M}{L_m} - \tfrac{L_m}{L_M} Bx^2 \right)$$

Proof. From (2.26), we have

$$\int_0^x \frac{1}{G(f)(t)}\,dt \geq \frac{L_m}{L_M}\frac{x}{I(f)(x)}$$

and taking into account (2.25), we have that (2.47) holds.

Theorem 2.2. *If* (2.42) *holds, then* (2.20) *has at least one solution* $\eta_0 < \frac{L_M}{L_m\sqrt{B}}$. *Moreover, if* $\beta(\frac{L_M}{L_m\sqrt{B}}) < 1$, *then* $\eta_0 \in \Omega$.

Proof. By Lemma 5, we have that there exists at least one solution η_0 of (2.46), which verifies $\eta_0 < x_0$ with $W_1(x_0) = 0$.

Taking into account Lemma 6, we have that $x_0 < x_1 = \frac{L_M}{L_m\sqrt{B}}$ where x_1 is the only positive root of $W_3(x)$. Then, if $\beta(x_1) < 1$, we have $\beta(\eta_0) < 1$ and $\eta_0 \in \Omega$. □

Thus, we have the following Theorem:

Theorem 2.3. *If* N *and* L *verify the conditions* (2.21), (2.22), (2.23), (2.42) *and* $\beta(\frac{L_M}{L_m\sqrt{B}}) < 1$, *then there exists at least one solution of the problem* (1.1)-(1.5) *where the free boundary* $s(t)$ *is given by* (2.9) *and the temperature is given by* $T(x,t) = T_m + (T_b - T_m)f(\eta)$, *with* $\eta = x/2\sqrt{\alpha_0 t}$ *where* f *is the unique solution of the integral equation* (2.15) *and* η_0 *is given by Theorem* 7.

III. Solution of the non-classical free boundary problem with a heat flux condition on the fixed face.

In this section, we consider the problem (1.1)-(1.5), but condition (1.2) will be replaced by condition (1.7) and the source term is given by (1.8). If we define the following transformation

$$\theta(x,t) = \frac{T(x,t) - T_m}{T_m} \qquad (T(x,t) = T_m + T_m\theta(x,t))\,, \tag{2.48}$$

then the problem to solve becomes

$$N(\theta)\theta_t = \alpha_0\left(L(\theta)\theta_x\right)_x - \frac{\lambda_0}{\rho_0 c_0 t}(\theta(0,t)+1)\ \ ,\ \ 0 < x < s(t) \tag{2.49}$$

$$k\left(T_m(\theta(0,t)+1)\right)\theta_x(0,t) = -\frac{q_0}{T_m\sqrt{t}} \tag{2.50}$$

$$\theta(s(t),t) = 0 \tag{2.51}$$

$$k(T_m)\theta_x(s(t),t) = \frac{-\rho_0\, l s'(t)}{T_m} \tag{2.52}$$

$$s(0) = 0. \tag{2.53}$$

Now, we assume a similarity type solution given by (2.8). Then, the free boundary conditions implies that the free boundary $s(t)$ must be of the type (2.9) where η_0 is a positive parameter to be determined later.

Therefore, the conditions (2.49)-(2.53) reduces to the following problem:

$$\big[L(f)f'(\eta)\big]' + 2\eta N(f)f'(\eta) = \frac{4}{\rho_0 c_0}\lambda_0(f(0)+1) \quad , \quad 0<\eta<\eta_0 \tag{2.54}$$

$$L(f(0))f'(0) = -q_0^* \tag{2.55}$$

$$f(\eta_0) = 0 \tag{2.56}$$

$$f'(\eta_0) = -M\eta_0, \tag{2.57}$$

where

$$M = \frac{2\alpha_0\rho_0 l}{k(T_m)T_m} \quad , \quad q_0^* = \frac{2\sqrt{\alpha_0}q_0}{k_0 T_m}$$

and $L(f(0))$ is given by (2.19).

We have that the problem (2.54)-(2.56) is equivalent to the following nonlinear integral equation of Volterra type:

$$f(\eta) = \chi(\eta_0, f)\Big(1 + Q\int_0^\eta \tfrac{w(f)(t)}{G(f)(t)}dt\Big) - \Psi(\eta, f), \ \eta > \eta_0 \tag{2.58}$$

where

$$Q = \frac{4\lambda_0}{\rho_0 c_0}, \tag{2.59}$$

$$\chi(\eta_0, f) = \frac{q_0^* - M\eta_0 L(f(0))\, G(f)(\eta_0)}{QL(f(0))\, w\,(f)\,(\eta_0)} \tag{2.60}$$

and

$$\Psi(\eta, f) = 1 + \frac{q_0^*}{L(f(0))}\int_0^\eta \frac{1}{G(f)(t)}\,dt, \tag{2.61}$$

the functions $G(f)$, $w(f)$ are defined in (2.17) and (2.18).

The condition (2.57) becomes

$$\begin{aligned} &Q\,w(f)(\eta_0)\Big(L(f(0)) + q_0^*\int_0^{\eta_0}\frac{1}{G(f)(t)}\,dt\Big) \\ &= \ (q_0^* - M\eta_0 L(f(0))\,G(f)(\eta_0))\Big(1 + Q\int_0^{\eta_0}\tfrac{w(f)(t)}{G(f)(t)}dt\Big). \end{aligned} \tag{2.62}$$

Similarly, as done in Section II, we will obtain some preliminary results to prove the existence of the solution of the system (2.58) and (2.62).

Lemma 2.6. *Let η_0 be a given positive real number. We suppose that the dimensionless thermal conductivity and specific heat verify conditions* (2.21), (2.22) *and* (2.23). *Then, for all $f,\ f^* \in C^0[0,\eta_0]$, $\forall \eta \in (0,\eta_0)$, we have*

$$|\chi(\eta_0,f)-\chi(\eta_0,f^*)| \le C_9 \|f^*-f\| \tag{2.63}$$

$$|\chi(\eta_0,f^*)| \le C_{10} \tag{2.64}$$

$$|\Psi(\eta,f)-\Psi(\eta,f^*)| \le C_{11} \|f^*-f\| \tag{2.65}$$

where

$$C_8 = \frac{L_M}{L_m^2}\left[\frac{\widetilde{L}}{L_m}\exp\left(\tfrac{N_M}{L_m}\eta_0^2\right)+C_3\right]$$

$$C_9 = \frac{L_M^2}{L_m^2}\exp\left(\tfrac{N_M}{L_m}\eta_0^2\right)\left\{\frac{q_0^*\eta_0}{L_m^2}\left(\widetilde{N}L_M+N_M\widetilde{L}\right)+M\frac{L_M^2}{L_m}\left(\frac{\eta_0}{2}C_3+L_MC_8\right)\right\}$$

$$C_{10} = \frac{L_M}{QL_m^2\eta_0}\left(q_0^*L_m+M\eta_0L_M^2\exp\left(\tfrac{N_M}{L_m}\eta_0^2\right)\right)$$

$$C_{11} = \frac{q_0^*\eta_0L_M^2}{L_m}\left\{\frac{1}{L_m^4}\exp\left(\tfrac{N_M}{L_m}\eta_0^2\right)\left[\frac{\eta_0^2L_M}{L_m^2}\left(\widetilde{N}L_M+N_M\widetilde{L}\right)+2\widetilde{L}\right]+2\widetilde{L}\right\}.$$

Proof. Taking into account previous lemmas, we have

$$\begin{aligned}
&|\chi(\eta_0,f)-\chi(\eta_0,f^*)| \le \frac{q_0^*|L(f^*(0))w(f^*)(\eta_0)-L(f(0))w(f)(\eta_0)|}{L(f(0))w\,(f)\,(\eta_0)L(f^*(0))w\,(f^*)\,(\eta_0)}\\
&+\frac{M\eta_0|L(f^*(0))w(f^*)(\eta_0)L(f(0))G(f)(\eta_0)-L(f(0))w(f)(\eta_0)L(f^*(0))G(f^*)(\eta_0)|}{L(f(0))w(f)(\eta_0)L(f^*(0))w(f^*)(\eta_0)}\\
&\le \tfrac{q_0^*L_M^2}{\eta_0^2L_m^2}\left|L(f^*(0))w\,(f^*)\,(\eta_0)-L(f(0))w\,(f)\,(\eta_0)\right|\\
&+\tfrac{ML_M^4}{\eta_0L_m^2}\left|w\,(f^*)\,(\eta_0)G(f)(\eta_0)-w\,(f)\,(\eta_0)G(f^*)(\eta_0)\right|\\
&\le \tfrac{q_0^*L_M^2}{\eta_0^2L_m^2}\int_0^{\eta_0}|I\,(f)\,(t)-I(f^*)(t)|\,dt\\
&+\tfrac{ML_M^4}{\eta_0L_m^2}\left[|w\,(f^*)\,(\eta_0)|\,|G(f)(\eta_0)-G(f^*)(\eta_0)|+G(f)(\eta_0)\,|w\,(f^*)\,(\eta_0)-w\,(f)\,(\eta_0)|\right]\\
&\le\left\{\tfrac{q_0^*L_M^2}{\eta_0L_m^2}C_1+\tfrac{ML_M^4}{\eta_0L_m^2}\left[\tfrac{\eta_0^2}{2L_m}\exp\left(\tfrac{N_M}{L_m}\eta_0^2\right)C_3+\tfrac{L_M}{L_m}\exp\left(\tfrac{N_M}{L_m}\eta_0^2\right)C_5\right]\right\}\|f^*-f\|\\
&=C_9\|f^*-f\|.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
|\chi(\eta_0,f^*)| &\le \left|\frac{q_0^*-M\eta_0L(f^*(0))\,G(f^*)(\eta_0)}{QL(f^*(0))\,w\,(f^*)\,(\eta_0)}\right|\\
&\le L_M\frac{q_0^*+M\eta_0L_M^2/L_m\exp(N_M\eta_0^2/L_m)}{QL_m\eta_0}=C_{10}.
\end{aligned}$$

Finally, by (2.16), we have

$$|\Psi(\eta,f)-\Psi(\eta,f^*)| = \Big|\frac{q_0^*}{L(f(0))}\int_0^\eta \frac{1}{G(f)(t)}\,dt - \frac{q_0^*}{L(f^*(0))}\int_0^\eta \frac{1}{G(f^*)(t)}\,dt\Big|$$

$$\le \frac{q_0^*}{L(f(0))}\int_0^\eta \Big|\frac{1}{G(f)(t)} - \frac{1}{G(f^*)(t)}\Big|dt + q_0^*\Big|\frac{1}{L(f(0))} - \frac{1}{L(f^*(0))}\Big|\int_0^\eta \frac{1}{G(f^*)(t)}\,dt$$

$$= \frac{q_0^*}{L(f(0))}\int_0^\eta \Big|\frac{G(f)(t)-G(f^*)(t)}{G(f)(t)G(f^*)(t)}\Big|dt + q_0^*\Big|\frac{L(f^*(0))-L(f(0)}{L(f(0))L(f^*(0))}\Big|\int_0^\eta \frac{1}{G(f^*)(t)}\,dt.$$

Taking into account (2.22), (2.26) and (2.33), we have

$$|\Psi(\eta,f)-\Psi(\eta,f^*)| \le \left\{\tfrac{q_0^*}{L_m}\eta_0 C_4 + \frac{q_0^*\widetilde{L}}{L_m^2}\frac{L_M}{L_m}\eta_0\right\}\|f^*-f\| = C_{11}\|f^*-f\|.$$

Theorem 2.4. *Let η_0 be a given positive real number. We suppose that* (2.21), (2.22), *and* (2.23) *hold. If η_0 satisfies the inequality*

$$\varepsilon(\eta_0) := C_9\left(1+Q\tfrac{\eta_0^3 L_M}{2L_m^2}\exp\left(\tfrac{N_M}{L_m}\eta_0^2\right)\right) + C_{10}Q\eta_0 C_6 + C_{11} < 1, \qquad (2.66)$$

then there exists a unique solution of the integral equation (2.58).

Proof. Let $U : C^0[0,\eta_0] \longrightarrow C^0[0,\eta_0]$ be the operator defined by

$$U(f)_{(\eta)} = \chi(\eta_0,f)\Big(1+Q\int_0^\eta \tfrac{w(f)(t)}{G(f)(t)}dt\Big) - \Psi(\eta,f), \qquad (2.67)$$

$f \in C^0[0,\eta_0]$, $0<\eta<\eta_0$. The solution of the equation (2.58) is the fixed point of the operator U, that is,

$$U(f(\eta)) = f(\eta) \quad , \quad 0<\eta<\eta_0 \qquad (2.68)$$

We note that the nonlinear operator W is, in fact, self mapping on $C^0[0,\eta_0]$ by the assumptions on the thermal coefficients.

Let $f, f^* \in C^0[0,\eta_0]$. Then, we obtain

$$|U(f)-U(f^*)| \le \Big|\chi(\eta_0,f)\Big(1+Q\int_0^\eta \tfrac{w(f)(t)}{G(f)(t)}dt\Big) - \chi(\eta_0,f^*)\Big(1+Q\int_0^\eta \frac{w(f^*)(t)}{G(f^*)(t)}dt\Big)\Big| + |\Psi(\eta,f)-\Psi(\eta,f^*)|$$

$$\le |\chi(\eta_0,f)-\chi(\eta_0,f^*)|\left(1+Q\int_0^\eta \frac{w\left(f\right)(t)}{G(f)(t)}dt\right)$$

$$+ |\chi(\eta_0, f^*)| Q \int_0^\eta \left| \frac{w(f)(t)}{G(f)(t)} - \frac{w(f^*)(t)}{G(f^*)(t)} \right| dt + |\Psi(\eta, f) - \Psi(\eta, f^*)| \,.$$

Then, taking into account Lemmas 1, 3, and 9, we obtain

$$\begin{aligned} & |U(f) - U(f^*)| \\ & \leq \left\{ C_9 \left(1 + Q \tfrac{\eta_0^3 L_M}{2 L_m^2} \exp\left(\tfrac{N_M}{L_m} \eta_0^2 \right) \right) + C_{10} Q \eta_0 C_6 + C_{11} \right\} \| f^* - f \| \end{aligned}$$

Finally, we have

$$\| U(f) - U(f^*) \| \leq \varepsilon(\eta_0) \| f^* - f \| \,.$$

Then, there exists a unique solution of the integral Eq.(2.58) if condition (2.66) is verified. (i.e., U is a contraction operator). □

Let Σ be the set defined by

$$\Sigma = \{ \eta_0 \in R^+ / \varepsilon(\eta_0) < 1 \} = \{ \eta_0 \in R^+ / \text{there exists a solution of (2.58)} \}.$$

Lemma 2.7. *Function $\varepsilon = \varepsilon(\eta)$ given by* (2.66)*, satisfies the following properties:*
(i) $\varepsilon(0) = \frac{L_M^6}{L_m^6} M \widetilde{L} (1 + 2 \frac{L_M}{L_m})$, *(ii)* $\varepsilon(+\infty) = +\infty$,
(iii) ε is an increasing function
(iv) If

$$\frac{L_M^6}{L_m^6} M \widetilde{L} \left(1 + 2 \frac{L_M}{L_m} \right) < 1, \tag{2.69}$$

then there exists $\widetilde{\eta} > 0$ such that $\varepsilon(\eta) < 1$ for all $\eta \in (0, \widetilde{\eta})$.

Next, we prove that the equation (2.62) has a unique solution. For this, we define for $x \in \Sigma$ the following functions:

$$V_1(x) := (q_0^* - M x L(f(0)) \, G(f)(x)) \left(1 + Q \int_0^x \frac{w(f)(t)}{G(f)(t)} dt \right), \tag{2.70}$$

and

$$V_2(x) := Q \, w\,(f)\,(x) \Big(L(f(0)) + q_0^* \int_0^x \frac{1}{G(f)(t)} \, dt \Big). \tag{2.71}$$

We have:

Lemma 2.8. *The functions V_1 and V_2 satisfy the following properties:*
(i) $V_1(0) = q_0^*$, $V_1(+\infty) = -\infty$,
(ii) $V_2(0) = 0$, $V_2(+\infty) = +\infty$ *and* $V_2(x) \geq 0$ *for all* $x > 0$.

Theorem 2.5. *If* (2.69) *holds, then* (2.62) *has at least one solution η_0. Moreover, if $\varepsilon(\frac{q_0^*}{M L_m}) < 1$, then $\varepsilon(\eta_0) < 1$.*

Proof. By the above Lemma, there exists at least one solution η_0 of (2.62) and it is satisfied $V_1(\eta_0) = V_2(\eta_0) > 0$. Let $x_0 = \min\{x > 0/V_1(x) = 0\} = \min\{x > 0/q_0^* - MxL(f(0))\ G(f)(x) = 0\} = \min\{x > 0/q_0^* = MxL(f(x))I(f)(x)\}$.

By properties of $L(f)$ and $I(f)$, we have $\eta_0 < x_0 \leq \frac{q_0^*}{ML_m}$. Then, if $\varepsilon(\frac{q_0^*}{ML_m}) < 1$, we have $\varepsilon(\eta_0) < 1$. □

Theorem 2.6. *If N and L verify the conditions* (2.21), (2.22), (2.23), (2.69) *and* $\varepsilon(\frac{q_0^*}{ML_m}) < 1$*, then the non-classical free boundary problem* (1.1), (1.3)–(1.5), (1.7), *and* (1.8) *has a unique solution given by* (2.9) *and* $T(x,t) = T_m + T_m f(\eta), \eta = x \big/ \left(2\sqrt{\alpha_0 t}\right)$ *where the function* $f = f(\eta)$ *is the unique solution of* (2.58) *and the coefficient* $\eta_0 > 0$ *is given by Theorem* 13.

Remark 2.2. In this paper, we have generalized the non-classical Stefan problems raised in [7] with the constant thermal coefficients and a source term given by (1.6) or (1.8). Moreover, if we consider null source term in the nonlinear Stefan problem (1.1)–(1.5) and in the nonlinear Stefan problem (1.1), (1.3)–(1.5), (1.7), we obtain the same solution given by [4].

Acknowledgments. This paper has been partially sponsored by the projects PIP No. 112-200801-00460 from CONICET-UA, Rosario (Argentina), ANPCYT PICTO 2008-00073 from Agencia (Argentina) and “Fondo de ayuda a la investigación” from Universidad Austral (Argentina).

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