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On a non-linear moving boundary problem for a diffusion–convection equation

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ABSTRACT

We study a one-dimensional free boundary problem for a non-linear diffusion–convection equation whose diffusivity is heterogeneous in space as well as being non-linear. Under the Bäcklund transformation the problem is reduced to an associated free boundary problem. We prove the existence and uniqueness, local in time, of the solution by using the Friedman Rubinstein integral representation method and the Banach contraction theorem.

1. Introduction

We study a non-linear free boundary problem for a semi-infinite region $x>0$ with a Dirichlet boundary condition at the fixed $x=0$ given by condition (2). It is required to determine the evolution of the moving phase separation $x=s(t)$ and the distribution $\theta(x,t)$. The modeling of this kind of systems is a problem with a great mathematical and industrial significance. Phase-change problems appear frequently in industrial processes and other problems of technological interest [1–10].

Owing to [13] we consider the following a free boundary problem with non-linear diffusion equation and a convective term

$$\frac{\partial\theta}{\partial t}=\frac{\partial}{\partial x}\left(D(\theta,x)\frac{\partial\theta}{\partial x}\right)-v(\theta)\frac{\partial\theta}{\partial x},\quad 0<x<s(t),\ t>0 \tag{1}$$

$$\theta(0,t)=g(t)\geq 0,\quad t>0 \tag{2}$$

$$D(\theta(s(t),t),s(t))\frac{\partial\theta}{\partial x}(s(t),t)=-\alpha\dot{s}(t),\quad t>0 \tag{3}$$

$$\theta(s(t),t)=0,\quad t>0,\ s(0)=B \tag{4}$$

$$\theta(x,0)=f(x)\geq 0,\quad 0\leq x\leq B \tag{5}$$

where the velocity $v(\theta)$ and the medium diffusivity $D(\theta,x)$ are given by

$$v(\theta)=\frac{d}{2(a+b\theta)^2},\quad D(\theta,x)=\frac{1+dx}{(a+b\theta)^2} \tag{6}$$

with positive parameters a,b,d and α. This kind of non-linear conductivity or diffusion coefficients was considered in numerous papers, e.g. [11–21]. In [18] it was founded that the non-linear diffusion equation can be transformed to the linear form that possesses a well-known analytical solution when the soil water diffusivity is given by $D(\theta)=a(b-\theta)^{-2}$, where a and b are constant. Moreover, they argued that the diffusivity functional form is plausible for soil water diffusivities. The non-linear transport equation (1) arises in connection with unsaturated flow in heterogeneous porous media. If we set $d=0$ and $b=0$ in the free boundary problem (1)–(6) then we retrieve the classical one-phase Lamé–Clapeyron–Stefan problem.

We follow [13] where it was studied a analogous problem but the boundary conditions allowed to use the similarity method.

The goal of this paper is to prove the local existence and uniqueness in time of the solution to the problem given by (1)–(6). First, under the Bäcklund transformation, we reduce the problem to an associated free boundary problem and we prove that the problem is equivalent to solve a system of Volterra integral equations (38) and (39) [22,23] following Friedman–Rubinstein's method given in [24,25]. Then we prove that the system of equations (38)–(39) has a unique local solution by using the Banach contraction theorem.

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2. Statement of the problem

We consider the free boundary problem (1)–(5). We will make the following assumptions on the initial and boundary data:

(i) Let $a,b,d,\alpha \in \mathbb{R}^+$ with $\alpha < a/b$.
(ii) Let $f \in C^1[0,B]$ be a non-negative function with $f'(0) > 0$.
(iii) Let $g \in C^1[0,1]$ be a non-negative function and $D = \max_{t\in[0,1]} g(t)$.
(iv) Compatibility conditions: $f(0)=g(0)$ and $f(B)=0$.

Taking into account (6) we can put our problem as

$$\frac{\partial\theta}{\partial t} = \frac{\partial}{\partial x}\left(\frac{1+dx}{(a+b\theta)^2}\frac{\partial\theta}{\partial x} + \frac{d}{2b(a+b\theta)}\right), \quad 0<x<s(t),\ t>0 \tag{7}$$

$$\theta(0,t) = g(t), \quad t>0 \tag{8}$$

$$\frac{1+ds(t)}{a^2}\frac{\partial\theta}{\partial x}(s(t),t) = -\alpha\dot{s}(t), \quad t>0 \tag{9}$$

$$\theta(s(t),t)=0, \quad t>0,\ s(0)=B \tag{10}$$

$$\theta(x,0)=f(x), \quad 0\le x\le B \tag{11}$$

If we define the transformations in the same way as in [11,13,21]

$$\begin{cases} y = \dfrac{2}{d}[(1+dx)^{1/2}-1] \\ \overline{\theta}(y,t) = \theta(x,t) \end{cases} \tag{12}$$

we obtain the following associated free boundary problem:

$$\frac{\partial\overline{\theta}}{\partial t} = \frac{\partial}{\partial y}\left(\frac{1}{(a+b\overline{\theta})^2}\frac{\partial\overline{\theta}}{\partial y}\right), \quad 0<y<\overline{S}(t),\ t>0 \tag{13}$$

$$\overline{\theta}(0,t)=g(t), \quad t>0 \tag{14}$$

$$\frac{1}{a^2}\frac{\partial\overline{\theta}}{\partial y}(\overline{S}(t),t) = -\alpha\overline{S}^\bullet(t), \quad t>0 \tag{15}$$

$$\overline{\theta}(\overline{S}(t),t)=0, \quad t>0,\ \overline{S}(0)=\overline{B} \tag{16}$$

$$\overline{\theta}(y,0)=\overline{f}(y), \quad 0\le y\le \overline{B}$$

where

$$\overline{B} = \frac{2}{d}[(1+dB)^{1/2}-1], \quad \overline{f}(y) = f\left(\frac{1}{d}\left[\left(\frac{d}{2}y+1\right)^2-1\right]\right) \tag{17}$$

and

$$\overline{S}(t) = \frac{2}{d}[(1+ds(t))^{1/2}-1] \tag{18}$$

is the free boundary.

By using the Bäcklund transformation we consider the new transformation

$$\begin{cases} y^* = y^*(y,t) = \int_{\overline{S}(t)}^{y}(a+b\overline{\theta}(\sigma,t))\,d\sigma + (-\alpha b+a)\overline{S}(t) \\ \theta^*(y^*,t) = \dfrac{1}{a+b\overline{\theta}(y,t)} \end{cases} \tag{19}$$

In order to obtain an alternative expression for y^* we compute

$$\begin{aligned}\frac{\partial y^*}{\partial t} &= -(a+b\overline{\theta}(\overline{S}(t),t))\overline{S}^\bullet(t) + \int_{\overline{S}(t)}^{y} b\frac{\partial\overline{\theta}}{\partial t}(\sigma,t)\,d\sigma + (-\alpha b+a)\overline{S}^\bullet(t) \\ &= -\alpha b\overline{S}^\bullet(t) + \int_{\overline{S}(t)}^{y} b\frac{\partial}{\partial\sigma}\left(\frac{1}{(a+b\overline{\theta}(\sigma,t))^2}\frac{\partial\overline{\theta}}{\partial\sigma}\right)d\sigma \\ &= -\alpha b\overline{S}^\bullet(t) + b\left(\frac{1}{(a+b\overline{\theta}(y,t))^2}\frac{\partial\overline{\theta}}{\partial y}(y,t) - \frac{1}{a^2}\frac{\partial\overline{\theta}}{\partial y}(\overline{S}(t),t)\right) \\ &= \frac{b}{(a+b\overline{\theta}(y,t))^2}\frac{\partial\overline{\theta}}{\partial y}(y,t) \\ &= \int_0^y \frac{\partial}{\partial\sigma}\left(\frac{b}{(a+b\overline{\theta}(\sigma,t))^2}\frac{\partial\overline{\theta}}{\partial\sigma}(\sigma,t)\right)d\sigma + \frac{b}{(a+bg(t))^2}\frac{\partial\overline{\theta}}{\partial y}(0,t)\end{aligned} \tag{20}$$

$$\frac{\partial y^*}{\partial t} = b\int_0^y \frac{\partial\overline{\theta}}{\partial t}(\sigma,t)\,d\sigma + \frac{b}{(a+bg(t))^2}\frac{\partial\overline{\theta}}{\partial y}(0,t) \tag{21}$$

and therefore the new expression for y^* is given by

$$\begin{aligned} y^*(y,t) &= \int_0^t\left(\int_0^y \frac{\partial}{\partial\sigma}(a+b\overline{\theta}(\sigma,\tau))\,d\sigma + \frac{b}{(a+bg(\tau))^2}\frac{\partial\overline{\theta}}{\partial y}(0,\tau)\right)d\tau \\ &\quad + \int_0^y (a+b\overline{\theta}(\sigma,0))\,d\sigma + \frac{b}{(a+bg(0))^2}\frac{\partial\overline{\theta}}{\partial y}(0,0) \\ &= \int_0^y (a+b\overline{\theta}(\sigma,t))\,d\sigma + \int_0^t \frac{b}{(a+bg(\tau))^2}\frac{\partial\overline{\theta}}{\partial y}(0,\tau)\,d\tau + M^* \end{aligned} \tag{22}$$

where

$$M^* = \frac{b}{(a+bg(0))^2}f'(0)$$

Now, applying (12) and (19) the problem (7)–(11) is transformed in a free boundary problem with a Dirichlet boundary condition given by

$$\frac{\partial\theta^*}{\partial t} = \frac{\partial^2\theta^*}{\partial y^{*2}}, \quad y_0^*(t)<y^*<S^*(t),\ t>0 \tag{23}$$

$$\theta^*(y_0^*(t),t) = g^*(t), \quad t>0,\ y_0^*(0)=M^* \tag{24}$$

$$\frac{\partial\theta^*}{\partial y^*}(y_0^*(t),t) = -g^*(t)\dot{y}_0^*(t), \quad t>0 \tag{25}$$

$$\frac{\partial\theta^*}{\partial y^*}(S^*(t),t) = \alpha^*\dot{S}^*(t), \quad t>0 \tag{26}$$

$$\theta^*(S^*(t),t) = \theta_f^*, \quad t>0 \tag{27}$$

$$\theta^*(y^*,0) = f^*(y^*), \quad M^*\le y^*\le B^* \tag{28}$$

$$S^*(0)=B^* \tag{29}$$

where

$$\alpha^* = \frac{\alpha b}{a(-\alpha b+a)}, \quad \theta_f^* = \frac{1}{a}, \quad B^* = (-\alpha b+a)\overline{B} \tag{30}$$

$$g^*(t) = \frac{1}{a+bg(t)} \ge \frac{1}{a+bD}, \quad f^*(y^*) = \frac{1}{a+b\overline{f}(y)} \tag{31}$$

and

$$y_0^*(t) = y^*|_{y=0} = \int_0^t \frac{b}{(a+bg(\tau))^2}\frac{\partial\overline{\theta}}{\partial y}(0,\tau)\,d\tau + M^* \tag{32}$$

$$S^*(t) = y^*|_{y=\overline{S}(t)} = (-\alpha b+a)\overline{S}(t) \tag{33}$$

are the free boundaries.

3. Existence and uniqueness of solutions

We have the following equivalence for the existence of solutions to the free boundary problem (23)–(29).

Theorem 1. *The solution to the free boundary problem* (23)–(29) *is given by the following expression*:

$$\theta^*(y^*,t)=\int_{M^*}^{B^*}G(y^*,t;\xi,0)f^*(\xi)\,d\xi+(\theta_f^*+\alpha^*)\int_0^{t^*}G(y^*,t;S^*(\tau),\tau)W(\tau)\,d\tau$$
$$-\theta_f^*\int_0^t G_\xi(y^*,t;S^*(\tau),\tau)\,d\tau+\int_0^t G_\xi(y^*,t;y_0^*(\tau),\tau)g^*(\tau)\,d\tau \tag{34}$$

and

$$y_0^*(t)=M^*-\int_0^t w(\tau)\,d\tau \tag{35}$$

$$S^*(t)=B^*+\int_0^t W(\tau)\,d\tau \tag{36}$$

where the functions w, W *are defined by*

$$w(t)=\frac{-1}{g^*(t)}\frac{\partial\theta^*}{\partial y^*}(y_0^*(t),t),\quad W(t)=\frac{1}{\alpha^*}\frac{\partial\theta^*}{\partial y^*}(S^*(t),t) \tag{37}$$

and they must satisfy the following system of two Volterra integral equations:

$$w(t)=\frac{1}{g^*(t)}\left\{\int_{M^*}^{B^*}N(y_0^*(t),t;\xi,0)f^{*'}(\xi)\,d\xi\right.$$
$$+(\theta_f^*+\alpha^*)\int_0^{t^*}G_{y^*}(y_0^*(t),t;S^*(\tau),\tau)W(\tau)\,d\tau$$
$$\left.-\int_0^t N(y_0^*(t),t;y_0^*(\tau),\tau)\dot{g}^*(\tau)\,d\tau\right\} \tag{38}$$

$$W(t)=\frac{2}{\alpha^*-\theta_f^*}\left\{\int_{M^*}^{B^*}N(S^*(t),t;\xi,0)f^{*'}(\xi)\,d\xi\right.$$
$$+(\theta_f^*+\alpha^*)\int_0^{t^*}G_{y^*}(S^*(t),t;S^*(\tau),\tau)W(\tau)\,d\tau$$
$$\left.-\int_0^t N(S^*(t),t;y_0^*(\tau),\tau)\dot{g}^*(\tau)\,d\tau\right\} \tag{39}$$

where G, N are the Green and Neumann functions and K is the fundamental solution to the heat equation, defined respectively by

$$G(x,t,\xi,\tau)=K(x,t,\xi,\tau)-K(-x,t,\xi,\tau) \tag{40}$$

$$N(x,t,\xi,\tau)=K(x,t,\xi,\tau)+K(-x,t,\xi,\tau) \tag{41}$$

$$K(x,t,\xi,\tau)=\begin{cases}\dfrac{1}{2\sqrt{\pi(t-\tau)}}\exp\left(-\dfrac{(x-\xi)^2}{4(t-\tau)}\right), & t>\tau\\ 0, & t\le\tau\end{cases} \tag{42}$$

and y_0^* *and* S^* *are given by* (35) *and* (36) *respectively*.

Proof. Let $\theta^*(y^*,t)$ be the solution to the problem (23)–(29). We integrate on the domain

$$D_{t^*,\varepsilon}=\{(\xi,\tau)/y_0^*(\tau)<\xi<S^*(\tau),\varepsilon<\tau<t-\varepsilon\}\quad(\varepsilon>0)$$

the Green identity

$$(G\theta_\xi^*-\theta^*G_\xi)_\xi-(G\theta^*)_\tau=0 \tag{43}$$

and we let $\varepsilon\to0$, to obtain the integral representation for $\theta^*(y^*,t)$ [24,25]

$$\theta^*(y^*,t)=\int_{M^*}^{B^*}G(y^*,t;\xi,0)f^*(\xi)\,d\xi+(\theta_f^*+\alpha^*)\int_0^t G(y^*,t;S^*(\tau),\tau)W(\tau)\,d\tau$$
$$-\theta_f^*\int_0^t G_\xi(y^*,t;S^*(\tau),\tau)\,d\tau+\int_0^t G_\xi(y^*,t;y_0^*(\tau),\tau)g^*(\tau)\,d\tau \tag{44}$$

by using the definitions of $w(t)$ and $W(t)$ given by (37). Moreover, if we differentiate (44) in variable y^* and we let $y^*\to y_0^{*+}(t)$ and $y^*\to S^{*-}(t)$, by the jump relations [24], we obtain the system of integral equations (38) and (39) for w and W.

Conversely the function $\theta^*(y^*,t)$ defined by (34), where w and W are the solutions of (38) and (39), satisfies the conditions (25), (26), (28) and (29). In order to prove the conditions (24) and (27) we define

$$\varphi_1(t)=\theta^*(S^*(t),t)-\theta_f^*\quad\text{and}\quad\varphi_2(t)=\theta^*(y_0^*(t),t)-g^*(t)$$

If we integrate the Green identity (43) over the domain $D_{t,\varepsilon}$ ($\varepsilon>0$) and we let $\varepsilon\to0$, we obtain that

$$\theta^*(y^*,t)=\int_{M^*}^{B^*}G(y^*,t^*;\xi,0)f^*(\xi)\,d\xi+\int_0^t G(y^*,t^*;S^*(\tau),\tau)\theta^*(S^*(\tau),\tau)W(\tau)\,d\tau$$
$$+\int_0^t[G(y^*,t;S^*(\tau),\tau)\alpha^*W(\tau)-\theta^*(S^*(\tau),\tau)G_\xi(y^*,t;S^*(\tau),\tau)]\,d\tau$$
$$+\int_0^t G(y^*,t;y_0^*(\tau),\tau)\theta^*(y_0^*(\tau),\tau)w(\tau)\,d\tau$$
$$-\int_0^t[G(y^*,t;y_0^*(\tau),\tau)g^*(\tau)w(\tau)-\theta^*(y_0^*(\tau),\tau)G_\xi(y^*,t;y_0^*(\tau),\tau)]\,d\tau \tag{45}$$

Then, if we compare this last expression (45) with (34) we deduce that

$$\int_0^t[G(y^*,t;S^*(\tau),\tau)W(\tau)-G_\xi(y^*,t;S^*(\tau),\tau)]\varphi_1(\tau)\,d\tau$$
$$+\int_0^t[G_\xi(y^*,t;y_0^*(\tau),\tau)-G(y^*,t;y_0^*(\tau),\tau)w(\tau)]\varphi_2(\tau)\,d\tau\equiv0 \tag{46}$$

We let $y^*\to S^{*-}(t)$ and $y^*\to y_0^{*+}(t)$ in (46) and we use the jump relations to obtain that φ_1 and φ_2 must satisfy the following system of Volterra integral equations:

$$\varphi_1(t)=-2\int_0^t\{[G(S^*(t),t;S^*(\tau),\tau)W(\tau)+G_\xi(S^*(t),t;S^*(\tau),\tau)]\varphi_1(\tau)$$
$$+[G_\xi(S^*(t),t;y_0^*(\tau),\tau)+G(S^*(t),t;y_0^*(\tau),\tau)w(\tau)]\varphi_2(\tau)\}\,d\tau \tag{47}$$

$$\varphi_2(t)=2\int_0^t\{[G(y_0^*(t),t;S^*(\tau),\tau)W(\tau)-G_\xi(y_0^*(t),t;S^*(\tau),\tau)]\varphi_1(\tau)$$
$$+[G_\xi(y_0^*(t),t;y_0^*(\tau),\tau)+G(y_0^*(t),t;y_0^*(\tau),\tau)w(\tau)]\varphi_2(\tau)\}\,d\tau \tag{48}$$

Following [23] it is easy to see that there exists a unique solution $\varphi_1\equiv\varphi_2\equiv0$ to the system of Volterra integral equations (47)–(48). Then (24) and (27) holds. □

Next, we use the Banach fixed point theorem in order to prove the local existence and uniqueness of solution $w,W\in C^0[0,\sigma]$ to the system of two Volterra integral equations (38) and (39) where σ is a positive small number to be determinate. We consider the Banach Space:

$$C_{R,\sigma}=\left\{\overrightarrow{w^*}=\begin{pmatrix}w\\W\end{pmatrix}\Big/ w,W:[0,\sigma]\to\mathbb{R},\ \text{continuous, with }\|\overrightarrow{w^*}\|_\sigma\le R\right\}$$

where

$$\|\overrightarrow{w^*}\|_\sigma:=\max_{t\in[0,\sigma]}|w(t)|+\max_{t\in[0,\sigma]}|W(t)|$$

We define the map $F:C_{R,\sigma}\longrightarrow C_{R,\sigma}$, such that

$$\overrightarrow{\tilde{w}^*}(t)=F(\overrightarrow{w^*}(t))=\begin{pmatrix}F_1(w(t),W(t))\\F_2(w(t),W(t))\end{pmatrix}$$

where

$$F_1(w(t),W(t)) = \frac{1}{g^*(t)}\left\{ \int_{M^*}^{B^*} N(y_0^*(t),t;\xi,0) f^{*'}(\xi)\, d\xi + (\theta_f^*+\alpha^*)\int_0^t G_{y^*}(y_0^*(t),t;S^*(\tau),\tau)W(\tau)\, d\tau - \int_0^t N(y_0^*(t),t;y_0^*(\tau),\tau)\dot{g}^*(\tau)\, d\tau \right\} \tag{49}$$

and

$$F_2(w(t),W(t)) = \frac{2}{\alpha^*-\theta_f^*}\left\{ \int_{M^*}^{B^*} N(S^*(t),t^*;\xi,0) f^{*'}(\xi)\, d\xi + (\theta_f^*+\alpha^*)\int_0^t G_{y^*}(S^*(t),t;S^*(\tau),\tau)W(\tau)\, d\tau - \int_0^t N(S^*(t),t^*;y_0^*(\tau),\tau)\dot{g}^*(\tau)\, d\tau \right\} \tag{50}$$

Lemma 2. *Let* $w, W \in C^0[0,\sigma]$, $\max_{t\in[0,\sigma]}|w(t)| \le R$, $\max_{t\in[0,\sigma]}|W(t)| \le R$ *and* $2R\sigma \le M^* < B^*/3$ *then* y_0^* *and* S^* *defined by* (35) *and* (36) *satisfies*

$$|y_0^*(t)-y_0^*(\tau)| \le R|t-\tau|, \quad \forall \tau,t\in[0,\sigma]$$

$$\frac{M^*}{2} \le y_0^*(t) \le \frac{3M^*}{2}, \quad \forall t\in[0,\sigma]$$

$$|S^*(t^*)-S^*(\tau)| \le R|t-\tau|, \quad \forall \tau,t\in[0,\sigma]$$

$$\frac{B^*}{2} \le S^*(t) \le \frac{3B^*}{2}, \quad \forall t\in[0,\sigma]. \qquad \square \tag{51}$$

To prove the following lemmas we need the classical inequality:

$$\frac{\exp\left(\frac{-x^2}{\alpha(t-\tau)}\right)}{(t-\tau)^{n/2}} \le \left(\frac{n\alpha}{2ex^2}\right)^{n/2}, \quad \alpha,x>0,\ t>\tau,\ n\in\mathbb{N}. \tag{52}$$

Lemma 3. *Let* $\sigma \le 1$. *Under the hypothesis of Lemma 2 we have the following properties*:

$$\int_{M^*}^{B} |f^{*'}(\xi)||N(y_0^*(t^*),t^*;\xi,0)|d\xi \le \|f^{*'}\| \tag{53}$$

$$\int_0^t |G_{y^*}(y_0^*(t),t;S^*(\tau),\tau)W(\tau)|\, d\tau \le C_1(R,B^*,M^*)t \tag{54}$$

$$\int_0^t |\dot{g}^*(\tau)||N(y_0^*(t),t;y_0^*(\tau),\tau)|\, d\tau \le 2\sqrt{\frac{t}{\pi}}\|\dot{g}^*\| \tag{55}$$

$$\int_{M^*}^{B} |f^{*'}(\xi)||N(S^*(t),t;\xi,0)|\, d\xi \le \|f^{*'}\| \tag{56}$$

$$\int_0^t |G_{y^*}(S^*(t),t;S^*(\tau),\tau)W(\tau)|\, d\tau \le C_2(R,B^*)\sqrt{t} \tag{57}$$

$$\int_0^t |\dot{g}^*(\tau)||N(S^*(t^*),t^*;y_0^*(\tau),\tau)|\, d\tau \le 2\sqrt{\frac{t}{\pi}}\|\dot{g}^*\| \tag{58}$$

where

$$C_1(R,B^*,M^*) = \frac{\sqrt{6}}{\sqrt{e\pi}}\left[\frac{1}{(B^*-3M^*)^2}+\frac{1}{(B^*+M^*)^2}\right]R$$

and

$$C_2(R,B^*) = \frac{R}{4\sqrt{\pi}}\left[2R+\frac{3}{B^{*2}}\left(\frac{2}{3e}\right)^{3/2}\right]$$

Proof. To prove (53) we consider

$$\int_{M^*}^{B} |f^{*'}(\xi)||N(y_0^*(t^*),t^*;\xi,0)|\, d\xi \le \|f^{*'}\|\int_0^\infty |N(y_0^*(t^*),t^*;\xi,0)|\, d\xi \le \|f^{*'}\|$$

We have

$$|G_{y^*}(y_0^*(t^*),t^*;S^*(\tau),\tau)| \le \frac{1}{4\sqrt{\pi}}\left\{ |y_0^*(t^*)-S^*(\tau)|\frac{\exp\left(\frac{-(y_0^*(t^*)-S^*(\tau))^2}{4(t-\tau)}\right)}{(t-\tau)^{3/2}} + |y_0^*(t^*)+S^*(\tau)|\frac{\exp\left(\frac{-(y_0^*(t^*)+S^*(\tau))^2}{4(t-\tau)}\right)}{(t-\tau)^{3/2}} \right\}$$

and by using Lemma 2 and (52) we obtain

$$|G_{y^*}(y_0^*(t^*),t^*;S^*(\tau),\tau)| \le \frac{\sqrt{6}}{\sqrt{e\pi}}\left[\frac{1}{(B^*-3M^*)^2}+\frac{1}{(B^*+M^*)^2}\right] \tag{59}$$

Then (54) holds.

To prove (55) by taking into account that

$$|N(y_0^*(t^*),t^*;y_0^*(\tau),\tau)| \le \frac{1}{\sqrt{\pi(t-\tau)}}$$

so, we obtain

$$\int_0^t |\dot{g}^*(\tau)||N(y_0^*(t^*),t^*;y_0^*(\tau),\tau)|\, d\tau \le 2\sqrt{\frac{t}{\pi}}\|\dot{g}^*\|$$

The inequality (56) is proved in the same way as (53).

In [26] it was proved

$$|G_{y^*}(S^*(t),t;S(\tau),\tau)| \le \frac{1}{4\sqrt{\pi}}\left[\frac{R}{\sqrt{t-\tau}}+\frac{3}{B^{*2}}\left(\frac{2}{3e}\right)^{3/2}\right]$$

then we have (57).

The inequality (58) is proved in the same way as (55). $\square$

Lemma 4. *Let* S_1^* *and* S_2^* *be the functions corresponding to* W_1 *and* W_2 *in* $C^0[0,\sigma]$ *respectively and* ${y_0}_1^*$ *and* ${y_0}_2^*$ *be the functions corresponding to* w_1 *and* w_2 *in* $C^0[0,\sigma]$ *respectively with*

$$\max_{t\in[0,\sigma]}|w_i(t)| \le R, \quad \max_{t\in[0,\sigma]}|W_i(t)| \le R,$$

then we have

$$\begin{cases} |S_2^*(t)-S_1^*(t)| \le t\|W_2-W_1\|_\sigma \\ |S_i^*(t)-S_i^*(\tau)| \le R|t-\tau|, \quad i=1,2 \\ \frac{B^*}{2} \le S_i^*(t) \le \frac{3B^*}{2}, \quad \forall t\in[0,\sigma],\ i=1,2 \end{cases} \tag{60}$$

and

$$\begin{cases} |y_{01}^*(t)-y_{02}^*(t)| \le t\|w_2-w_1\|_\sigma \\ |y_{0i}^*(t)-y_{0i}^*(\tau)| \le R|t-\tau|, \quad i=1,2 \\ \frac{M^*}{2} \le {y_0}_i^*(t) \le \frac{3M^*}{2}, \quad \forall t\in[0,\sigma],\ i=1,2. \end{cases} \qquad \square \tag{61}$$

Lemma 5. *If we take $R^2\sigma \le 1$, $\sigma \le 1$ and $2R\sigma \le M^* < B^*/3$ then we have*

$$\int_0^t \left| W_2(\tau)G_{y^*}(y^*_{0_2}(t),t;S^*_2(\tau),\tau) - W_1(\tau)G_{y^*}(y^*_{0_1}(t),t;S^*_1(\tau),\tau)\right| d\tau \le [P_3(M^*,B^*)\|W_1-W_2\| + RP_1(M^*,B^*)\|w_1-w_2\| + RP_2(M^*,B^*)\|W_1-W_2\|]\sqrt{t} \tag{62}$$

$$\int_{M^*}^{B^*} |f^{*'}(\xi)|\left|N(y^*_{0_2}(t),t;\xi,0) - N(y^*_{0_1}(t),t;\xi,0)\right| d\xi \le \frac{2\|f^{*'}\|}{\sqrt{\pi}}\|w_1-w_2\|\sqrt{\sigma} \tag{63}$$

$$\int_0^t |\dot{g}^*(\tau)|\left|N(y^*_{0_2}(t),t;y^*_{0_2}(\tau),\tau) - N(y^*_{0_1}(t),t,y^*_{0_1}(\tau),\tau)\right| d\tau \le P_4(M^*,R)\|\dot{g}^*\|\|w_1-w_2\|\sqrt{t} \tag{64}$$

$$\int_{M^*}^{B^*} |f^{*'}(\xi)|\left|N(S^*_2(t),t;\xi,0) - N(S^*_1(t),t;\xi,0)\right| d\xi \le \|f^{*'}\|\frac{2}{\sqrt{\pi}}\|W_1-W_2\|\sqrt{t} \tag{65}$$

$$\int_0^t \left| W_2(\tau)G_{y^*}(S^*_2(t),t;S^*_2(\tau),\tau) - W_1(\tau)G_{y^*}(S^*_1(t),t;S^*_1(\tau),\tau)\right| d\tau \le P_5(R,B^*)\|W_1-W_2\|t \tag{66}$$

$$\int_0^t |g^{*'}(\tau)|\left|N(S^*_2(t),t;y^*_{0_2}(\tau),\tau) - N(S^*_1(t),t,y^*_{0_1}(\tau),\tau)\right| d\tau \le \|\dot{g}^*\|P_6(M^*,B^*)(\|w_1-w_2\| + \|W_1-W_2\|)\sqrt{t} \tag{67}$$

where

$$P_1(M^*,B^*) = \frac{1}{\sqrt{\pi}e^{3/2}}\left[\frac{\sqrt{6}(3B^*-M^*)^2}{16(B^*-3M^*)^3} + \frac{27\sqrt{3}}{4} + \frac{12\sqrt{6}}{(B^*-3M^*)^3} + \frac{6\sqrt{3}}{(B^*+M^*)^3}\right] \tag{68}$$

$$P_2(M^*,B^*) = \frac{12\sqrt{6}}{\sqrt{\pi}e^{3/2}}\left[\frac{1}{(B^*-3M^*)^3} + \frac{9}{8} + \frac{(3B^*-M^*)^2}{8(B^*-3M^*)^3} + \frac{1}{(B^*+M^*)^2}\right] \tag{69}$$

$$P_3(M^*,B^*) = \frac{\sqrt{6}}{\sqrt{\pi e}}\left[\frac{1}{(B^*-3M^*)^2} + \frac{1}{(B^*+M^*)^2}\right] \tag{70}$$

$$P_4(M^*,B^*) = \frac{R^2}{\sqrt{\pi}} + \left(\frac{6}{e}\right)^{3/2}\frac{3R}{M^{*2}\sqrt{\pi}}$$

$$P_5(R,B^*) = \frac{1}{4\sqrt{\pi}}\left[6R + \frac{3}{B^{*2}}\left(\frac{2}{3e}\right)^{3/2} + \frac{6R}{B^{*2}}\left(\frac{6}{e}\right)^{3/2}\right] \tag{71}$$

$$P_6(M^*,B^*) = \frac{6^{3/2}}{\sqrt{\pi}e^{3/2}}\left[\frac{3B^*-M^*}{(B^*-3M^*)^3} + \frac{3}{(B^*+M^*)^2}\right] \tag{72}$$

Proof. To prove (62) we have

$$\left|W_2(\tau)G_{y^*}(y^*_{0_2}(t),t;S^*_2(\tau),\tau) - W_1(\tau)G_{y^*}(y^*_{0_1}(t),t;S^*_1(\tau),\tau)\right| \le |W_1(\tau)-W_2(\tau)|\left|G_{y^*}(y^*_{0_2}(t),t;S^*_2(\tau),\tau)\right| + |W_1(\tau)|\left|G_{y^*}(y^*_{0_2}(t),t;S^*_2(\tau),\tau) - G_{y^*}(y^*_{0_1}(t),t;S^*_1(\tau),\tau)\right|$$

Taking into account that

$$\left|G_{y^*}(y^*_{0_2}(t),t;S^*_2(\tau),\tau) - G_{y^*}(y^*_{0_1}(t),t;S^*_1(\tau),\tau)\right| \le \left|G_{y^*}(y^*_{0_2}(t),t;S^*_2(\tau),\tau) - G_{y^*}(y^*_{0_1}(t),t;S^*_2(\tau),\tau)\right| + \left|G_{y^*}(y^*_{0_1}(t),t;S^*_2(\tau),\tau) - G_{y^*}(y^*_{0_1}(t),t;S^*_1(\tau),\tau)\right| \tag{73}$$

and by using the mean value theorem we have that there exists $m = m(t)$ between $y^*_{0_1}(t)$ and $y^*_{0_2}(t)$ such that

$$\left|G_{y^*}(y^*_{0_2}(t),t;S^*_2(\tau),\tau) - G_{y^*}(y^*_{0_1}(t),t;S^*_2(\tau),\tau)\right| = \left|G_{y^*y^*}(m(t),t;S^*_2(\tau),\tau)\right|\left|y^*_{01}(t) - y^*_{02}(t)\right| = \left|G_t(m(t),t;S^*_2(\tau),\tau)\right|\left|y^*_{0_1}(t) - y^*_{0_2}(t)\right|$$

and there exists $n = n(\tau)$ between $S^*_1(\tau)$ and $S^*_2(\tau)$ such that

$$\left|G_{y^*}(y^*_{0_1}(t),t;S^*_2(\tau),\tau) - G_{y^*}(y^*_{0_1}(t),t;S^*_1(\tau),\tau)\right| = \left|G_{y^*\xi}(y^*_{0_1}(t),t;n(\tau),\tau)\right|\left|S^*_1(t^*) - S^*_2(t^*)\right| = \left|N_\tau(y^*_{0_1},t;n(\tau),\tau)\right|\left|S^*_1(t^*) - S^*_2(t^*)\right|.$$

Taking into account that

$$|G_t(m(t),t;S^*_2(\tau),\tau)| \le \frac{1}{8\sqrt{\pi}(t-\tau)^{3/2}}\left\{(m(t)-S^*_2(\tau))^2\exp\left(\frac{-(m(t)-S^*_2(\tau))^2}{4(t-\tau)}\right) + (m(t)+S^*_2(\tau))^2\exp\left(\frac{-(m(t)+S^*_2(\tau))^2}{4(t-\tau)}\right)\right\} + \frac{1}{4\sqrt{\pi}(t-\tau)^{3/2}}\left\{\exp\left(\frac{-(m(t)-S^*_2(\tau))^2}{4(t-\tau)}\right) + \exp\left(\frac{-(m(t)+S^*_2(\tau))^2}{4(t-\tau)}\right)\right\}$$

and

$$|N_\tau(y^*_{0_1},t;n(\tau),\tau)| \le \frac{1}{8\sqrt{\pi}(t-\tau)^{3/2}}\left\{(y^*_{0_1}-n(\tau))^2\exp\left(\frac{-(y^*_{0_1}-n(\tau))^2}{4(t-\tau)}\right) + (y^*_{0_1}+n(\tau))^2\exp\left(\frac{-(y^*_{0_1}+n(\tau))^2}{4(t-\tau)}\right)\right\} + \frac{1}{4\sqrt{\pi}(t-\tau)^{3/2}}\left\{\exp\left(\frac{-(y^*_{0_1}-n(\tau))^2}{4(t-\tau)}\right) + \exp\left(\frac{-(y^*_{0_1}+n(\tau))^2}{4(t-\tau)}\right)\right\}$$

from (52) and Lemma 2 we have

$$\left|G_{y^*}(y^*_{0_2}(t),t;S^*_2(\tau),\tau) - G_{y^*}(y^*_{0_1}(t),t;S^*_1(\tau),\tau)\right| \le P_1(M^*,B^*)t\|w_1-w_2\|_t + P_2(M^*,B^*)\tau\|W_1-W_2\|_t$$

Moreover, by (59)

$$\left|G_{y^*}(y^*_{0_2}(t),t;S^*_2(\tau),\tau)\right| \le \frac{\sqrt{6}}{\sqrt{e\pi}}\left[\frac{1}{(B^*-3M^*)^2} + \frac{1}{(B^*+M^*)^2}\right] = P_3(M^*,B^*)$$

then, we obtain

$$\left|W_2(\tau)G_{y^*}(y^*_{0_2}(t),t;S^*_2(\tau),\tau) - W_1(\tau)G_{y^*}(y^*_{0_1}(t),t;S^*_1(\tau),\tau)\right| \le P_3(M^*,B^*)\|W_1-W_2\|_t + [P_1(M^*,B^*)t\|w_1-w_2\|_t + P_2(M^*,B^*)\tau\|W_1-W_2\|_t]R$$

and (62) hold.

Following [26] we obtain (63) and (65).

To prove (66) we use estimates obtained in [27].

To finish the thesis, we shall prove (67):

$$\int_0^t |\dot{g}^*(\tau)|\left|N(S^*_2(t),t,y^*_{0_2}(\tau),\tau) - N(S^*_1(t),t,y^*_{0_1}(\tau),\tau)\right| d\tau \le \|\dot{g}^*\|\int_0^t \{\left|N(S^*_2(t),t,y^*_{0_2}(\tau),\tau) - N(S^*_2(t),t,y^*_{0_1}(\tau),\tau)\right|$$

$$+|N(S_2^*(t),t,y_{0_1}^*(\tau),\tau)-N(S_1^*(t),t,y_{0_1}^*(\tau),\tau)|\}\,d\tau \tag{74}$$

By the mean value theorem there exists $c(\tau)\in(y_{0_1}^*(\tau),y_{0_2}^*(\tau))$ such that

$$\begin{aligned}&|N(S_2^*(t),t,y_{0_2}^*(\tau),\tau)-N(S_2^*(t),t,y_{0_1}^*(\tau),\tau)|\\&=|N_\xi(S_2^*(t),t,c(\tau),\tau)|\,|y_{0_1}^*(\tau)-y_{0_2}^*(\tau)|\\&\le\frac{1}{4\sqrt{\pi}}\left\{|S^*(t)-c(\tau)|\frac{\exp\left(\frac{-(S_2^*(t)-c(\tau))^2}{4(t-\tau)}\right)}{(t-\tau)^{3/2}}\right.\\&\left.+|S^*(t)+c(\tau)|\frac{\exp\left(\frac{-(S^*(t)+c(\tau))^2}{4(t-\tau)}\right)}{(t-\tau)^{3/2}}\right\}|y_{0_1}^*(\tau)-y_{0_2}^*(\tau)|\end{aligned}$$

then taking into account (52) and Lemma 2 we have

$$|N(S_2^*(t),t,y_{0_2}^*(\tau),\tau)-N(S_2^*(t),t,y_{0_1}^*(\tau),\tau)|\le P_5(M^*,B^*)|y_{0_1}^*(\tau)-y_{0_2}^*(\tau)|$$

Furthermore, in a similar way we have that exists $C(t)\in(S_1^*(t^*),S_2^*(t^*))$ such that

$$\begin{aligned}&|N(S_2^*(t),t,y_{0_1}^*(\tau),\tau)-N(S_1^*(t),t,y_{0_1}^*(\tau),\tau)|\\&=|N_{y^*}(C(t),t,y_{0_1}^*(\tau),\tau)|\,|S_2^*(t)-S_1^*(t)|\le P_6(M^*,B^*)|S_2^*(t)-S_1^*(t)|\end{aligned}$$

then (67) is obtained. □

Theorem 6. *The map $F:C_{R,\sigma}\longrightarrow C_{R,\sigma}$ is well defined and it is a contraction map if σ satisfies the following inequalities*:

$$\sigma\le 1,\quad 2R\sigma\le M^*<\frac{B^*}{3},\quad R^2\sigma\le 1 \tag{75}$$

$$H_1(\|\dot{g}^*\|_\sigma,a,b,B^*,\theta_f^*,\alpha^*,D,M^*,R,\sigma)\le 1 \tag{76}$$

$$H_2(\|f^*\|,\|f^{*'}\|a,b,B^*,\theta_f^*,\alpha^*,D,M^*,R,\sigma)\le 1 \tag{77}$$

where R is given by

$$R=1+\|f^{*'}\|\left(a+bD+\frac{2}{|\alpha^*-\theta_f^*|}\right) \tag{78}$$

and

$$\begin{aligned}&H_1(\|\dot{g}^*\|_\sigma,a,b,B^*,\theta_f^*,\alpha^*,D,M^*,R,\sigma)\\&=(a+bD)\left\{(\theta_f^*+\alpha^*)C_1(B^*,R,M^*)\sigma+\frac{2}{\sqrt{\pi}}\|\dot{g}^*\|_\sigma\sqrt{\sigma}\right\}\\&+|\frac{1}{\alpha^*-\theta_f^*}|\left\{(\theta_f^*+\alpha^*)\frac{R^2}{2\sqrt{\pi}}C_2(B^*,R)\sqrt{\sigma}+4\sqrt{\frac{\sigma}{\pi}}\|\dot{g}^*\|_\sigma\right\}\end{aligned} \tag{79}$$

$$\begin{aligned}&H_2(\|\dot{g}^*\|_\sigma,a,b,B^*,\theta_f^*,\alpha^*,D,M^*,R,\sigma)\\&=(a+bD)\left\{\frac{2}{\sqrt{\pi}}\|f^{*'}\|+(\theta_f^*+\alpha^*)\left(P_3(M^*,B^*)+RP_1(M^*,B^*)+\frac{RP_2(M^*,B^*)}{2}\right)\right.\\&\left.+\|\dot{g}^*\|_\sigma P_4(M^*,R)\right\}+\frac{2}{|\alpha^*-\theta_f^*|}\left(\frac{2\|f^*\|}{\sqrt{\pi}}+(\theta_f^*+\alpha^*)P_5(M^*,B^*)\right.\\&\left.+\|\dot{g}^*\|_\sigma P_6(M^*,B^*)\right)\end{aligned} \tag{80}$$

Then there exists a unique solution on $C_{R,\sigma}$ to the system of integral equations (38) *and* (39).

Proof. Firstly we demonstrate that F maps $C_{R,\sigma}$ into itself, that is

$$\|F(\overrightarrow{w^*})\|_\sigma=\max_{t\in[0,\sigma]}|F_1(w(t),W(t))|+\max_{t\in[0,\sigma]}|F_2(w(t),W(t))|\le R$$

Taking into account Lemma 3 we have

$$|F_1(w(t),W(t))|\le(a+bD)\left\{\|f^{*'}\|+(\theta_f^*+\alpha^*)C_1(B^*,R,M^*)t+2\sqrt{\frac{t}{\pi}}\|\dot{g}^*\|\right\}$$

$$|F_2(w(t),W(t))|\le\frac{2}{|\alpha^*-\theta_f^*|}\left\{\|f^{*'}\|+(\theta_f^*+\alpha^*)\frac{R}{4\sqrt{\pi}}C_2(B^*,R)\sqrt{t}+2\sqrt{\frac{t}{\pi}}\|\dot{g}^*\|\right\}$$

and then

$$\|F(\overrightarrow{w^*})\|_\sigma\le\|f^{*'}\|\left(a+bD+\frac{2}{|\alpha^*-\theta_f^*|}\right)+H_1(\|\dot{g}^*\|_\sigma,a,b,B^*,\theta_f^*,\alpha^*,D,M^*,R,\sigma)$$

where H_1 is given by (79). Selecting R by (78) and σ such that (75) hold, we obtain $\|F(w^*)\|_\sigma\le R$. Now, we will prove that

$$\|F(\overrightarrow{w_2^*})-F(\overrightarrow{w_1^*})\|_\sigma\le H_2(\|\dot{g}^*\|_\sigma,a,b,B^*,\theta_f^*,\alpha^*,D,M^*,R,\sigma)\|\overrightarrow{w_2^*}-\overrightarrow{w_1^*}\|_\sigma$$

where $\overrightarrow{w_1^*}=\binom{w_1}{W_1},\overrightarrow{w_2^*}=\binom{w_2}{W_2}\in C_{R,\sigma}$. Taking into account Lemma 5 we have

$$\begin{aligned}&\|F(\overrightarrow{w_2^*})-F(\overrightarrow{w_1^*})\|_\sigma=\max_{t\in[0,\sigma]}|F_1(w_2(t),W_2(t))-F_1(w_1(t),W_1(t))|\\&+\max_{t\in[0,\sigma]}|F_2(w_2(t),W_2(t))-F_2(w_1(t),W_1(t))|\\&\le\left\{(a+bD)\left[\frac{2}{\sqrt{\pi}}\|f^{*'}\|+(\theta_f^*+\alpha^*)\left(P_3(M^*,B^*)+RP_1(M^*,B^*)+\frac{RP_2(M^*,B^*)}{2}\right)\right.\right.\\&\left.+\|\dot{g}^*\|_\sigma P_4(M^*,R)\right]+\frac{2}{|\alpha^*-\theta_f^*|}\left[\frac{2\|f^*\|}{\sqrt{\pi}}+(\theta_f^*+\alpha^*)P_5(M^*,B^*)\right.\\&\left.\left.+\|\dot{g}^*\|P_6(M^*,B^*)\right]\right\}\|\overrightarrow{w_2^*}-\overrightarrow{w_1^*}\|_\sigma\sqrt{\sigma}\\&\le H_2(\|\dot{g}^*\|_\sigma,a,b,B^*,\theta_f^*,\alpha^*,D,M^*,R,\sigma)\|\overrightarrow{w_2^*}-\overrightarrow{w_1^*}\|_\sigma\end{aligned} \tag{81}$$

By hypothesis (75)–(78) we have that F is a contraction. □

4. Conclusions

A one-dimensional free boundary problem is studied for a non-linear diffusion–convection equation whose diffusivity is heterogeneous in space as well as being non-linear. After using some non-linear transformations that simplify the transport equation to a form that is both linear and homogeneous in space, the transformed problem now has two free boundaries. It is shown that the problem is equivalent to a system of two integral equations. Then sufficient conditions for data was found applying the Banach contraction theorem, to prove existence and uniqueness, local in time of the solution.

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