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THE STEFAN PROBLEM WITH TEMPERATURE-DEPENDENT THERMAL CONDUCTIVITY AND A CONVECTIVE TERM WITH A CONVECTIVE CONDITION AT THE FIXED FACE

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Abstract. We study a one-phase Stefan problem for a semi-infinite material with temperature-dependent thermal conductivity and a convective term with a convective boundary condition at the fixed face $x = 0$. We obtain sufficient conditions for data in order to have a parametric representation of the solution of the similarity type for $t \geq t_0 > 0$ with t_0 an arbitrary positive time. We obtain explicit solutions through the unique solution of a Cauchy problem with the time as a parameter and we also give an algorithm in order to compute the explicit solution.

1. **Introduction.** We consider a Stefan problem for a semi-infinite region $x > 0$ with temperature-dependent thermal conductivity and a convective term with phase change temperature $\theta_f = 0$ and a new boundary condition at the fixed face $x = 0$ given by the convective condition convective (2) with respect to the one considered in [20], [22], [24]. It is required to determine the evolution of the moving phase separation $x = s(t)$ and the temperature distribution $\theta(x, t)$. The modeling of this kind of systems is a problem with a great mathematical and industrial significance. Phase-change problems appear frequently in industrial processes an other problems of technological interest [1], [2], [8]–[16], [19]. A large bibliography on the subject was given in [26].

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We consider a one-phase Stefan problem in fusion process with nonlinear heat conduction equations. Owing to [20], [22], [24] we consider the following free boundary (fusion process) problem

$$\rho c \frac{\partial \theta}{\partial t} = \frac{\partial}{\partial x}\left(k(\theta, x)\frac{\partial \theta}{\partial x}\right) - v(\theta)\frac{\partial \theta}{\partial x} \ , \ 0 < x < s(t) \ , \ t > 0 \tag{1}$$

$$k(\theta(0,t),0)\frac{\partial \theta}{\partial x}(0,t) = \frac{h_0}{\sqrt{t}}\left(\theta\left(0,t\right) - \theta_0\right) \ , \ h_0 > 0 \ , \ \theta_0 > 0, t > 0 \tag{2}$$

$$k\left(\theta\left(s(t),t\right), s(t)\right)\frac{\partial \theta}{\partial x}\left(s(t),t\right) = -\rho l \ \overset{\bullet}{s}(t) \ , \ t > 0 \tag{3}$$

$$\theta\left(s(t),t\right) = 0 \ , \ t > 0 \ , \ s(0) = 0 \tag{4}$$

where the thermal conductivity $k(\theta, x)$ and the velocity $v(\theta)$ are given by

$$v(\theta) = \rho c \frac{d}{2\left(a + b\theta\right)^2} \quad , \quad k(\theta, x) = \rho c \frac{1 + dx}{\left(a + b\theta\right)^2} \tag{5}$$

and c, ρ and l are the specific heat, the density and the latent heat of fusion of the medium respectively, all of them are assumed to be constant with positive parameters a, d and real parameter b. The coefficient h_0 is a positive constant which caracterizes the thermal transfer coefficient $h_0/\sqrt{t}$ in (2), $\theta_0 > 0$ is the temperature of the external medium and we assume $a + b\theta_0 > 0$. This kind of nonlinear thermal conductivity or diffusion coefficients was considered in numerous papers, e.g. [3]- [7] and [17], [21], [23], [25]. In [17] it was founded that the nonlinear diffusion equation can be transformed to the linear form that possesses a well-known analytical solution when the soil water diffusivity is given by $D(\theta) = a(b - \theta)^{-2}$, where a and b are constant. Moreover, they argued that the diffusivity functional form is plausible for soil water diffusivities. The nonlinear transport Eq.(1) arises in connection with unsaturated flow in heterogeneous porous media. If we set $d = 0$ and $b = 0$ in the free boundary problem (1) $-$ (5) then we retrieve the classical one-phase Lamé-Clapeyron-Stefan problem. The first explicit solution for the one-phase Stefan problem was given in [18]. We will determine which conditions on the parameters of the problem must be verified in order to have an instantaneous phase-change process.

In [22] we improve the results given in [20] and [24] by obtaining an explicit solution through the unique solution of a Cauchy problem where the time is a parameter and giving an algorithm in order to compute the explicit solution. We prove the existence and uniqueness of the evolution of the moving phase separation $x = s(t)$. The coefficient which characterize the free boundary is the unique solution of a transcendental equation. Besides, we consider the same problem given in [24] but the heat flux boundary condition is replaced by a constant temperature condition at $x = 0$.

In this paper we obtain a unique solution of the problem (1) $-$ (5) following the method developed in [22] and [24] but we change the heat flux condition by a convective boundary condition on the fixed face $x = 0$ which is given by the expression (2) (see [27], [28], [29]).

In Section II we consider the free boundary problem (1) $-$ (4) with the nonlinear heat coefficients (5) under the hypotheses $b > 0$ and $a > bl/c$; $b > 0$ and $a \le bl/c$; or $b < 0$. We follow [22] in the sense that the existence of the explicit solution of the problem (1) $-$ (5) is obtained through the unique solution of the Cauchy problem (69) $-$ (70) in the spatial variable and the time is a parameter for $t \ge t_0 > 0$ with

t_0 is a arbitrary positive time. This explicit solution can be obtained as a function of a parameter which is given as the unique solution of the transcendental Eq.(47). Moreover, we also give an algorithm in order to compute the explicit solution for the temperature $\theta = \theta(x,t)$ and the free boundary $x = s(t)$.

2. **Solution of the free boundary problem with convective boundary condition on the fixed face.** We consider the free boundary problem (1) − (4). Taking into account (5) we can put our problem as

$$\frac{\partial\theta}{\partial t} = \frac{\partial}{\partial x}\left(\frac{1+dx}{(a+b\theta)^2}\frac{\partial\theta}{\partial x} + \frac{d}{2b\,(a+b\theta)}\right)\ ,\ 0<x<s(t)\ ,\ t>0 \tag{6}$$

$$k(\theta(0,t),0)\frac{\partial\theta}{\partial x}(0,t) = \frac{h_0}{\sqrt{t}}\,(\theta(0,t)-\theta_0)\ ,\ t>0 \tag{7}$$

$$\frac{1+ds(t)}{a^2}\frac{\partial\theta}{\partial x}\,(s(t),t) = -\alpha\,\dot{s}(t)\ ,\ t>0 \tag{8}$$

$$\theta\,(s(t),t) = 0\ ,\ t>0\ ,\ s(0)=0 \tag{9}$$

where $\alpha = \frac{l}{c}$ and $a,d\in\mathbb{R}^+, b\in\mathbb{R}$.

If we define the transformations in the same way as in [22], [24]

$$\begin{cases} y = \dfrac{2}{d}\left[(1+dx)^{\frac{1}{2}}-1\right] \quad , \quad \overline{S}(t) = \dfrac{2}{d}\left[(1+ds(t))^{\frac{1}{2}}-1\right] \\ \overline{\theta}(y,t) = \theta(x,t) \end{cases} \tag{10}$$

we obtain the following free boundary problem

$$\frac{\partial\overline{\theta}}{\partial t} = \frac{\partial}{\partial y}\left(\frac{1}{\left(a+b\overline{\theta}\right)^2}\frac{\partial\overline{\theta}}{\partial y}\right)\ ,\ 0<y<\overline{S}(t)\ ,\ t>0 \tag{11}$$

$$\frac{1}{\left(a+b\,\overline{\theta}\,(0,t)\right)^2}\frac{\partial\overline{\theta}}{\partial y}(0,t) = \frac{h_0^*}{\sqrt{t}}\,\left(\overline{\theta}(0,t)-\theta_0\right)\ ,\ t>0 \tag{12}$$

$$\frac{1}{a^2}\frac{\partial\overline{\theta}}{\partial y}\left(\overline{S}(t),t\right) = -\alpha\,\dot{\overline{S}}\,(t)\ ,\ t>0 \tag{13}$$

$$\overline{\theta}\left(\overline{S}(t),t\right) = 0\ ,\ t>0 \quad , \quad \overline{S}(0)=0 \tag{14}$$

where $h_0^* = \frac{h_0}{\rho c}$.

Then, we define the new transformation

$$\begin{cases} y^* = y^*(y,t) = \displaystyle\int_{\overline{S}(t)}^{y}\left(a+b\overline{\theta}\,(\sigma,t)\right)d\sigma + (-\alpha b+a)\,\overline{S}(t) \\ \theta^*(y^*,t^*) = \dfrac{1}{a+b\overline{\theta}(y,t)} \quad , \quad t^*=t \\ S^*(t^*) = y^*\Big|_{y=\overline{S}(t)} = (-\alpha b+a)\,\overline{S}(t). \end{cases} \tag{15}$$

In order to obtain an alternative expression for y^* we compute

$$
\begin{aligned}
\frac{\partial y^*}{\partial t} &= -\left(a+b\overline{\theta}\left(\overline{S}(t),t\right)\right)\dot{\overline{S}}(t)+\int_{\overline{S}(t)}^{y} b\frac{\partial\overline{\theta}}{\partial t}(\sigma,t)\,d\sigma+(-\alpha b+a)\,\dot{\overline{S}}(t) \\
&= -\alpha b\,\dot{\overline{S}}(t)+\int_{\overline{S}(t)}^{y} b\frac{\partial}{\partial\sigma}\left(\frac{1}{\left(a+b\overline{\theta}(\sigma,t)\right)^2}\frac{\partial\overline{\theta}}{\partial\sigma}\right)d\sigma \\
&= -\alpha b\,\dot{\overline{S}}(t)+b\left(\frac{1}{\left(a+b\overline{\theta}(y,t)\right)^2}\frac{\partial\overline{\theta}}{\partial y}(y,t)-\frac{1}{a^2}\frac{\partial\overline{\theta}}{\partial y}(\overline{S}(t),t)\right) \\
&= \frac{b}{\left(a+b\overline{\theta}(y,t)\right)^2}\frac{\partial\overline{\theta}}{\partial y}(y,t) \\
&= \int_0^y \frac{\partial}{\partial\sigma}\left(\frac{b}{\left(a+b\overline{\theta}(\sigma,t)\right)^2}\frac{\partial\overline{\theta}}{\partial\sigma}(\sigma,t)\right)d\sigma+\frac{b}{\left(a+b\overline{\theta}(0,t)\right)^2}\frac{\partial\overline{\theta}}{\partial y}(0,t) \qquad (16) \\
&= b\int_0^y \frac{\partial\overline{\theta}}{\partial t}(\sigma,t)\,d\sigma+\frac{bh_0^*}{\sqrt{t}}\left(\overline{\theta}(0,t)-\theta_0\right), \qquad (17)
\end{aligned}
$$

and therefore the new expression for y^* is given by

$$
\begin{aligned}
y^*(y,t) &= \int_0^t\left(\int_0^y \frac{\partial}{\partial\sigma}\left(a+b\overline{\theta}(\sigma,\tau)\right)d\sigma+\frac{bh_0^*}{\sqrt{t}}\left(\overline{\theta}(0,t)-\theta_0\right)\right)d\tau \\
&\quad+\int_0^y\left(a+b\overline{\theta}(\sigma,0)\right)d\sigma \\
&= \int_0^y\left(a+b\overline{\theta}(\sigma,t)\right)d\sigma+\int_0^t\frac{bh_0^*}{\sqrt{\tau}}\left(\overline{\theta}(0,\tau)-\theta_0\right)d\tau\ . \qquad (18)
\end{aligned}
$$

Now, applying (10) and (15) the problem (6) – (9) is transformed in a Stefan-like problem with a convective boundary condition given by [27]

$$
\frac{\partial\theta^*}{\partial t^*}=\frac{\partial^2\theta^*}{\partial y^{*2}}\ ,\ bh_0^*\int_0^t\frac{\overline{\theta}(0,\tau)-\theta_0}{\sqrt{\tau}}\,d\tau<y^*<S^*(t^*)\ ,\ t^*>0 \qquad (19)
$$

$$
\begin{aligned}
&\frac{\partial\theta^*}{\partial y^*}\left(bh_0^*\int_0^t\frac{\overline{\theta}(0,\tau)-\theta_0}{\sqrt{\tau}}\,d\tau,t^*\right) \\
&= -\frac{bh_0^*}{\sqrt{\tau}}\left(\overline{\theta}(0,t)-\theta_0\right)\ \theta^*\left(bh_0^*\int_0^t\frac{\overline{\theta}(0,\tau)-\theta_0}{\sqrt{\tau}}\,d\tau,t^*\right)\ ,\ t^*>0 \qquad (20)
\end{aligned}
$$

$$
\frac{\partial\theta^*}{\partial y^*}\left(S^*(t^*),t^*\right)=\alpha^*\ \dot{S}^*(t^*)\ ,\ t^*>0 \qquad (21)
$$

$$
\theta^*\left(S^*(t^*),t^*\right)=\theta_f^*\ ,\ t^*>0\quad ,\quad S^*(0)=0 \qquad (22)
$$

where

$$
\alpha^*=\frac{\alpha b}{a\left(-\alpha b+a\right)}\qquad ,\qquad \theta_f^*=\frac{1}{a}\,. \qquad (23)
$$

Then, if we introduce the similarity variable:

$$
\xi^*=\frac{y^*}{\sqrt{2\gamma^* t^*}} \qquad (24)
$$

where γ^* is a dimensionless positive constant to be determined, and the solution is sought of the type

$$\theta^*(y^*,t^*) = \Theta^*(\xi^*) \quad , \quad S^*(t^*) = \sqrt{2\gamma^* t^*}\,, \tag{25}$$

then, we get that (19) − (22) yields

$$\frac{d^2\Theta^*}{d\xi^{*2}} + \gamma^*\xi^*\frac{d\Theta^*}{d\xi^*} = 0\,,\ \frac{bh_0^*}{\sqrt{2\gamma^* t^*}}\int_0^{t^*}\frac{\overline{\theta}(0,\tau)-\theta_0}{\sqrt{\tau}}\,d\tau < \xi^* < 1 \tag{26}$$

$$\begin{aligned}&\frac{d\Theta^*}{d\xi^*}\left(\frac{bh_0^*}{\sqrt{2\gamma^* t^*}}\int_0^{t^*}\frac{\overline{\theta}(0,\tau)-\theta_0}{\sqrt{\tau}}\,d\tau\right)\\ &= -bh_0^*\sqrt{2\gamma^*}\left(\overline{\theta}(0,\tau)-\theta_0\right)\ \Theta^*\left(\frac{bh_0^*}{\sqrt{2\gamma^* t^*}}\int_0^{t^*}\frac{\overline{\theta}(0,\tau)-\theta_0}{\sqrt{\tau}}\,d\tau\right)\end{aligned} \tag{27}$$

$$\frac{d\Theta^*}{d\xi^*}(1) = \alpha^*\,\gamma^* \tag{28}$$

$$\Theta^*(1) = \theta_f^*. \tag{29}$$

Then, in order to obtain a similarity type solution to the problem (26) − (29) we assume that

$$\overline{\theta}(0,t) = \overline{\theta}_0\ ,\ t>0 \tag{30}$$

is a constant to be determined . We define

$$\xi_1^* = \frac{bh_0^*}{\sqrt{2\gamma^* t^*}}\int_0^{t^*}\frac{\overline{\theta}(0,\tau)-\theta_0}{\sqrt{\tau}}\,d\tau = \frac{\sqrt{2}bh_0^*\left(\overline{\theta}_0-\theta_0\right)}{\sqrt{\gamma^*}}\,. \tag{31}$$

Therefore, we get that the problem (26) − (29) yields

$$\frac{d^2\Theta^*}{d\xi^{*2}} + \gamma^*\xi^*\frac{d\Theta^*}{d\xi^*} = 0\,,\ \xi_1^* < \xi^* < 1 \tag{32}$$

$$\frac{d\Theta^*}{d\xi^*}(\xi_1^*) = -\xi_1^*\gamma^*\Theta^*(\xi_1^*) \tag{33}$$

$$\frac{d\Theta^*}{d\xi^*}(1) = \alpha^*\,\gamma^* \tag{34}$$

$$\Theta^*(1) = \theta_f^* \tag{35}$$

and we get that the condition (30) is equivalent to:

$$\Theta^*(\xi_1^*) = \frac{\sqrt{2}h_0^*}{(a+b\theta_0)\sqrt{2}h_0^* + \xi_1^*\sqrt{\gamma^*}}. \tag{36}$$

The solution of the differential equation (32) is given by

$$\Theta^*(\xi^*) = A\operatorname{erf}\left(\sqrt{\frac{\gamma^*}{2}}\xi^*\right) + B \tag{37}$$

where A and B are two unknown coefficients to be determined. From (34) and (35) we get

$$A = \sqrt{\frac{\pi\gamma^*}{2}}\alpha^* \exp\left(\frac{\gamma^*}{2}\right) \tag{38}$$

$$B = \frac{1}{a} - \sqrt{\frac{\pi\gamma^*}{2}}\alpha^* \exp\left(\frac{\gamma^*}{2}\right) \operatorname{erf}\left(\sqrt{\frac{\gamma^*}{2}}\right). \tag{39}$$

The unknown constants γ^* and ξ_1^* are determined by the remaining boundary conditions (33) and (36) which yield the following system of equations

$$\operatorname{erf}(w) - \operatorname{erf}(z) = \frac{1}{\alpha^*\sqrt{\pi}}\left(\frac{\alpha^* \exp(-z^2)}{z} + \frac{\exp(-w^2)}{aw}\right) \tag{40}$$

$$\operatorname{erf}(z) - \operatorname{erf}(w) = \left(\frac{h_0^*}{(a+b\theta_0)\,h_0^* + z} - \frac{1}{a}\right)\frac{\exp(-w^2)}{\alpha^* w\sqrt{\pi}} \tag{41}$$

where we have defined the new unknowns given by

$$w = \sqrt{\frac{\gamma^*}{2}} > 0 \text{ and } z = \xi_1^* w. \tag{42}$$

If we defined

$$P(x) = x\exp(x^2)\ ,\ R_1(x) = \frac{-P(x)h_0^*}{\alpha^*\left[(a+b\theta_0)\,h_0^* + x\right]} \tag{43}$$

the equations (40) and (41) can be written as

$$P(w) = R_1(z)\ , \tag{44}$$

$$\operatorname{erf}(z) - \operatorname{erf}(w) = \frac{h_0^* b\theta_0 + z}{h_0^* P(z)}. \tag{45}$$

Lemma 2.1. *(For case $b > 0$ and $a > bl/c$) If $a, b, c, d \in \mathbb{R}^+$ and $a > bl/c$ then the system of equations (44) and (45) has a unique solution $z = \widetilde{z}_1, w = w(\widetilde{z}_1)$ where $z_0 < \widetilde{z}_1 < 0$ with $z_0 = -(a+b\theta_0)\,h_0^*$.*

Proof. In order to find the unknowns z and w, we can define from (44) that

$$w = w(z) = P^{-1}(R_1(z)). \tag{46}$$

Taking into account $(31), (42) - (44)$ and $b > 0$ we have $z_0 < z < 0$. We replace $w(z)$, given by (46), in the equation (45) and we obtain

$$\operatorname{erf}(z) - \operatorname{erf}(w(z)) = \frac{b\theta_0 h_0^* + z}{ah_0^*\sqrt{\pi}P(z)}\ ,\ z_0 < z < 0. \tag{47}$$

In order to obtain the solution of equation (47) we define the functions I and E_1 given by

$$I(z) = \operatorname{erf}(z) - \operatorname{erf}(w(z)) \tag{48}$$

$$E_1(z) = \frac{b\theta_0 h_0^* + z}{ah_0^*\sqrt{\pi}P(z)}. \tag{49}$$

It's easy to see that the functions defined above have the following properties:

(i) $R_1(z) > 0$ if $z > z_0$; $R_1(z_0) = +\infty$; $R_1(0) = 0$; $R_1(+\infty) = +\infty$; $R_1'(z) < 0$, $\forall z_0 < z < 0$.

(ii) $w(z_0) = +\infty$; $w'(z) < 0$ for $z_0 < z < 0$ and $w(0) = 0$.

(iii) $I(z) < 0\ \forall z \in (z_0, 0)$; $I(z_0) = \operatorname{erf}(z_0) - 1$; $I'(z) > 0$, for $z_0 < z < 0$.

(iv) $E_1(z_0) = \dfrac{-1}{\sqrt{\pi}P(z_0)} > 0$; $E_1(z_1) = 0$ where $z_1 = -b\theta_0 h_0^* > z_0$; $E_1(0) = -\infty$; $E_1(z) > 0\ \forall z \in (z_0, z_1)$; $E_1(z) < 0\ \forall\ z_1 < z < 0$.

Then, we have that there exists a unique $\widetilde{z}_1 \in (z_1, 0)$ such that $I(\widetilde{z}_1) = E_1(\widetilde{z}_1)$. Moreover the system of equations has a unique solution $z = \widetilde{z}_1$ and $w = w(\widetilde{z}_1) = P^{-1}(R_1(\widetilde{z}_1))$. □

Lemma 2.2. *(For case $b < 0$ and $a > -bl/c$) If $a, c, d \in \mathbb{R}^+$, $b < 0$ and $a > -\frac{bl}{c}$ then the system of equations (44) and (45) has a unique solution $\widetilde{z}_1$, $w(\widetilde{z}_1)$ with $\widetilde{z}_1 > 0$.*

Proof. In this case, from (31), (42) − (44) and $b < 0$ we have $z > 0$. We replace $w(z)$ given by (46) in the equation (45) and we obtain

$$\operatorname{erf}(z) - \operatorname{erf}(w(z)) = \frac{b\theta_0 h_0^* + z}{a h_0^* \sqrt{\pi} P(z)},\ z > 0 \tag{50}$$

which is equivalent to the equation

$$\Lambda_1(z) = \Lambda_2(z),\ z > 0$$

where

$$\Lambda_1(z) = \operatorname{erf}(z) - \frac{b\theta_0 h_0^* + z}{a h_0^* \sqrt{\pi} P(z)},\ z > 0$$

and

$$\Lambda_2(z) = \operatorname{erf}(w(z)),\ z > 0.$$

Taking into account the above hypothesis we have that functions R_1, w, Λ_1 and Λ_2 satisfy the following properties:

(i) $\Lambda_1(0) = +\infty$; $\Lambda_1(+\infty) = 1$.

(ii) There exists $z_4 > 0$ such that $\Lambda_1'(z_4) = 0$; $\Lambda_1'(z) < 0\ \forall z \in (0, z_4)$ and $\Lambda_1'(z) > 0\ \forall z \in (z_4, +\infty)$.

(iii) $R_1(0) = 0$; $R_1(+\infty) = +\infty$ and $R_1'(z) > 0$, $\forall\ z > 0$.

(iv) $w(0) = 0$; $w(+\infty) = +\infty$ and $w'(z) > 0\ \forall z > 0$.

(v) $\Lambda_2(0) = 0$; $\Lambda_2(+\infty) = 1$ and $\Lambda_2'(z) > 0\ \forall z > 0$.

(vi) $\Lambda_1(z) < \Lambda_2(z)$ for $z \to +\infty$.

Then the system of equations (44) − (45) has a unique solution $z = \widetilde{z}_1 > 0$ and $w = w(\widetilde{z}_1) > 0$. □

Lemma 2.3. *(For case $b > 0$ and $a < \frac{bl}{c}$) If $a, b, c, d \in \mathbb{R}^+$ and $a < \frac{bl}{c}$ there does not exist any solution to the equations (44) − (45).*

Proof. Taking into account (31), (42) − (44) and $\alpha^* < 0$ we have

$$z < z_0 = -(a + b\theta_0)\, h_0^* < 0$$

The equation

$$I(z) = E_1(z),\ z < z_0$$

hasn't any solution because $I(z) < 0$ and $E_1(z) > 0\ \forall z < z_0 = -(a + b\theta_0)\, h_0^*$. □

Theorem 2.4. *(General case) Let $a, c, d \in \mathbb{R}^+$. If $b > 0$ and $a > bl/c$, or $b < 0$ and $a > -\frac{bl}{c}$ then the free boundary problem (1) − (4) has a unique solution of the* *similarity type which is given by*

$$\begin{aligned} \theta(\xi) &= \frac{1}{b}\left[\frac{1}{Aerf\left(\sqrt{\frac{\gamma^*}{2}}\xi^*\right)+B}-a\right] \\ \xi &= \frac{y}{\sqrt{2\gamma t}} = \frac{\frac{2}{d}\left[(1+dx)^{\frac{1}{2}}-1\right]}{\sqrt{2\gamma t}} \\ s(t) &= \frac{1}{d}\left[\left(1+\frac{d}{2}\sqrt{2\gamma t}\right)^2-1\right] \end{aligned} \tag{51}$$

with

$$\xi = (-\alpha b + a)\int_{\xi_1^*}^{\xi^*}\left[Aerf\left(\sqrt{\frac{\gamma^*}{2}}\sigma\right)+B\right]d\sigma\ , \tag{52}$$

the coefficients A, B *are given by* (38) , (39),

$$\gamma = \frac{\gamma^*}{(-\alpha b + a)^2} \tag{53}$$

where γ^* *and* ξ_1^* *are defined by*

$$\gamma^* = 2\left[P^{-1}(R_1(\widetilde{z}_1))\right]^2\ ,\ \xi_1^* = \frac{\widetilde{z}_1}{P^{-1}(R_1(\widetilde{z}_1))} \tag{54}$$

with $\widetilde{z}_1$ *the unique solution of the equation* (47).

Remark 1. The particular case $b > 0$ and $a = bl/c$ can not be studied through a similar method developed above because the transformation (15) is not useful due to the definition of the free boundary $S^*(t^*)$ as a function of the $\overline{S}(t)$.

In order to obtain the explicit solution for the case $b > 0$ and $a > \frac{bl}{c}$, or for the case $b < 0$ and $a > -\frac{bl}{c}$ we can follow the process:
(i) There exists a unique solution $\widetilde{z}_1$ of the Eq.(47) .
(ii) We have

$$\gamma^* = 2w^2(\widetilde{z}_1) = 2\left[P^{-1}(R_1(\widetilde{z}_1))\right]^2 \text{ and } \xi_1^* = \frac{\widetilde{z}_1}{w} \tag{55}$$

with

$$w = w(\widetilde{z}_1) = P^{-1}(R_1(\widetilde{z}_1)) \tag{56}$$

are the unique solutions of the system of equations

$$\operatorname{erf}\left(\sqrt{\frac{\gamma^*}{2}}\right) - \operatorname{erf}(\xi_1^*\sqrt{\frac{\gamma^*}{2}}) = \frac{1}{\alpha^*\sqrt{\pi}}\left(\frac{\alpha^*\exp(-\xi_1^{*2}\frac{\gamma^*}{2})}{\xi_1^* w} + \frac{\exp(-\frac{\gamma^*}{2})}{a\sqrt{\frac{\gamma^*}{2}}}\right) \tag{57}$$

$$\operatorname{erf}(\xi_1^*\sqrt{\frac{\gamma^*}{2}}) - \operatorname{erf}\left(\sqrt{\frac{\gamma^*}{2}}\right) = \left(\frac{h_0^*}{(a+b\theta_0)\,h_0^* + \xi_1^*\sqrt{\frac{\gamma^*}{2}}} - \frac{1}{a}\right)\frac{\exp(-\frac{\gamma^*}{2})}{\alpha^*\sqrt{\frac{\gamma^*}{2}}} \tag{58}$$

and

$$A = \alpha^*\sqrt{\pi}P(w)\ ,\ B = \theta_f^* - \sqrt{\pi}\alpha^* P(w)\operatorname{erf}(w) \tag{59}$$

where

$$\alpha^* = \frac{\alpha b}{a(a-\alpha b)}\ ,\ \theta_f^* = \frac{1}{a}\ ,\ \alpha = \frac{l}{c}. \tag{60}$$

(iii) We have

$$\theta^*(y^*,t^*) = \Theta^*(\xi^*) = A\mathrm{erf}\left(\sqrt{\frac{\gamma^*}{2}}\xi^*\right) + B \tag{61}$$

with $\xi^* = \dfrac{y^*}{\sqrt{2\gamma^* t^*}}$ and

$$S^*(t) = \sqrt{2\gamma^* t} = 2w\sqrt{t}\,. \tag{62}$$

(iv) We compute

$$y^*(y,t) = \int_0^y \left(a + b\overline{\theta}(\sigma,t)\right) d\sigma + bh_0^* 2\sqrt{t}\left(\overline{\theta}_0 - \theta_0\right)\ ,\ h_0^* = h_0/\rho c. \tag{63}$$

where

$$\overline{\theta}_0 = \frac{1}{b}\left(\frac{1}{A\mathrm{erf}\left(\sqrt{\frac{\gamma^*}{2}}\xi_1^*\right) + B} - a\right) = \theta_0 + \frac{\widetilde{z}_1}{bh_0^* w^2} < \theta_0.$$

(v) We have

$$\overline{S}(t) = \frac{S^*(t)}{a-\alpha b} = \frac{\sqrt{2\gamma^* t}}{a-\alpha b} = \frac{2w\sqrt{t}}{a-\alpha b}\ , \tag{64}$$

$$\begin{aligned} \frac{1}{a+b\overline{\theta}(y,t)} &= \theta^*(y^*,t) = \Theta^*(\xi^*) = A\mathrm{erf}\left(\frac{y^*}{2\sqrt{t}}\right) + B \\ &= B + A\mathrm{erf}\left[\frac{1}{2\sqrt{t}}\int_0^y \left(a+b\overline{\theta}(\sigma,t)\right)d\sigma + bh_0^*\left(\overline{\theta}_0 - \theta_0\right)\right] \end{aligned} \tag{65}$$

which is an integral equation for $\overline{\theta} = \overline{\theta}(y,t)$ where $t > 0$ is a parameter.

(vi) The free boundary $s = s(t)$ is given by:

$$\begin{aligned} s(t) &= \frac{1}{d}\left[\left(1 + \frac{d}{2}\overline{S}(t)\right)^2 - 1\right] = \frac{1}{d}\left[\left(1 + \frac{d}{a-\alpha b}\sqrt{\frac{\gamma^*}{2}}\sqrt{t}\right)^2 - 1\right] \\ &= \frac{w}{a-\alpha b}\left[2\sqrt{t} + \frac{dw}{a-\alpha b}t\right]. \end{aligned} \tag{66}$$

(vii) If we define

$$Y(y,t) = bh_0^*\left(\overline{\theta}_0 - \theta_0\right) + \frac{1}{2\sqrt{t}}\int_0^y \left(a + b\overline{\theta}(\sigma,t)\right) d\sigma \tag{67}$$

then we obtain

$$\frac{\partial Y}{\partial y}(y,t) = \frac{a + b\overline{\theta}(y,t)}{2\sqrt{t}} \tag{68}$$

that is $Y = Y(y,t)$ satisfies the following Cauchy problem in variable y :

$$\frac{\partial Y}{\partial y}(y,t) = \frac{1}{2\sqrt{t}}\left(\frac{1}{B + A\mathrm{erf}(Y(y,t))}\right), 0 < y < \overline{S}(t), t > 0 \tag{69}$$

$$Y(0,t) = bh_0^*\left(\overline{\theta}_0 - \theta_0\right)\ , \tag{70}$$

where $t > 0$ is a parameter.

$(viii)$ The temperature $\overline{\theta} = \overline{\theta}\,(y,t)$ is given by:

$$\overline{\theta}\,(y,t) = \frac{1}{b}\left[\frac{1}{B + A\mathrm{erf}(Y(y,t))} - a\right]\ ,\ 0 < y < \overline{S}(t)\ ,\ t > 0 \tag{71}$$

as a function of Y.

(ix) The temperature $\theta = \theta(x,t)$ is given by:

$$\begin{aligned}\theta(x,t) &= \overline{\theta}\left(\frac{2}{d}\left(\sqrt{1+dx}-1\right),t\right)\\ &= \frac{1}{b}\left[\frac{1}{B + A\mathrm{erf}(Y(\frac{2}{d}\left(\sqrt{1+dx}-1\right),t))} - a\right], 0 < x < s(t), t > 0\end{aligned} \tag{72}$$

as a function of Y where $s(t)$ as defined in (66).

Theorem 2.5. *Let us consider the hypothesis $a, c, d \in \mathbb{R}^+$ with $b > 0$ and $a > \frac{bl}{c}$, or $b < 0$ and $a > -\frac{bl}{c}$. Let $\widetilde{z}_1$ be the unique solution of the Eq. (47) and $\gamma^* = 2w^2$ and $\xi_1^* = \dfrac{\widetilde{z}_1}{w}$ (with $w = P^{-1}\left(R_1\left(\widetilde{z}_1\right)\right)$) the unique solutions of the system of equations (57) − (58) or (44) − (45). Let A and B be the coefficients defined by (59). Then we have:*

(i) There exists a unique solution $Y = Y(y,t)$ of the Cauchy problem (69) − (70) for all $t \geq t_0 > 0$ with t_0 is an arbitrary positive time.

(ii) There exists a unique solution $\theta = \theta(x,t)$ and $s = s(t)$ of the free boundary problem (1) − (4) given by (72) and (66) respectively for $t \geq t_0 > 0$ with t_0 is an arbitrary positive time.

Proof. It is sufficient to prove that the Cauchy problem (69) − (70) has a unique solution for $t \geq t_0 > 0$. The ordinary differential equation, with parameter $t > 0$, can be written as

$$\frac{\partial Y}{\partial y}(y,t) = G(y,t) \tag{73}$$

where

$$G(y,t) = \frac{1}{2\sqrt{t}}\left(\frac{1}{B + A\mathrm{erf}(Y(y,t))}\right) \tag{74}$$

satisfies the condition

$$\left|\frac{\partial G}{\partial y}(y,t)\right| \leq Const \quad \forall t \geq t_0 > 0 \tag{75}$$

with $t_0 > 0$ an arbitrary positive time. □

In order to have an algorithm to compute the explicit solution we can give the following process:

Remark 2. (Algorithm in order to compute the explicit solution) For the cases $b > 0$ and $a > \frac{bl}{c}$, or $b < 0$ and $a > -\frac{bl}{c}$ we can obtain the explicit solution $\theta = \theta(x,t)$ and $s = s(t)$ of the free boundary problem (1) − (4) by the following process:

(i) Compute $\widetilde{z}_1$ the unique solution of the Eq. (47).

(ii) Compute

$$w = P^{-1}\left(R_1\left(\widetilde{z}_1\right)\right) \tag{76}$$

$$\gamma^* = 2w^2 = 2\left[P^{-1}\left(R_1\left(\widetilde{z}_1\right)\right)\right]^2 \text{ and } \xi_1^* = \frac{\widetilde{z}_1}{w} \tag{77}$$

and

$$A = \alpha^* \sqrt{\pi} P(w) \ , \ B = \theta_f^* - \sqrt{\pi}\alpha^* P(w) \mathrm{erf}(w) \tag{78}$$

where α^*, θ_f^* are defined in (23).

(iii) Fix t_0 as an arbitrary positive time and compute $Y = Y(y,t)$ as the unique solution of the Cauchy problem $(69) - (70)$ for $t \geq t_0 > 0$.

(iv) Compute the free boundary $s = s(t)$ by the explicit expression (66).

(v) Compute the temperature $\theta = \theta(x,t)$ by the explicit expression (72).

(vi) Compute the constant temperature $\theta(0,t)$ at the fixed face by the expression

$$\theta(0,t) = \overline{\theta}_0 = \theta_0 + \frac{\widetilde{z}_1}{b h_0^* w^2} < \theta_0, \ \forall t \geq t_0.$$

Remark 3. The particular case $d = 0$ can not be solve through a similar method developed above because the transformation (15) is the identity when $d \to 0$. The free boundary problem $(1), (3) - (5)$ for the particular case $d = 0$ with temperature or a heat flux at the fixed face $x = 0$ was solved in [21]; the free boundary problem $(1) - (5)$ for the case $d = 0$ (with the convective boundary condition (2) at the fixed face $x = 0$) is an open problem.

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