EXISTENCE AND UNIQUENESS FOR ONE-PHASE STEFAN PROBLEMS OF NON-CLASSICAL HEAT EQUATIONS WITH TEMPERATURE BOUNDARY CONDITION AT A FIXED FACE

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Abstract. We prove the existence and uniqueness, local in time, of a solution for a one-phase Stefan problem of a non-classical heat equation for a semi-infinite material with temperature boundary condition at the fixed face. We use the Friedman-Rubinstein integral representation method and the Banach contraction theorem in order to solve an equivalent system of two Volterra integral equations.

1. Introduction

The one-phase Stefan problem for a semi-infinite material for the classical heat equation requires the determination of the temperature distribution u of the liquid phase (melting problem) or of the solid phase (solidification problem), and the evolution of the free boundary $x = s(t)$. Phase-change problems appear frequently in industrial processes and other problems of technological interest [2, 3, 6, 8, 9, 10, 11, 12, 18, 29]. A large bibliography on the subject was given in [25].

Non-classical heat conduction problem for a semi-infinite material was studied in [4, 7, 17, 27, 28], e.g. problems of the type

$$u_t - u_{xx} = -F(u_x(0,t)), \quad x > 0, \ t > 0,$$
$$u(0,t) = 0, \quad t > 0 \tag{1.1}$$
$$u(x,0) = h(x), \quad x > 0$$

where $h(x), x > 0$, and $F(V), V \in \mathbb{R}$, are continuous functions. The function F, henceforth referred as control function, is assumed to satisfy the condition

(H1) $F(0) = 0$.

As observed in [27, 28], the heat flux $w(x,t) = u_x(x,t)$ for problem (1.1) satisfies a classical heat conduction problem with a nonlinear convective condition at $x = 0$,

which can be written in the form

$$
\begin{aligned}
w_t - w_{xx} &= 0, \quad x > 0,\ t > 0, \\
w_x(0,t) &= F(w(0,t)), \quad t > 0, \\
w(x,0) &= h'(x) \geq 0, \quad x > 0.
\end{aligned}
\tag{1.2}
$$

The literature concerning problem (1.2) has increased rapidly since the publication of the papers [19, 21, 22]. Related problems have been also studied; see for example [1, 14, 16]. In [26], a one-phase Stefan problem for a non-classical heat equation for a semi-infinite material was presented. There the free boundary problem consists in determining the temperature $u = u(x,t)$ and the free boundary $x = s(t)$ with a control function F which depends on the evolution of the heat flux at the extremum $x = 0$ is given by the conditions

$$
\begin{aligned}
u_t - u_{xx} &= -F(u_x(0,t)), \quad 0 < x < s(t),\ 0 < t < T, \\
u(0,t) &= f(t) \geqslant 0, \quad 0 < t < T, \\
u(s(t),t) &= 0, \quad u_x(s(t),t) = -\dot{s}(t), \quad 0 < t < T, \\
u(x,0) &= h(x) \geq 0, \quad 0 \leqslant x \leqslant b = s(0) \quad (b > 0).
\end{aligned}
\tag{1.3}
$$

The goal in this paper is to prove the existence and uniqueness, local in time, of a solution to the one-phase Stefan problem (1.3) for a non-classical heat equation with temperature boundary condition at the fixed face $x = 0$. First, we prove that problem (1.3) is equivalent to a system of two Volterra integral equations (2.4)-(2.5) following the Friedman-Rubinstein's method given in [13, 23]. Then, we prove that the problem (2.4)-(2.5) has a unique local solution by using the Banach contraction theorem.

2. Existence and Uniqueness of Solutions

We have the following equivalence for the existence of solutions to the non-classical free boundary problem (1.3).

Theorem 2.1. *The solution of the free-boundary problem* (1.3) *is*

$$
\begin{aligned}
u(x,t) = {}&\int_0^b G(x,t;\xi,0)h(\xi)d\xi + \int_0^t G_\xi(x,t;0,\tau)f(\tau)\,d\tau \\
&+ \int_0^t G(x,t;s(\tau),\tau)v(\tau)\,d\tau - \iint_{D(t)} G(x,t;\xi,\tau)F(V(\tau))d\xi\,d\tau,
\end{aligned}
\tag{2.1}
$$

$$
s(t) = b - \int_0^t v(\tau)\,d\tau,
\tag{2.2}
$$

where $D(t) = \{(x,\tau) : 0 < x < s(\tau), 0 < \tau < t\}$, *with* $f \in C^1[0,T]$, $h \in C^1[0,b]$, $h(b) = 0$, $h(0) = f(0)$, F *is a Lipschitz function over* $C^0[0,T]$, *and the functions* $v \in C^0[0,T]$, $V \in C^0[0,T]$ *defined by*

$$
v(t) = u_x(s(t),t), \quad V(t) = u_x(0,t)
\tag{2.3}
$$

must satisfy the following system of Volterra integral equations

$$v(t) = 2 \int_0^b N(s(t), t; \xi, 0) h'(\xi) d\xi - 2 \int_0^t N(s(t), t; 0, \tau) \dot{f}(\tau) \, d\tau$$

$$+ 2 \int_0^t G_x(s(t), t; s(\tau), \tau) v(\tau) \, d\tau \tag{2.4}$$

$$+ 2 \int_0^t [N(s(t), t; s(\tau), \tau) - N(s(t), t; 0, \tau)] F(V(\tau)) \, d\tau.$$

$$V(t) = \int_0^b N(0, t; \xi, 0) h'(\xi) d\xi$$

$$- \int_0^t N(0, t; 0, \tau) \dot{f}(\tau) \, d\tau + \int_0^t G_x(0, t; s(\tau), \tau) v(\tau) \, d\tau \tag{2.5}$$

$$+ \int_0^t [N(0, t; s(\tau), \tau) - N(0, t; 0, \tau)] F(V(\tau)) \, d\tau,$$

where G, N are the Green and Neumann functions and K is the fundamental solution of the heat equation, defined respectively by

$$G(x, t, \xi, \tau) = K(x, t, \xi, \tau) - K(-x, t, \xi, \tau),$$

$$N(x, t, \xi, \tau) = K(x, t, \xi, \tau) + K(-x, t, \xi, \tau),$$

$$K(x, t, \xi, \tau) = \begin{cases} \frac{1}{2\sqrt{\pi(t-\tau)}} \exp\left(-\frac{(x-\xi)^2}{4(t-\tau)} \right) & t > \tau \\ 0 & t \le \tau \, , \end{cases}$$

where $s(t)$ is given by (2.2),

Proof. Let $u(x, t)$ be the solution to (1.3). We integrate, on the domain $D_{t,\varepsilon} = \{(\xi, \tau) : 0 < \xi < s(\tau), \varepsilon < \tau < t - \varepsilon\}$, the Green identity

$$(Gu_\xi - uG_\xi)_\xi - (Gu)_\tau = GF(u_\xi(0, \tau)) \, . \tag{2.6}$$

Now we let $\varepsilon \to 0$, to obtain the following integral representation for $u(x, t)$,

$$u(x, t) = \int_0^b G(x, t; \xi, 0) h(\xi) d\xi + \int_0^t G_\xi(x, t; 0, \tau) f(\tau) \, d\tau$$

$$+ \int_0^t G(x, t; s(\tau), \tau) \, u_\xi(s(\tau), \tau) \, d\tau - \iint_{D(t)} G(x, t; \xi, \tau) F(u_\xi(0, \tau)) d\xi \, d\tau \, .$$

From the definition of $v(t)$ and $V(t)$ by (2.3), we obtain (2.1) and (2.2). If we differentiate $u(x, t)$ in variable x and we let $x \to 0^+$ and $x \to s(t)$, by using the jump relations, we obtain the integral equations for v and V given by (2.4) and (2.5).

　　Conversely, the function $u(x, t)$ defined by (2.1) where v and V are the solutions of (2.4)and (2.5), satisfy the conditions (1.3) (i),(ii),(iv) and (v). In order to prove condition (1.3) (iii) we define $\psi(t) = u(s(t), t)$. Taking into account that u satisfy the conditions (1.3) (i),(ii),(iv) and (v), if we integrate the Green identity (2.6) over

the domain $D_{t,\varepsilon}$, $(\varepsilon > 0)$ and we let $\varepsilon \to 0$ we obtain that

$$u(x,t) = \int_0^b G(x,t;\xi,0)h(\xi)d\xi + \int_0^t G(x,t;s(\tau),\tau)v(\tau)\,d\tau$$
$$+ \int_0^t \psi(\tau)[G_x(x,t;s(\tau),\tau) - G(x,t;s(\tau),\tau)v(\tau)]\,d\tau$$
$$+ \int_0^t G_\xi(x,t;0,\tau)f(\tau)\,d\tau - \iint_{D(t)} G(x,t;\xi,\tau)F(V(\tau))d\xi\,d\tau.$$

Then, if we compare this last expression with (2.1), we deduce that

$$M(x,t) = \int_0^t \psi(\tau)[G_x(x,t;s(\tau),\tau) - G(x,t;s(\tau),\tau)v(\tau)]\,d\tau \equiv 0 \qquad (2.7)$$

for $0 < x < s(t)$, $0 < t < \sigma$. We let $x \to s(t)$ in (2.7) and by using the jump relations we have that ψ satisfy the integral equation

$$\frac{1}{2}\psi(t) + \int_0^t \psi(\tau)[G_x(s(t),t;s(\tau),\tau) - G(s(t),t;s(\tau),\tau)v(\tau)]\,d\tau = 0.$$

Then we deduce that

$$|\psi(t)| \leq C \int_0^t \frac{|\psi(\tau)|}{\sqrt{t-\tau}}\,d\tau$$
$$\leq C^2 \int_0^t \frac{d\tau}{\sqrt{t-\tau}} \int_0^\tau \frac{|\psi(\eta)|}{\sqrt{\tau-\eta}}d\eta$$
$$= C^2 \int_0^t |\psi(\eta)|d\eta \int_\eta^t \frac{d\tau}{[(t-\tau)(\tau-\eta)]^{1/2}}$$
$$= \pi C^2 \int_0^t |\psi(\eta)|d\eta$$

where $C = C(t)$; therefore by using the Gronwall inequality we have that $\psi(t) = 0$ over $[0,\sigma]$. $\qquad\square$

Next, we use the Banach fixed point theorem in order to prove the local existence and uniqueness of solution $v, V \in C^0[0,\sigma]$ to the system of two Volterra integral equations (2.4)-(2.5) where σ is a positive small number. Consider the Banach space

$$C_{M,\sigma} = \big\{\vec{w} = \begin{pmatrix} v \\ V \end{pmatrix} : v, V : [0,\sigma] \to \mathbb{R}, \text{ continuous, with } \|\vec{w}\|_\sigma \leq M\big\}$$

with

$$\|\vec{w}\|_\sigma := \|v\|_\sigma + \|V\|_\sigma := \max_{t\in[0,\sigma]} |v(t)| + \max_{t\in[0,\sigma]} |V(t)|$$

We define $A : C_{M,\sigma} \longrightarrow C_{M,\sigma}$, such that

$$\vec{\tilde{w}}(t) = A(\vec{w}(t)) = \begin{pmatrix} A_1(v(t),V(t)) \\ A_2(v(t),V(t)) \end{pmatrix}$$

where

$$A_1(v(t),V(t)) = F_0(v(t)) + 2\int_0^t [N(s(t),t,s(\tau),\tau) - N(s(t),t,0,\tau)]F(V(\tau))\,d\tau$$
$$(2.8)$$

with

$$F_0(v(t)) = 2 \int_0^b N(s(t), t, \xi, 0) h'(\xi) d\xi - 2 \int_0^t N(s(t), t, 0, \tau) \dot{f}(\tau) \, d\tau$$
$$+ 2 \int_0^t G_x(s(t), t, s(\tau), \tau) v(\tau) \, d\tau$$

and

$$A_2(v(t), V(t)) = \int_0^b N(0, t, \xi, 0) h'(\xi) d\xi - \int_0^t N(0, t, 0, \tau) \dot{f}(\tau) \, d\tau$$
$$+ \int_0^t G_x(0, t, s(\tau), \tau) v(\tau) \, d\tau \tag{2.9}$$
$$+ \int_0^t [N(0, t, s(\tau), \tau) - N(0, t, 0, \tau)] F(V(\tau)) \, d\tau.$$

Lemma 2.2. *If* $v \in C^0[0, \sigma]$, $\max_{t \in [0,\sigma]} |v(t)| \leq M$ *and* $2M\sigma \leq b$ *then* $s(t)$ *defined by (2.2) satisfies*

$$|s(t) - s(\tau)| \leq M|t - \tau| \quad |s(t) - b| \leq \frac{b}{2}, \quad \forall t, \tau \in [0, \sigma].$$

To prove the following Lemmas we need the inequality

$$\exp\left(\frac{-x^2}{\alpha(t-\tau)}\right) / (t-\tau)^{n/2} \leq \left(\frac{n\alpha}{2ex^2}\right)^{n/2}, \quad \alpha, x > 0, \ t > \tau, \ n \in \mathbb{N}. \tag{2.10}$$

Lemma 2.3. *Let* $\sigma \leq 1$, $M \geq 1$, $f \in C^1[0, T]$, $h \in C^1[0, b]$, F *a Lipschitz function over* $C^0[0, T]$. *Under the hypothesis of Lemma 2.2, we have the following properties:*

$$\int_0^t |N(s(t), t, 0, \tau)||\dot{f}(\tau)| \, d\tau \leq \|\dot{f}\|_t C_1(b) t \tag{2.11}$$

$$\int_0^t |G_x(s(t), t, s(\tau), \tau)||v(\tau)| \, d\tau \leq M^2 C_2(b) \sqrt{t} \tag{2.12}$$

$$\int_0^b |N(s(t), t, \xi, 0)||h'(\xi)| d\xi \leq \|h'\| \tag{2.13}$$

$$\int_0^t |N(s(t), t, s(\tau), \tau) - N(s(t), t, 0, \tau)||F(V(\tau))| \, d\tau \leq C_4(L) M \sqrt{t} \tag{2.14}$$

$$\int_0^b |N(0, t, \xi, 0)||h'(\xi)| d\xi \leq \|h'\| \tag{2.15}$$

$$\int_0^t |N(0, t, 0, \tau)||\dot{f}(\tau)| \, d\tau \leq \frac{2\|\dot{f}\|_\sigma}{\sqrt{\pi}} \sqrt{t} \tag{2.16}$$

$$\int_0^t |G_x(0, t, s(\tau), \tau)||v(\tau)| \, d\tau \leq C_3(b) M t \tag{2.17}$$

$$\int_0^t |N(0, t, s(\tau), \tau) - N(0, t, 0, \tau)||F(V(\tau))| \, d\tau \leq C_4(L) M \sqrt{t} \tag{2.18}$$

where L is the Lipschitz constant of F and

$$C_1(b) = (\frac{8}{eb^2})^{1/2}\frac{1}{\sqrt{\pi}}, \quad C_2(b) = \frac{1}{2\sqrt{\pi}} + \frac{3b}{4\sqrt{\pi}}(\frac{2}{3eb^2})^{3/2}$$

$$C_3(b) = \frac{3b}{8\sqrt{\pi}}(\frac{24}{eb^2})^{3/2}, \quad C_4(L) = \frac{4L}{\sqrt{\pi}}.$$

(2.19)

Proof. To prove (2.11), we have

$$|N(s(t),t,0,\tau)| = |K(s(t),t,0,\tau) + K(-s(t),t,0,\tau)| = 2K(s(t),t,0,\tau)$$

$$= \exp\left(\frac{-s^2(t)}{4(t-\tau)}\right)\frac{(t-\tau)^{-1/2}}{\sqrt{\pi}}$$

$$\leq \exp\left(\frac{-b^2}{16(t-\tau)}\right)\frac{(t-\tau)^{-1/2}}{\sqrt{\pi}}$$

$$\leq (\frac{8}{eb^2})^{1/2}\frac{1}{\sqrt{\pi}} = C_1(b)$$

then (2.11) holds. To prove (2.12), we have

$$|G_x(s(t),t,s(\tau),\tau)| = \left|K_x(s(t),t,s(\tau),\tau) + K_x(-s(t),t,s(\tau),\tau)\right|$$

$$= \frac{(t-\tau)^{-3/2}}{4\sqrt{\pi}}\left|(s(t)-s(\tau))\exp\left(\frac{-(s(t)-s(\tau))^2}{4(t-\tau)}\right)\right.$$

$$\left. - (s(t)+s(\tau))\exp\left(\frac{-(s(t)+s(\tau))^2}{4(t-\tau)}\right)\right|$$

$$\leq \frac{(t-\tau)^{-3/2}}{4\sqrt{\pi}}\left(M(t-\tau) + 3b\exp\left(\frac{-9b^2}{4(t-\tau)}\right)\right)$$

$$\leq \frac{1}{4\sqrt{\pi}}\left(M(t-\tau)^{-1/2} + 3b(\frac{2}{3eb^2})^{3/2}\right).$$

Then

$$\int_0^t |G_x(s(t),t,s(\tau),\tau)||v(\tau)|\,d\tau \leq \frac{M}{4\sqrt{\pi}}\left(2M\sqrt{t} + 3b(\frac{2}{3eb^2})^{3/2}t\right)$$

$$\leq M^2\sqrt{t}\left(\frac{1}{2\sqrt{\pi}} + \frac{3b}{M4\sqrt{\pi}}(\frac{2}{3eb^2})^{3/2}\right)$$

$$\leq M^2C_2(b)\sqrt{t},$$

which implies (2.12). To prove (2.13), we have

$$\int_0^b |N(s(t),t,\xi,0)||h'(\xi)|d\xi \leq \|h'\|\int_0^\infty |N(s(t),t,\xi,0)|d\xi \leq \|h'\|$$

because

$$\int_0^\infty |N(s(t),t,\xi,0)|d\xi \leq 1.$$

To prove (2.14), by taking into account that

$$|N(s(t),t,s(\tau),\tau) - N(s(t),t,0,\tau)| \leq \frac{2}{\sqrt{\pi(t-\tau)}}$$

we obtain

$$\int_0^t |N(s(t),t,s(\tau),\tau) - N(s(t),t,0,\tau)||F(V(\tau))|\,d\tau \leq LM \int_0^t \frac{2}{\sqrt{\pi(t-\tau)}}\,d\tau$$
$$= C_4(L)M\sqrt{t}.$$

The inequality (2.15) is prove in the same way as (2.13). To prove (2.16), we have

$$\int_0^t |N(0,t,0,\tau)||\dot{f}(\tau)|\,d\tau \leq \|\dot{f}\|_\sigma \int_0^t |N(0,t,0,\tau)|\,d\tau$$
$$= \|\dot{f}\|_\sigma \int_0^t \frac{1}{\sqrt{\pi(t-\tau)}}\,d\tau$$
$$= \frac{\|\dot{f}\|_\sigma}{\sqrt{\pi}}2\sqrt{t}.$$

To prove (2.17), we have

$$|G_x(0,t,s(\tau),\tau)| = \frac{(t-\tau)^{-3/2}}{4\sqrt{\pi}}\,s(\tau)\exp\left(\frac{-(s(\tau))^2}{4(t-\tau)}\right)$$
$$\leq \frac{3b}{8\sqrt{\pi}}(t-\tau)^{-3/2}\exp\left(\frac{-b^2}{16(t-\tau)}\right)$$
$$\leq \frac{3b}{8\sqrt{\pi}}\left(\frac{24}{eb^2}\right)^{3/2}.$$

To prove (2.18), as in (2.14), we prove that

$$|N(0,t,s(\tau),\tau) - N(0,t,0,\tau)| \leq \frac{2}{\sqrt{\pi(t-\tau)}}$$

and therefore (2.18) holds. □

Lemma 2.4. *Let s_1, s_2 be the functions corresponding to v_1, v_2 in $C^0[0,\sigma]$, respectively, with $\max_{t\in[0,\sigma]} |v_i(t)| \leq M$, $i = 1,2$, Then we have*

$$\begin{aligned}
|s_2(t) - s_1(t)| &\leq t\|v_2 - v_1\|_t, \\
|s_i(t) - s_i(\tau)| &\leq M|t-\tau|, \quad i = 1,2, \\
\frac{b}{2} \leq s_i(t) &\leq \frac{3b}{2}, \quad \forall t \in [0,\sigma], \ i = 1,2.
\end{aligned} \tag{2.20}$$

Lemma 2.5. *Let $f \in C^1[0,T]$, $h \in C^1[0,b]$, F a Lipschitz function in $C^0[0,T]$. We have*

$$|F_0(v_2(t)) - F_0(v_1(t))| \leq E(b,h,f)\sqrt{t}\|v_2 - v_1\|_t; \qquad (2.21)$$

$$\int_0^t |N(s_2(t),t,s_2(\tau),\tau) - N(s_2(t),t,0,\tau)||F(V_2(\tau)) - F(V_1(\tau))|\, d\tau \qquad (2.22)$$
$$\leq C_4(L)\sqrt{t}\|V_2 - V_1\|_t;$$

$$\int_0^t |N(s_2(t),t,0,\tau) - N(s_1(t),t,0,\tau)||F(V_1(\tau))|\, d\tau \qquad (2.23)$$
$$\leq C_5(b,L,M)t\|v_2 - v_1\|_t;$$

$$\int_0^t |N(s_2(t),t,s_2(\tau),\tau) - N(s_1(t),t,s_1(\tau),\tau)||F(V_1(\tau))|\, d\tau \qquad (2.24)$$
$$\leq [C_6(L,M)\sqrt{t} + C_7(b,L,M)t]\|v_2 - v_1\|_t;$$

$$\int_0^t |G_x(0,t,s_2(\tau),\tau)||v_2(\tau) - v_1(\tau)|\, d\tau \leq C_8(b)t\|v_2 - v_1\|_t; \qquad (2.25)$$

$$\int_0^t |G_x(0,t,s_2(\tau),\tau)v_2(\tau) - G_x(0,t,s_1(\tau),\tau)v_1(\tau)|\, d\tau \qquad (2.26)$$
$$\leq (C_8(b)t + C_9(b,M)t^2)\|v_2 - v_1\|_t;$$

$$\int_0^t \Big| [N(0,t,s_2(\tau),\tau) - N(0,t,0,\tau)]F(V_2(\tau))$$
$$- [N(0,t,s_1(\tau),\tau) - N(0,t,0,\tau)]F(V_1(\tau)) \Big|\, d\tau \qquad (2.27)$$
$$\leq C_4(L)\sqrt{t}\|V_2 - V_1\|_t + C_5(b,L,M)t^2\|v_2 - v_1\|_t,$$

where the constants are defined by

$$C_4(L) = \frac{4L}{\sqrt{\pi}}, \quad C_5(b,L,M) = LM\frac{3b}{8\sqrt{\pi}}\Big(\frac{24}{eb^2}\Big)^{3/2},$$

$$C_6(L,M) = \frac{LM^3}{\sqrt{\pi}}, \quad C_7(b,L,M) = \Big(\frac{6}{eb^2}\Big)^{3/2}\frac{3bLM^2}{2\sqrt{\pi}}, \qquad (2.28)$$

$$C_8(b) = \frac{3}{4\sqrt{\pi}}\Big(\frac{24}{eb^2}\Big)^{3/2}, \quad C_9(b,M) = \Big[\Big(\frac{40}{eb^2}\Big)^{\frac{5}{2}}\frac{9b^2}{16\sqrt{\pi}} + \frac{1}{2\sqrt{\pi}}\Big(\frac{24}{eb^2}\Big)^{3/2}\Big]\frac{M}{2}.$$

Proof. The proof of (2.21) can be found in [13]. To prove (2.22), we have

$$|N(s_2(t),t,s_2(\tau),\tau) - N(s_2(t),t,0,\tau)| \leq \frac{2}{\sqrt{\pi(t-\tau)}}.$$

Then

$$\int_0^t |N(s_2(t),t,s_2(\tau),\tau) - N(s_2(t),t,0,\tau)||F(V_2(\tau)) - F(V_1(\tau))|\, d\tau$$
$$\leq \frac{4L}{\sqrt{\pi}}\sqrt{t}\|V_2 - V_1\|_t$$

To prove (2.23), we use the mean value theorem: There exists $c = c(t, \tau)$ between $s_1(t)$ and $s_2(t)$ such that

$$
\begin{aligned}
&|N(s_2(t), t, 0, \tau) - N(s_1(t), t, 0, \tau)||F(V_1(\tau))| \\
&= |N_x(c, t, 0, \tau)||s_2(\tau) - s_1(\tau)||F(V_1(\tau))| \\
&\leq |c| \exp\big(-\frac{c^2}{4(t-\tau)}\big) \frac{(t-\tau)^{-3/2}}{2\sqrt{\pi}} LM\tau|v_2(\tau) - v_1(\tau)| \\
&\leq \frac{3b}{4\sqrt{\pi}} \exp\big(-\frac{b^2}{16(t-\tau)}\big)(t-\tau)^{-3/2} LM\tau|v_2(\tau) - v_1(\tau)| \\
&\leq \frac{3b}{4\sqrt{\pi}} \big(\frac{24}{eb^2}\big)^{3/2} LM\tau|v_2(\tau) - v_1(\tau)|.
\end{aligned}
$$

Then

$$
\begin{aligned}
&\int_0^t |N(s_2(t), t, 0, \tau) - N(s_1(t), t, 0, \tau)||F(V_1(\tau))| \, d\tau \\
&\leq \frac{3b}{8\sqrt{\pi}} \big(\frac{24}{eb^2}\big)^{\frac{3}{2}} LMt\|v_2 - v_1\|_t = C_5(b, L, M)t\|v_2 - v_1\|_t.
\end{aligned}
$$

To prove (2.24), we have

$$
\begin{aligned}
&N(s_2(t), t, s_2(\tau), \tau) - N(s_1(t), t, s_1(\tau), \tau) \\
&= K(s_2(t), t, s_2(\tau), \tau) - K(s_1(t), t, s_1(\tau), \tau) \\
&\quad + K(-s_2(t), t, s_2(\tau), \tau) - K(-s_1(t), t, s_1(\tau), \tau).
\end{aligned}
$$

As in [24], for each (t, τ), $0 < \tau < t$, we define

$$
f_{t,\tau}(x) = \exp\big(\frac{-x^2}{4(t-\tau)}\big).
$$

Then we have

$$
\begin{aligned}
&K(s_2(t), t, s_2(\tau), \tau) - K(s_1(t), t, s_1(\tau), \tau) \\
&= \frac{(t-\tau)^{-1/2}}{2\sqrt{\pi}} \Big[\exp\big(-\frac{(s_2(t) - s_2(\tau))^2}{4(t-\tau)}\big) - \exp\big(-\frac{(s_1(t) - s_1(\tau))^2}{4(t-\tau)}\big)\Big] \\
&= \frac{(t-\tau)^{-1/2}}{2\sqrt{\pi}} \big[f_{t,\tau}(s_2(t) - s_2(\tau)) - f_{t,\tau}(s_1(t) - s_1(\tau))\big]
\end{aligned}
$$

and

$$
\begin{aligned}
&K(-s_2(t), t, s_2(\tau), \tau) - K(-s_1(t), t, s_1(\tau), \tau) \\
&= \frac{(t-\tau)^{-1/2}}{2\sqrt{\pi}} \Big[\exp\big(-\frac{(s_2(t) + s_2(\tau))^2}{4(t-\tau)}\big) - \exp\big(-\frac{(s_1(t) + s_1(\tau))^2}{4(t-\tau)}\big)\Big] \\
&= \frac{(t-\tau)^{-1/2}}{2\sqrt{\pi}} \big[f_{t,\tau}(s_2(t) + s_2(\tau)) - f_{t,\tau}(s_1(t) + s_1(\tau))\big]
\end{aligned}
$$

By the mean value theorem there exists $c = c(t, \tau)$ between $s_2(t) - s_2(\tau)$ and $s_1(t) - s_1(\tau)$ such that

$$f_{t,\tau}(s_2(t) - s_2(\tau)) - f_{t,\tau}(s_1(t) - s_1(\tau))$$
$$= f'_{t,\tau}(c)(s_2(t) - s_2(\tau) - s_1(t) + s_1(\tau))$$
$$= \frac{-c}{2(t-\tau)} \exp(-\frac{c^2}{4(t-\tau)})(s_2(t) - s_2(\tau) - s_1(t) + s_1(\tau))$$

Taking into account that

$$|c| \leq \max\{|s_i(t) - s_i(\tau)|, i = 1, 2\} \leq M(t - \tau)$$

it results

$$|f_{t,\tau}(s_2(t) - s_2(\tau)) - f_{t,\tau}(s_1(t) - s_1(\tau))| \leq \frac{M}{2}[|s_2(t)) - s_1(t)| + |s_2(\tau) - s_1(\tau)|]$$
$$\leq M^2\|v_2 - v_1\|_t.$$

Then we have

$$|K(s_2(t), t, s_2(\tau), \tau) - K(s_1(t), t, s_1(\tau), \tau)| \leq \frac{M^2}{2\sqrt{\pi(t-\tau)}}\|v_2 - v_1\|_t.$$

In the same way we have

$$f_{t,\tau}(s_2(t) + s_2(\tau)) - f_{t,\tau}(s_1(t) + s_1(\tau))$$
$$= f'_{t,\tau}(c^*)(s_2(t) + s_2(\tau) - s_1(t) - s_1(\tau))$$
$$= \frac{-c^*}{2(t-\tau)} \exp(-\frac{c^{*2}}{4(t-\tau)})(s_2(t) + s_2(\tau) - s_1(t) - s_1(\tau))$$

where $c^* = c^*(t, \tau)$ is between $s_2(t) + s_2(\tau)$ and $s_1(t) + s_1(\tau)$. Since $s_1(t) + s_1(\tau) \leq c^* \leq s_2(t) + s_2(\tau)$, (or viceversa), we deduce that $b \leq c^* \leq 3b$, that is $\exp(-c^{*2}/4(t-\tau)) \leq \exp(-b^2/4(t-\tau))$. Then we obtain

$$|K(-s_2(t), t, s_2(\tau), \tau) - K(-s_1(t), t, s_1(\tau), \tau)|$$
$$= \frac{(t-\tau)^{-1/2}}{2\sqrt{\pi}}|f_{t,\tau}(s_2(t) + s_2(\tau)) - f_{t,\tau}(s_1(t) + s_1(\tau))|$$
$$\leq \frac{3b}{4\sqrt{\pi}(t-\tau)^{3/2}} \exp\left(-\frac{b^2}{4(t-\tau)}\right)2M\|v_2 - v_1\|_t$$
$$\leq (\frac{6}{eb^2})^{3/2}\frac{3bM}{2\sqrt{\pi}}\|v_2 - v_1\|_t$$

and

$$|N(s_2(t), t, s_2(\tau), \tau) - N(s_1(t), t, s_1(\tau), \tau)|$$
$$\leq (\frac{M^2}{2\sqrt{\pi(t-\tau)}} + (\frac{6}{eb^2})^{3/2}\frac{3bM}{2\sqrt{\pi}})\|v_2 - v_1\|_t.$$

Therefore,

$$\int_0^t |N(s_2(t), t, s_2(\tau), \tau) - N(s_1(t), t, s_1(\tau), \tau)||F(V_1(\tau))|\, d\tau$$

$$\leq \int_0^t \Big(\frac{M^2}{2\sqrt{\pi(t-\tau)}} + \big(\frac{6}{eb^2}\big)^{3/2}\frac{3bM}{2\sqrt{\pi}}\Big)\|v_2 - v_1\|_t |F(V_1(\tau))|\, d\tau$$

$$\leq LM\Big(\frac{M^2\sqrt{t}}{\sqrt{\pi}} + \big(\frac{6}{eb^2}\big)^{3/2}\frac{3bM}{2\sqrt{\pi}}t\Big)\|v_2 - v_1\|_t$$

$$= (C_6(L, M)\sqrt{t} + C_7(L, M, b)t)\|v_2 - v_1\|_t.$$

To prove (2.25), we take into account (2.10):

$$G_x(0, t, s_2(\tau), \tau) = K(0, t, s_2(\tau), \tau)\frac{s_2(\tau)}{t - \tau}$$

$$= \exp\Big(-\frac{s_2^2(\tau)}{4(t-\tau)}\Big)\frac{(t-\tau)^{-3/2}}{2\sqrt{\pi}}s_2(\tau)$$

$$\leq \frac{1}{2\sqrt{\pi}}\big(\frac{24}{eb^2}\big)^{3/2}s_2(\tau) \leq \frac{3b}{4\sqrt{\pi}}\big(\frac{24}{eb^2}\big)^{\frac{3}{2}} = C_8(b).$$

To prove (2.26), we have

$$|G_x(0, t, s_2(\tau), \tau)v_2(\tau) - G_x(0, t, s_1(\tau), \tau)v_1(\tau)|$$

$$\leq |G_x(0, t, s_2(\tau), \tau)||v_2(\tau) - v_1(\tau)|$$

$$+ |G_x(0, t, s_2(\tau), \tau) - G_x(0, t, s_1(\tau), \tau)||v_1(\tau)|.$$

Using the mean value theorem there exists $c = c(\tau)$ between $s_2(\tau)$ and $s_1(\tau)$ such that $G_x(0, t, s_2(\tau), \tau) - G_x(0, t, s_1(\tau), \tau) = G_{x\xi}(0, t, c, \tau)(s_2(\tau) - s_1(\tau))$. Taking into account the following properties

$$G_{x\xi}(0, t, c, \tau) = \frac{K(0, t, c, \tau)}{t - \tau}\big(\frac{c^2}{2(t-\tau)} + 1\big),$$

$$\frac{K(0, t, c, \tau)}{t - \tau} = \frac{1}{2\sqrt{\pi}}\exp\big(-\frac{c^2}{4(t-\tau)}\big)(t-\tau)^{-3/2} \leq \frac{1}{2\sqrt{\pi}}\big(\frac{24}{eb^2}\big)^{3/2},$$

$$K(0, t, c, \tau)\frac{c^2}{2(t-\tau)^2} = \frac{1}{4\sqrt{\pi}}\exp\big(-\frac{c^2}{4(t-\tau)}\big)(t-\tau)^{-\frac{5}{2}}c^2 \leq \frac{9b^2}{16\sqrt{\pi}}\big(\frac{40}{eb^2}\big)^{5/2}$$

we have

$$|G_x(0, t, s_2(\tau), \tau) - G_x(0, t, s_1(\tau), \tau)||v_1(\tau)|$$

$$\leq \big(\frac{1}{2\sqrt{\pi}}\big(\frac{24}{eb^2}\big)^{\frac{3}{2}} + \frac{9b^2}{16\sqrt{\pi}}\big(\frac{40}{eb^2}\big)^{\frac{5}{2}}\big)|s_2(\tau) - s_1(\tau)||v_1(\tau)|$$

$$\leq M\big(\frac{1}{2\sqrt{\pi}}\big(\frac{24}{eb^2}\big)^{\frac{3}{2}} + \frac{9b^2}{16\sqrt{\pi}}\big(\frac{40}{eb^2}\big)^{\frac{5}{2}}\big)\tau|v_2(\tau) - v_1(\tau)|\,.$$

Then

$$\int_0^t |G_x(0, t, s_2(\tau), \tau) - G_x(0, t, s_1(\tau), \tau)||v_1(\tau)|\, d\tau$$

$$\leq \big(\frac{1}{2\sqrt{\pi}}\big(\frac{24}{eb^2}\big)^{\frac{3}{2}} + \frac{9b^2}{16\sqrt{\pi}}\big(\frac{40}{eb^2}\big)^{\frac{5}{2}}\big)\frac{Mt^2}{2}\|v_2 - v_1\|_t.$$

Then (2.26) holds by using (2.25). To prove (2.27), we have

$$
\begin{aligned}
&[N(0,t,s_2(\tau),\tau) - N(0,t,0,\tau)]F(V_2(\tau)) \\
&- [N(0,t,s_1(\tau),\tau) - N(0,t,0,\tau)]F(V_1(\tau)) \\
&= [N(0,t,s_2(\tau),\tau) - N(0,t,0,\tau)][F(V_2(\tau)) - F(V_1(\tau))] \\
&+ [N(0,t,s_2(\tau),\tau) - N(0,t,s_1(\tau),\tau)]F(V_1(\tau))
\end{aligned}
\tag{2.29}
$$

Using $|N(0,t,s_2(\tau),\tau) - N(0,t,0,\tau)| \leq \frac{2}{\sqrt{\pi(t-\tau)}}$ we get

$$
|N(0,t,s_2(\tau),\tau) - N(0,t,0,\tau)||F(V_2(\tau)) - F(V_1(\tau))| \leq \frac{2}{\sqrt{\pi(t-\tau)}}L|V_2(\tau) - V_1(\tau)|,
$$

and

$$
\begin{aligned}
&\int_0^t |N(0,t,s_2(\tau),\tau) - N(0,t,0,\tau)||F(V_2(\tau)) - F(V_1(\tau))|\, d\tau \\
&\leq \frac{4\sqrt{t}}{\sqrt{\pi}}L\|V_2 - V_1\|_t = C_4(L)\sqrt{t}\|V_2 - V_1\|_t.
\end{aligned}
\tag{2.30}
$$

Furthermore,

$$
|N(0,t,s_2(\tau),\tau) - N(0,t,s_1(\tau),\tau)| = |N_\xi(0,t,c,\tau)||s_2(\tau) - s_1(\tau)|
$$

where $c = c(\tau)$ is between $s_2(\tau)$ and $s_1(\tau)$ and

$$
\begin{aligned}
|N_\xi(0,t,c,\tau)||s_2(\tau) - s_1(\tau)| = |-G_x(0,t,c,\tau)||s_2(\tau) - s_1(\tau)| \\
\leq \frac{|c|}{2\sqrt{\pi}}(\frac{24}{eb^2})^{3/2}\tau|v_2(\tau) - v_1(\tau)| \\
\leq \frac{3b}{4\sqrt{\pi}}(\frac{24}{eb^2})^{3/2}\tau|v_2(\tau) - v_1(\tau)|.
\end{aligned}
$$

Then

$$
\begin{aligned}
&\int_0^t |N(0,t,s_2(\tau),\tau) - N(0,t,s_1(\tau),\tau)||F(V_1(\tau))|\, d\tau \\
&\leq LM\frac{3b}{4\sqrt{\pi}}(\frac{24}{eb^2})^{3/2}\frac{t^2}{2}\|v_2 - v_1\|_t = C_5(L,M,b)t^2\|v_2 - v_1\|_t
\end{aligned}
\tag{2.31}
$$

Therefore, by (2.29), (2.30), and (2.31)), the inequality (2.27) holds.   □

Theorem 2.6. *The map $A : C_{M,\sigma} \to C_{M,\sigma}$ is well defined and is a contraction map if σ satisfies the following inequalities:*

$$
\sigma \leq 1\,,\, 2M\sigma \leq b
\tag{2.32}
$$

$$
(2\|\dot{f}\|_\sigma C_1(b) + MC_3(b))\sigma + (2M^2 C_2(b) + \frac{2\|\dot{f}\|_\sigma}{\sqrt{\pi}} + 3MC_4(L))\sqrt{\sigma} \leq 1
\tag{2.33}
$$

$$
D(b,f,h,L,M)\sqrt{\sigma} < 1,
\tag{2.34}
$$

where

$$
M = 1 + 3\|h'\|
\tag{2.35}
$$

and

$$D_1(b, f, h, L, M) = E(b, f, h) + 2C_6(L, M) + 3C_4(L)$$
$$D_2(b, L, M) = 2[C_5(b, L, M) + 2C_7(b, L, M) + C_8(b)]$$
$$D_3(b, L, M) = C_9(b, M) + C_5(b, L, M)$$
$$D(b, f, h, L, M) = D_1(b, f, h, L, M) + D_2(b, L, M) + D_3(b, L, M).$$

Then there exists a unique solution on $C_{M,\sigma}$ to the system of integral equations (2.4), (2.5).

Proof. Firstly we demonstrate that A maps $C_{\sigma,M}$ into itself, that is

$$\|A(\vec{w})\|_\sigma = \max_{t \in [0,\sigma]} |A_1(v(t), V(t))| + \max_{t \in [0,\sigma]} |A_2(v(t), V(t))| \le M \qquad (2.36)$$

Using the Lemmas 2.3, 2.4 and the definitions (2.8)-(2.9), we have

$$|A_1(v(t), V(t))| \le 2\|\dot{f}\|_\sigma C_1(b)t + 2M^2 C_2(b)\sqrt{t} + 2\|h'\| + 2C_4(L)M\sqrt{t},$$

$$|A_2(v(t), V(t))| \le \|h'\| + \left(\frac{2\|\dot{f}\|_\sigma}{\sqrt{\pi}} + C_4(L)M\right)\sqrt{t} + C_3(b)Mt.$$

Then

$$\|A(\vec{w})\|_\sigma = \max_{t \in [0,\sigma]} |A_1(v(t), V(t))| + \max_{t \in [0,\sigma]} |A_2(v(t), V(t))|$$
$$\le 3\|h'\| + (2\|\dot{f}\|_\sigma C_1(b) + C_3(b)M)\sigma$$
$$+ \left(2M^2 C_2(b) + \frac{2\|\dot{f}\|_\sigma}{\sqrt{\pi}} + 3MC_4(L)\right)\sqrt{\sigma}.$$

Selecting M by (2.35) and σ such that (2.32) and (2.33) hold, we obtain (2.36).

Now, we prove that

$$\|A(\vec{w_2}) - A(\overrightarrow{w_1})\|_\sigma \le D(b, h, f, L, M)\sqrt{\sigma}\|\vec{w_2} - \vec{w_1}\|_\sigma$$

where $\vec{w_1} = \binom{v_1}{V_1}$, $\vec{w_2} = \binom{v_2}{V_2}$. By selecting σ such that (2.34) holds, A becomes a contraction mapping on $C_{\sigma,M}$ and therefore it has a unique fixed point. To prove this assertion we consider

$$A(\vec{w_1})(t) - A(\vec{w_2})(t) = \begin{pmatrix} A_1(v_2(t), V_2(t)) - A_1(v_1(t), V_1(t)) \\ A_2(v_2(t), V_2(t)) - A_2(v_1(t), V_1(t)) \end{pmatrix}$$

where

$$A_1(v_2(t), V_2(t)) - A_1(v_1(t), V_1(t))$$
$$= F_0(v_2(t)) - F_0(v_1(t)) + 2\int_0^t [N(s_2(t), t, s_2(\tau), \tau) - N(s_2(t), t, 0, \tau)]F(V_2(\tau))\,d\tau$$
$$- 2\int_0^t [N(s_1(t), t, s_1(\tau), \tau) - N(s_1(t), t, 0, \tau)]F(V_1(\tau))\,d\tau$$

and

$$A_2(v_2(t), V_2(t)) - A_2(v_1(t), V_1(t))$$
$$= \int_0^t [G_x(0, t, v_2(\tau), \tau)v_2(\tau) - G_x(0, t, v_1(\tau), \tau)v_1(\tau)] \, d\tau$$
$$+ \int_0^t \big\{ [N(0, t, s_2(\tau), \tau) - N(0, t, 0, \tau)]F(V_2(\tau))$$
$$- [N(0, t, s_1(\tau), \tau) - N(0, t, 0, \tau)]F(V_1(\tau)) \big\} \, d\tau \,.$$

Taking into account the Lemmas 2.4 and 2.5 it results

$$|A_1(v_2(t), V_2(t)) - A_1(v_1(t), V_1(t))|$$
$$\leq E(b, h, f)\sqrt{t}\|v_2 - v_1\|_t + 2C_4(L)\sqrt{t}\|V_2 - V_1\|_t$$
$$+ 2C_5(b, L, M)t\|v_2 - v_1\|_t + 2[C_6(L, M)\sqrt{t} + C_7(b, L, M)t]\|v_2 - v_1\|_t,$$

and

$$|A_2(v_2(t), V_2(t)) - A_2(v_1(t), V_1(t))|$$
$$\leq (C_8(b)t + C_9(b, M)t^2)\|v_2 - v_1\|_t$$
$$+ C_4(L)\sqrt{t}\|V_2 - V_1\|_t + C_5(b, L, M)t^2\|v_2 - v_1\|_t \,.$$

Therefore,

$$\|A(\vec{w_2}) - A(\vec{w_1})\|_\sigma$$
$$\leq \max_{t\in[0,\sigma]} |A_1(v_2(t), V_2(t)) - A_1(v_1(t), V_1(t))|$$
$$+ \max_{t\in[0,\sigma]} |A_2(v_2(t), V_2(t)) - A_2(v_1(t), V_1(t))|$$
$$\leq \{D_1(b, f, h, L, M)\sqrt{\sigma} + D_2(b, L, M)\sigma + D_3(b, L, M)\sigma^2\}\|\vec{w_2} - \vec{w_1}\|_\sigma$$
$$\leq D(b, f, h, L, M)\sqrt{\sigma}\|\vec{w_2} - \vec{w_1}\|_\sigma \,.$$

By hypothesis (2.34) we have that A is a contraction. $\qquad\square$

Remark. If F satisfies the conditions

  (H2) $F(V) > 0$, for all $V \neq 0$ and $F(0) = 0$,

then by the maximum principle [5], u is a sub-solution for the same problem with $F \equiv 0$, that is

$$u(x, t) \leq u_0(x, t), \quad s(t) \leq s_0(t)$$

where $u_0(x, t)$ and $s_0(t)$ solve the classical Stefan problem

$$u_{0t} - u_{0xx} = 0, \quad 0 < x < s_0(t), \, 0 < t < T,$$
$$u_0(0, t) = f(t) \geqslant 0, \quad 0 < t < T,$$
$$u_0(s_0(t), t) = 0, \quad u_{0x}(s_0(t), t) = -\dot{s}_0(t).0 < t < T,$$
$$u_0(x, 0) = h(x) \quad 0 \leqslant x \leqslant b = s_0(0) \,.$$

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