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Explicit solution of a free boundary problem for a nonlinear absorption model of mixed saturated–unsaturated flow

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In wet soils, zones of saturation naturally develop in the vicinity of impermeable strata, surface ponds and subterranean cavities. Hydrology must be then concerned with transient flow through coexisting unsaturated and saturated zones. The models of advancing saturated zones necessarily involve a nonlinear free boundary problem.

A closed-form analytic solution is presented for a nonlinear diffusion model under conditions of ponding at the surface. The soil water diffusivity is restricted to the special functional form $D(\theta)=a/(b-\theta)^2$, where θ is the water content field to be determined and a, b are positive constants. The explicit solution depends on a parameter C (determined by the data of the problem), according to two cases: $1<C<C_1$ or $C\geq C_1$, where C_1 is a constant which is obtained as the unique solution of an equation. This result complements the study given in P. Broadbridge, *Water Resources Research*, 1990, **26**, 2435–2443, in order to established when the explicit solution is available. The behavior of the bifurcation parameter C_1 as a function of the driving potential is studied with the corresponding limits for small and large values. Moreover, the sorptivity is proven to be continuously differentiable function of the variable C.

Key words: free boundary problem, mixed saturated–unsaturated flow, nonlinear absorption model.

1 INTRODUCTION

Following refs [1,6], we consider a homogeneous soil which initially has some uniform volumetric water content θ_n. At times $t>0$, water is supplied at the surface $x=0$ under pressure head Ψ_0. Then, a mixed saturated–unsaturated flow problem representing absorption of water by a soil with a constant pond depth at the surface is presented. At every time t the zone of saturation extends from $x=0$ to $x=s(t)$ (the free boundary), and the unsaturated zone extends for $x>s(t)$. By assuming the Darcy's law and neglecting the gravity, the water flux is given by

$$v=-K(\Psi)\frac{\partial\Psi}{\partial x}, \tag{1}$$

where Ψ is the soil water matric potential and K is the hydraulic conductivity.

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In the saturated zone we have[1]

$$\Psi(x,t)=\Psi_0-\frac{\Psi_0-\Psi_s}{s(t)}x;\ 0<x<s(t); \tag{2}$$

and, the following free boundary problem eqns (3)–(7) arises for the unsaturated zone[7]:

$$\theta(s(t)^+,t)=\theta_s,\ t>0, \tag{3}$$

$$\frac{\partial\theta}{\partial t}=\frac{\partial}{\partial x}\left[D(\theta)\frac{\partial\theta}{\partial x}\right],\ x>s(t),\ t>0, \tag{4}$$

$$-D(\theta)\frac{\partial\theta}{\partial x}\left(s(t)^+,t\right)=K_s\frac{\Psi_0-\Psi_s}{s(t)},\ t>0, \tag{5}$$

$$\theta(x,0)=\theta(+\infty,t)=\theta_n,\ x>s(t),\ t>0, \tag{6}$$

$$s(0)=0 \tag{7}$$

where

x	spatial coordinate
t	time
θ	volumetric water content
θ_n	initial volumetric water content
θ_s	volumetric water content at saturation
Ψ	soil water matric potential
Ψ_0	pond depth
Ψ_s	soil water potential at $x = s(t)$, $\Psi_s < \Psi < \Psi_0$
K	hydraulic conductivity
K_s	hydraulic conductivity at saturation
D	soil water diffusivity $\left(D = K\dfrac{d\Psi}{d\theta}\right)$

From now on we consider the free boundary problem eqns (3)–(7), where the position $s(t)$ of the free boundary and the water field $\theta(x,t)$ must be determined. We restrict our attention to the special functional form of the soil water diffusivity expressed by

$$D(\theta) = \frac{a}{(b-\theta)^2} \tag{8}$$

where a and b are positive constants. With this form of diffusivity, the nonlinear diffusion eqn (4) may be transformed in a linear one. Following ref. [2], we normalize the water content variable as follows

$$\Theta = \frac{\theta - \theta_n}{\theta_s - \theta_n} \tag{9}$$

and we consider

$$\begin{cases} C = \dfrac{b-\theta_n}{\theta_s - \theta_n} > 1 \text{ parameter;} \\ \lambda_s = \dfrac{a}{(\theta_s-\theta_n)C(C-1)K_s} \text{ length scale;} \\ t_s = \dfrac{a}{C(C-1)K_s^2} \text{ time scale;} \\ x_* = \dfrac{x}{\lambda_s} \text{ dimensionless length;} \\ t_* = \dfrac{t}{t_s} \text{ dimensionless time.} \end{cases} \tag{10}$$

Then, problem eqns (3)–(7) is transformed into problem eqns (11)–(15)

$$\frac{\partial\Theta}{\partial t_*} = \frac{\partial}{\partial x_*}\left[\frac{C(C-1)}{(C-\Theta)^2}\frac{\partial\Theta}{\partial x_*}\right], \; x_* > s_*(t_*), \; t_* > 0, \tag{11}$$

$$s_*(0) = 0, \tag{12}$$

$$\Theta(x_*, 0) = \Theta(+\infty, t_*) = 0, \; x_* > s_*(t_*), \; t_* > 0, \tag{13}$$

$$\Theta(s_*(t_*)^+, \; t_*) = 1, \; t_* > 0, \tag{14}$$

$$-\frac{C(C-1)}{(C-\Theta)^2}\frac{\partial\Theta}{\partial x_*}\left(s_*(t_*)^+, t_*\right) = \frac{\Psi_{0*} - \Psi_{s*}}{s_*(t_*)}, \; t_* > 0, \tag{15}$$

where

$$s_*(t_*) = \frac{s(t)}{\lambda_s} = \frac{s(t_s t_*)}{\lambda_s} \tag{16}$$

is the position of the free boundary.

Now we define a dimensionless depth coordinate moving with the saturated–unsaturated interface

$$y_* = x_* - s_*(t_*) > 0, \; t_* = t_* > 0; \tag{17}$$

hence, we have the dimensionless free boundary problem eqns (18)–(22)

$$\frac{\partial\Theta}{\partial t_*} = \frac{\partial}{\partial y_*}\left[\frac{C(C-1)}{(C-\Theta)^2}\frac{\partial\Theta}{\partial y_*}\right] + \frac{ds_*}{dt_*}\frac{\partial\Theta}{\partial y_*}, \; y_* > 0, \; t_* > 0, \tag{18}$$

$$s_*(0) = 0, \tag{19}$$

$$\Theta(y_*, 0) = \Theta(+\infty, t_*) = 0, \; y_* > 0, \; t_* > 0, \tag{20}$$

$$\Theta(0, t_*) = 1, \; t_* > 0, \tag{21}$$

$$-\frac{C(C-1)}{(C-\Theta)^2}\frac{\partial\Theta}{\partial y_*}\left(0^+, t_*\right) = \frac{\Psi_{o*} - \Psi_{s*}}{s_*(t_*)}, \; t_* > 0, \tag{22}$$

where

$\Psi_{0*} = \dfrac{\Psi_0}{\lambda_s}$ dimensionless pond depth

$\Psi_{s*} = \dfrac{\Psi_s}{\lambda_s}$ dimensionless soil water potential at the moving saturated – unsaturated interface.

The goal of the paper is to solve the dimensionless free boundary problem eqns (18)–(22). We will show an explicit to this problem which depends on a parameter C, according to two cases: $1 < C < C_1$ or $C \geq C_1$, where C_1 is a constant (the bifurcation parameter) obtained as the unique solution of the following equation:

$$Q\left(\frac{\delta}{2}\sqrt{C-1}\right) = \frac{2}{C}, \; C > 1, \tag{23}$$

where Q is a real function defined by

$$Q(x) = \sqrt{\pi}x\exp(x^2)\text{erfc}(x), \; x > 0, \tag{24}$$

and $\delta > 0$ is a parameter defined in eqn (43).

2 CLOSED-FORM ANALYTIC SOLUTION OF THE FREE BOUNDARY EQNS (18)–(22).

In order that the two boundary conditions eqns (3) and (5) are compatible, $s(t)$ must be of the form

$$s(t) = m\sqrt{t}, \tag{25}$$

m being an unknown constant. By eqns (2) and (5), the unknown m is related to the unknown sorptivity S by the following expression

$$m=\frac{2K_s(\Psi_0-\Psi_s)}{S}, \tag{26}$$

and $v(s(t),t)=S/2\sqrt{t}$ is the infiltration rate, where v is related to Ψ through the Darcy eqn (1). The sorptivity S is a basic hydraulic property relating cumulative intake $I(t)$ (expressed as a length) to the square root of time for a one-dimensional sorption into a soil without gravity, i.e. $I(t)=S\sqrt{t}$ (Ref. 8). It has been shown in[4,5] that the dominant parameter governing the dynamics of infiltration at small times is the sorptivity S. Since S ia measure of the capillary uptake or removal of water, is essentially a property of the medium with some resemblance to permeability. When v is in cm s^{-1} and t in s, the unit of S is cm s$^{-\frac{1}{2}}$ (Ref. 4).

Then, in terms of dimensionless variables we have

$$s_*(t_*)=m_*\sqrt{t_*} \tag{27}$$

where

$$m_*=\frac{2K_s(\Psi_0-\Psi_s)(\theta_s-\theta_n)}{S}\sqrt{\frac{C(C-1)}{a}}=\frac{m}{\lambda_s}\sqrt{t_s}. \tag{28}$$

To linearize the diffusion eqn (4) we define the variables[3]

$$\begin{cases}\mu=\dfrac{C(C-1)}{C-\Theta},\\ \chi=\dfrac{1}{\sqrt{C(C-1)}}\displaystyle\int_0^{y_*}\left(C-\Theta(\nu,t_*)\right)d\nu,\\ \tau=t_*;\end{cases} \tag{29}$$

and we obtain the problem eqns (30)–(33)

$$\frac{\partial\mu}{\partial\tau}=\frac{\partial^2\mu}{\partial\chi^2}+\frac{\gamma}{2\sqrt{\tau}}\frac{\partial\mu}{\partial\chi},\ \chi>0,\ \tau>0, \tag{30}$$

$$\mu(0^+,\tau)=C,\ \tau>0, \tag{31}$$

$$\frac{-\sqrt{C(C-1)}}{\mu}\frac{\partial\mu}{\partial\chi}(0^+,\tau)=\frac{S_*}{2\sqrt{\tau}},\ \tau>0, \tag{32}$$

$$\mu(\chi,0)=C-1=\mu(+\infty,\tau),\ \chi>0,\ \tau>0, \tag{33}$$

where

$$\gamma=\frac{S}{\sqrt{a}}+\frac{2\sqrt{a}}{CS}(\Psi_{0*}-\Psi_{s*}),\ S_*=S\sqrt{\frac{C(C-1)}{a}}. \tag{34}$$

Now we assume a similarity solution

$$\mu=g(\phi),\ \phi=\frac{\chi}{\sqrt{\tau}}. \tag{35}$$

Then the problem eqns (30)–(33) reduces to the problem eqns (36)–(39)

$$\frac{1}{2}g'(\phi)(\phi+\gamma)+g''(\phi)=0,\ \phi>0, \tag{36}$$

$$g(+\infty)=C-1, \tag{37}$$

$$-\sqrt{C(C-1)}g'(0^+)=\frac{CS_*}{2}=\frac{CS}{2}\sqrt{\frac{C(C-1)}{a}}, \tag{38}$$

$$g(0^+)=C. \tag{39}$$

The solution to the conditions eqns (36)–(38) is given by

$$g(\phi)=C-1+\frac{CS}{2}\sqrt{\frac{\pi}{a}}\exp\left(\frac{\gamma^2}{4}\right)\operatorname{erfc}\left(\frac{\phi+\gamma}{2}\right),\ \phi>0 \tag{40}$$

where the coefficient γ is unknown.

The extra boundary condition eqn (39) is consistent with this solution provided that

$$\frac{1}{C}=\frac{S}{2}\sqrt{\frac{\pi}{a}}\exp\left(\frac{\gamma^2}{4}\right)\operatorname{erfc}\left(\frac{\gamma}{2}\right). \tag{41}$$

Since S and γ verify the following relation (another method is given in Remark 3 and Appendix A)

$$S=\frac{\sqrt{a}}{2}\left(\gamma\pm\sqrt{\gamma^2-\gamma_0^2(C)}\right),\ \gamma\geq\gamma_0(C),\ C>1 \tag{42}$$

where

$$\begin{cases}\gamma_0^2(C)=\dfrac{8}{C}(\Psi_{0*}-\Psi_{s*})=\delta^2(C-1),\\ \delta=\sqrt{\dfrac{8K_s(\Psi_0-\Psi_s)(\theta_s-\theta_n)}{a}}\end{cases} \tag{43}$$

we have that the above eqn (41) in variable $\gamma=\gamma(C)$ is given by

$$\frac{1}{C}=\frac{1}{2}\left(1\pm\sqrt{1-\left(\frac{\gamma_0(C)}{\gamma}\right)^2}\right)Q\left(\frac{\gamma}{2}\right),\ \gamma\geq\gamma_0(C),\quad C>1. \tag{44}$$

In studying eqn (44), we shall consider two cases respectively corresponding to choose the sign (+) or (−).

Case 1: (sign + in the expression of S as a function of γ) The eqn (44) may be written as

$$\frac{1}{C}=H_1(\gamma,C)Q\left(\frac{\gamma}{2}\right),\ \gamma\geq\gamma_0(C),\ C>1. \tag{45}$$

where H_1 is defined by

$$H_1(\gamma,C)=\frac{1}{2}\left(1+\sqrt{1-\left(\frac{\gamma_0(C)}{\gamma}\right)^2}\right),\ \gamma\geq\gamma_0(C),\quad C>1. \tag{46}$$

The function H_1 satisfies the following properties

$$\begin{cases}\text{(i)}\quad H_1(\gamma_0(C),C)=\dfrac{1}{2},\ C>1,\\ \text{(ii)}\quad H_1(+\infty,C)=1,\ C>1,\\ \text{(iii)}\quad \dfrac{\partial H_1}{\partial\gamma}(\gamma,C)>0,\ \gamma>\gamma_0(C),\ C>1\end{cases} \tag{47}$$

Now we define the real function

$$F_1(\gamma, C) = \frac{1}{CH_1(\gamma, C)}, \; \gamma \geq \gamma_0(C), \; C > 1. \tag{48}$$

which satisfies the following properties

$$\begin{cases} \text{(i)} & F_1(\gamma_0(C), C) = \frac{2}{C}, \; C > 1, \\ \text{(ii)} & F_1(+\infty, C) = \frac{1}{C}, \; C > 1, \\ \text{(iii)} & \frac{\partial F_1}{\partial \gamma}(\gamma, C) < 0, \; \gamma > \gamma_0(C), \; C > 1 \end{cases} \tag{49}$$

Then, we have that the eqn (45) is equivalent to

$$F_1(\gamma, C) = Q\left(\frac{\gamma}{2}\right), \; \gamma \geq \gamma_0(C), \; C > 1. \tag{50}$$

Since Q satisfies the properties

$$\begin{cases} \text{(i)} & Q(0) = 0, \\ \text{(ii)} & Q(+\infty) = 1, \\ \text{(iii)} & Q'(0) = \pi, \; Q'(x) > 0, \; x > 0, \\ \text{(iv)} & Q''(x) < 0, \; x > 0 \end{cases} \tag{51}$$

we conclude that eqn (50) admits a unique solution in the variable γ if and only if

$$F_1(\gamma_0(C), C) = \frac{2}{C} \geq Q\left(\frac{\gamma_0(C)}{2}\right) \Leftrightarrow M(C) \leq 2$$

where the real function M is defined by

$$M(C) = CQ\left(\frac{\gamma_0(C)}{2}\right) = CQ\left(\frac{\delta}{2}\sqrt{C-1}\right), \; C > 1. \tag{52}$$

The function M satisfies the following properties

$$\begin{cases} \text{(i)} & M'(C) > 0, \; C > 1, \\ \text{(ii)} & M(1) = Q(0) = 0, \\ \text{(iii)} & M(+\infty) = +\infty, \\ \text{(iv)} & M(C) < C, \; C > 1, \\ \text{(v)} & \lim_{C \to +\infty} \frac{M(C)}{C} = 1. \end{cases} \tag{53}$$

Therefore, there exist a unique constant $C_1 > 1$ such that

$$M(C_1) = C_1 Q\left(\frac{\delta}{2}\sqrt{C_1 - 1}\right) = 2 \tag{54}$$

and

$$M(C) \leq 2 \Leftrightarrow KC \leq C_1.$$

Moreover, by using eqn (53)iv we deduce

$$C_1 > 2 \tag{55}$$

Case 2: (sign $-$ in the expression of S as a function of γ)

The eqn (44) may be written as

$$\frac{1}{C} = H_2(\gamma, C) Q\left(\frac{\gamma}{2}\right), \; \gamma \geq \gamma_0(C), \; C > 1. \tag{56}$$

where H_2 is defined by

$$H_2(\gamma, C) = \frac{1}{2}\left(1 - \sqrt{1 - \left(\frac{\gamma_0(C)}{\gamma}\right)^2}\right), \; \gamma \geq \gamma_0(C), \; C > 1. \tag{57}$$

which satisfies the following properties

$$\begin{cases} \text{(i)} & H_2(\gamma_0(C), C) = \frac{1}{2}, \; C > 1, \\ \text{(ii)} & H_2(+\infty, C) = 0, \; C > 1, \\ \text{(iii)} & \frac{\partial H_2}{\partial \gamma}(\gamma, C) < 0, \; \gamma > \gamma_0(C), \; C > 1 \end{cases} \tag{58}$$

Now we define the real function

$$F_2(\gamma, C) = \frac{1}{CH_2(\gamma, C)}, \; \gamma \geq \gamma_0(C), \; C > 1. \tag{59}$$

which satisfies the following properties

$$\begin{cases} \text{(i)} & F_2(\gamma_0(C), C) = \frac{2}{C}, \; C > 1, \\ \text{(ii)} & F_2(+\infty, C) = +\infty, \; C > 1, \\ \text{(iii)} & \frac{\partial F_2}{\partial \gamma}(\gamma, C) > 0, \; \gamma > \gamma_0(C), \; C > 1 \end{cases} \tag{60}$$

Hence, the eqn (56) is equivalent to

$$F_2(\gamma, C) = Q\left(\frac{\gamma}{2}\right), \; \gamma \geq \gamma_0(C), \; C > 1. \tag{61}$$

Taking into account the properties of functions Q and F_2, we deduce that eqn (59) admits a unique solution in the variable γ if and only if

$$F_2(\gamma_0(C), C) = \frac{2}{C} \leq Q\left(\frac{\gamma_0(C)}{2}\right) \Leftrightarrow$$

$$M(C) \geq 2, C > 1 \Leftrightarrow C \geq C_1.$$

Then, we have obtained the following result:

Theorem 1. *Assume that* $C = (b - \theta_n)/(\theta_s - \theta_n) > 1$. *Then, there exists a bifurcation parameter* $C_1 = C_1(\delta) = C_1(a, K_s, \Psi_0 - \Psi_s, \theta_s - \theta_n) > 1$, *which is the unique solution of the eqn (23). We have:*

I) *If* $1 < C \leq C_1$: *There exist a unique* $\gamma_1(C) \geq \gamma_0(C)$ *such that*

$$\frac{1}{C} = \frac{1}{2}\left(1 + \sqrt{1 - \left(\frac{\gamma_0(C)}{\gamma_1(C)}\right)^2}\right) Q\left(\frac{\gamma_1(C)}{2}\right); \tag{62}$$

and, the solution of the problem eqns (36)–(39) is given by

$$g_1(\phi) = C - 1 + \frac{S_1(C)C}{2}\sqrt{\frac{\pi}{a}}\exp\left(\frac{\gamma_1^2(C)}{4}\right) \times \operatorname{erfc}\left(\frac{\phi + \gamma_1(C)}{2}\right), \; \phi > 0, \tag{63}$$

where

$$S_1(C) = \frac{\sqrt{a}}{2}\left(\gamma_1(C) + \sqrt{\gamma_1^2(C) - \gamma_0^2(C)}\right). \tag{64}$$

II) *If $C \geq C_1$: There exist a unique $\gamma_2(C) \geq \gamma_0(C)$ such that*

$$\frac{1}{C} = \frac{1}{2}\left(1 - \sqrt{1 - \left(\frac{\gamma_0(C)}{\gamma_2(C)}\right)^2}\right) \mathrm{Q}\left(\frac{\gamma_2(C)}{2}\right); \quad (65)$$

and, the solution of the problem eqns (36)–(39) is given by

$$g_2(\phi) = C - 1 + \frac{S_2(C)C}{2}\sqrt{\frac{\pi}{a}}\exp\left(\frac{\gamma_2^2(C)}{4}\right) \times \operatorname{erfc}\left(\frac{\phi + \gamma_2(C)}{2}\right), \ \phi > 0, \quad (66)$$

where

$$S_2(C) = \frac{\sqrt{a}}{2}\left(\gamma_2(C) - \sqrt{\gamma_2^2(C) - \gamma_0^2(C)}\right). \quad (67)$$

Remark 1. *For the case $C = C_1$, we have that $M(C_1) = 2$, that is*

$$\frac{1}{C_1} = \frac{1}{2}\mathrm{Q}\left(\frac{\gamma_0(C_1)}{2}\right). \quad (68)$$

Then, $\gamma_0(C_1)$ satisfies the two equations eqns (45) and (55) because

$$H_1(\gamma_0(C), C) = H_2(\gamma_0(C), C) = \frac{1}{2}, \ C > 1.$$

Remark 2. *For $C = C_1$ both solutions g_1 and g_2 coincide because*

$$\gamma_1(C_1) = \gamma_2(C_1) = \gamma_0(C_1) = \delta\sqrt{C_1 - 1}, \quad (69)$$

and

$$S(C_1) = S_1(C_1) = S_2(C_1) = \frac{\sqrt{a}}{2}\gamma_0(C_1) = \frac{\delta}{2}\sqrt{a(C_1 - 1)} = \sqrt{2(\Psi_0 - \Psi_s)(\theta_s - \theta_n)(C_1 - 1)}. \quad (70)$$

Therefore, the solution of the problem eqns (36)–(39) is given by

$$g(\phi) = C_1 - 1 + \frac{S(C_1)C_1}{2}\sqrt{\frac{\pi}{a}}\exp\left(\frac{\gamma_0^2(C_1)}{4}\right) \times \operatorname{erfc}\left(\frac{\phi + \gamma_0(C_1)}{2}\right), \ \phi > 0. \quad (71)$$

The sorptivity S as a function of the variable C is given by

$$S(C) = \begin{cases} S_1(C) = \frac{\sqrt{a}}{2}\left(\gamma_1(C) + \sqrt{\gamma_1^2(C) - \gamma_0^2(C)}\right), & 1 < C_1 \\ \frac{\sqrt{a}}{2}\gamma_0(C_1), & C = C_1 \\ S_2(C) = \frac{\sqrt{a}}{2}\left(\gamma_2(C) - \sqrt{\gamma_2^2(C) - \gamma_0^2(C)}\right), & C > C_1 \end{cases}$$

where $\gamma_0(C)$ is defined in eqn (43), and $\gamma_1(C)$ and $\gamma_2(C)$ are defined by eqns (62) and (65) respectively.

The function $S = S(C)$ is continuously differentiable. Moreover, we have

$$S(1^+) = +\infty, \ S(+\infty) = 0. \quad (72)$$

Proof. We get $S_1(C_1^-) = S_2(C_1^+)$ because of eqn (70). By elementary but tedious computations we obtain

$$\frac{\partial S_1}{\partial C}(C_1^-) = \frac{\partial S_2}{\partial C}(C_1^+) = \frac{\delta}{2C_1}\sqrt{a(C_1 - 1)} \times \left(\frac{\delta^2}{8}C_1(C_1 - 2) - 1\right)$$

On the other hand, by elementary computations we get eqn (72).

Remark 3. *An alternative method to prove Theorem 1 was suggested by an anonymous referee and it is shown in the Appendix.*

Finally, we invert the relations eqns (35), (29), (10) and (9) to obtain the parametric solution to the problem eqns (3)–(7), which depends on C.

Corollary 2. *There exists a bifurcation parameter $C_1 = C_1(\delta) = C_1(a, K_s, \Psi_0 - \Psi_s, \theta_s - \theta_n) > 1$, for the solution of problem (3)–(7) which is given by:*

(I) *Case $1 < C \leq C_1$. We have*

$$\theta_1(\chi, \tau) = (\theta_s - \theta_n)C\left(1 - \frac{(C-1)}{g_1\left(\frac{\chi}{\sqrt{\tau}}\right)}\right) + \theta_n, \ \chi > 0, \ \tau > 0, \quad (73)$$

$$x = \lambda_s y_{1*}(\chi, \tau) + m_1\sqrt{t_s\tau}, \ \chi > 0, \ \tau > 0, \quad (74)$$

$$t = t_s\tau, \ \chi > 0, \ \tau > 0, \quad (75)$$

$$s_1(\chi, \tau) = m_1\sqrt{t_s\tau}, \ \chi > 0, \ \tau > 0, \ \text{(the free boundary)} \quad (76)$$

with

$$m_1 = \frac{2K_s(\Psi_0 - \Psi_s)(\theta_s - \theta_n)}{S_1(C)}\sqrt{\frac{C(C-1)}{a}}\frac{\lambda_s}{\sqrt{t_s}} = \frac{2K_s(\Psi_0 - \Psi_s)}{S_1(C)}, \quad (77)$$

$$y_{1*}(\chi, \tau) = \frac{1}{\sqrt{C(C-1)}}\int_0^{\chi} g_1\left(\frac{\nu}{\sqrt{\tau}}\right)d\nu = \frac{\sqrt{\tau}}{\sqrt{C(C-1)}}\left\{(C-1)\frac{\chi}{\sqrt{\tau}} + S_1(C)C\sqrt{\frac{\pi}{a}} \times \exp\left(\frac{\gamma_1^2(C)}{4}\right).\left[\left(\frac{\chi}{2\sqrt{\tau}} + \frac{\gamma_1(C)}{2}\right)\right.\right.$$

$$\times \operatorname{erfc}\left(\frac{\chi}{2\sqrt{\tau}}+\frac{\gamma_1(C)}{2}\right) - \frac{1}{\sqrt{\pi}}\exp\left(-\left(\frac{\chi}{2\sqrt{\tau}}+\frac{\gamma_1(C)}{2}\right)^2\right) - \frac{\gamma_1(C)}{2}\operatorname{erfc}\left(\frac{\gamma_1(C)}{2}\right) + \frac{1}{\sqrt{\pi}}\exp\left(-\frac{\gamma_1^2(C)}{4}\right)\Bigg]\Bigg\}, \ \chi>0,\ \tau>0. \tag{78}$$

(II) *Case* $C \geq C_1$. *We have*

$$\theta_2(\chi,\tau) = (\theta_s-\theta_n)C\left(1-\frac{(C-1)}{g_2\left(\frac{\chi}{\sqrt{\tau}}\right)}\right)+\theta_n,\ \chi>0,\ \tau>0, \tag{79}$$

$$x=\lambda_s y_{2*}(\chi,\tau)+m_2\sqrt{t_s\tau},\ \chi>0,\ \tau>0, \tag{80}$$

$$t=t_s\tau,\ \chi>0,\ \tau>0, \tag{81}$$

$$s_2(\chi,\tau)=m_2\sqrt{t_s\tau},\ \chi>0,\ \tau>0,\ \text{(the free boundary)} \tag{82}$$

with

$$m_2=\frac{2K_s(\Psi_0-\Psi_s)(\theta_s-\theta_n)}{S_2(C)}\sqrt{\frac{C(C-1)}{a}}\frac{\lambda_s}{\sqrt{t_s}} = \frac{2K_s(\Psi_0-\Psi_s)}{S_2(C)}, \tag{83}$$

$$y_{2*}(\chi,\tau)=\frac{1}{\sqrt{C(C-1)}}\int_0^\chi g_2\left(\frac{\nu}{\sqrt{\tau}}\right)d\nu = \frac{\sqrt{\tau}}{\sqrt{C(C-1)}}\Bigg\{(C-1)\frac{\chi}{\sqrt{\tau}}+S_2(C)C\sqrt{\frac{\pi}{a}} \times \exp\left(\frac{\gamma_2^2(C)}{4}\right)\cdot\Bigg[\left(\frac{\chi}{2\sqrt{\tau}}+\frac{\gamma_2(C)}{2}\right) \times \operatorname{erfc}\left(\frac{\chi}{2\sqrt{\tau}}+\frac{\gamma_2(C)}{2}\right)-\frac{1}{\sqrt{\pi}} \times \exp\left(-\left(\frac{\chi}{2\sqrt{\tau}}+\frac{\gamma_2(C)}{2}\right)^2\right)-\frac{\gamma_2(C)}{2} \times \operatorname{erfc}\left(\frac{\gamma_2(C)}{2}\right)+\frac{1}{\sqrt{\pi}}\exp\left(-\frac{\gamma_2^2(C)}{4}\right)\Bigg]\Bigg\},\ \chi>0,\ \tau>0. \tag{84}$$

Remark 4. *For the case* $C=C_1$, *the two parametric solutions coincide one each other, that is*

$$\theta_1(\chi,\tau)=\theta_2(\chi,\tau)=(\theta_s-\theta_n)C_1\left(1-\frac{(C_1-1)}{g\left(\frac{\chi}{\sqrt{\tau}}\right)}\right)+\theta_n,\ \chi>0,\ \tau>0, \tag{85}$$

$$x=\lambda_s y_*(\chi,\tau)+m\sqrt{t_s\tau},\ \chi>0,\ \tau>0, \tag{86}$$

$$t=t_s\tau,\ \chi>0,\ \tau>0, \tag{87}$$

$$s(\chi,\tau)=m\sqrt{t_s\tau},\ \chi>0,\ \tau>0,\ \text{(the free boundary)} \tag{88}$$

with

$$m=m_1=m_2=\frac{2K_s(\Psi_0-\Psi_s)(\theta_s-\theta_n)}{S(C_1)}\times\sqrt{\frac{C_1(C_1-1)}{a}}\frac{\lambda_s}{\sqrt{t_s}}=\frac{2K_s(\Psi_0-\Psi_s)}{S(C_1)} = \sqrt{\frac{2K_s(\Psi_0-\Psi_s)}{(\theta_s-\theta_n)(C_1-1)}}, \tag{89}$$

$$y_*(\chi,\tau)=\frac{1}{\sqrt{C_1(C_1-1)}}\int_0^\chi g\left(\frac{\nu}{\sqrt{\tau}}\right)d\nu = \frac{\sqrt{\tau}}{\sqrt{C_1(C_1-1)}}\Bigg\{(C_1-1)\frac{\chi}{\sqrt{\tau}}+S(C_1)C_1\sqrt{\frac{\pi}{a}} \times \exp\left(\frac{\gamma_0^2(C_1)}{4}\right)\cdot\Bigg[\left(\frac{\chi}{2\sqrt{\tau}}+\frac{\gamma_0(C_1)}{2}\right) \times \operatorname{erfc}\left(\frac{\chi}{2\sqrt{\tau}}+\frac{\gamma_0(C_1)}{2}\right) - \frac{1}{\sqrt{\pi}}\exp\left(-\left(\frac{\chi}{2\sqrt{\tau}}+\frac{\gamma_0(C_1)}{2}\right)^2\right) - \frac{\gamma_0(C_1)}{2}\operatorname{erfc}\left(\frac{\gamma_0(C_1)}{2}\right) + \frac{1}{\sqrt{\pi}}\exp\left(-\frac{\gamma_0^2(C_1)}{4}\right)\Bigg]\Bigg\},\ \chi>0,\ \tau>0. \tag{90}$$

3 BEHAVIOR OF THE BIFURCATION PARAMETER C_1 UPON THE DATA

We shall study the bifurcation parameter C_1, the unique

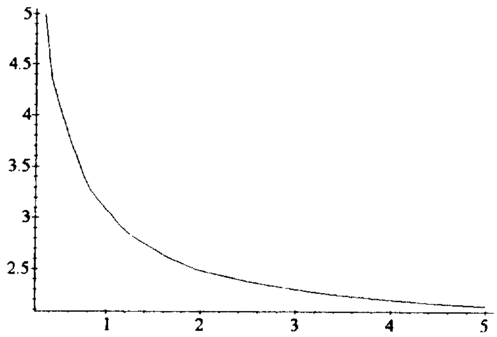

Fig. 1. The bifurcation parameter C_1 *versus* variable which are related by the eqn (23).

solution of the eqn (23), as a function of the variable δ defined by eqn (43). See Fig. 1 and Table 1.

Table 1. Values for C_l as a function of δ

δ	C_1
0.0001	799.77
0.001	173.23
0.005	60.027
0.01	38.256
0.1	9.1978
1	3.037
2	2.4751
3	2.2771
4	2.1809
7	2.0713
8	2.0561
10	2.0371

Lemma 3. *We have that $C_1 = C_1(\delta)$ satisfies the following properties:*

$$\begin{cases} \text{(i)} \quad C_1 > 2; & \text{(ii)} \quad \dfrac{\partial C_1}{\partial \delta} < 0, \ \forall \delta > 0; \\ \text{(iii)} \quad \lim\limits_{\delta \to 0^+} C_1(\delta) = +\infty; & \text{(iv)} \quad \lim\limits_{\delta \to +\infty} C_1(\delta) = 2. \end{cases} \tag{91}$$

Moreover, we have that the inverse function $\delta = \delta(C_1)$ is given explicitly by

$$\delta = \frac{2}{\sqrt{C_1 - 1}} Q^{-1}\left(\frac{2}{C_1}\right), \ C_1 > 2,$$

where Q^{-1} is the inverse function of Q.

Proof.

(i) is the condition eqn (55).

(ii) By using eqn (51) we have

$$\frac{dC_1}{d\delta}(\delta) = \frac{1}{2} \times \frac{-C_1(\delta)\sqrt{C_1(\delta) - 1}\,Q'\left(\dfrac{\delta\sqrt{C_1(\delta) - 1}}{2}\right)}{Q\left(\dfrac{\delta\sqrt{C_1(\delta) - 1}}{2}\right) + \dfrac{C_1(\delta)\delta}{4\sqrt{C_1(\delta) - 1}} Q'\left(\dfrac{\delta\sqrt{C_1(\delta) - 1}}{2}\right)}$$

$< 0, \ \delta > 0.$

(iii) If the limit of $C_1(\delta)$ is finite when $\delta \to 0^+$ we have a contradiction with eqn (23) because its left hand side goes to 0 and its right hand side goes to a positive number. Therefore eqn (91)iii holds.

(iv) If $\lim_{\delta \to +\infty} C_1(\delta) = +\infty$ then we have a contradiction with eqn (23) because its left hand side goes to 1 when δ goes to $+\infty$ (because of eqn (51)ii) while its right hand side goes to 0. Then, the limit of $C_1(\delta)$ is finite (≥ 2) when δ goes to $+\infty$. Then, by eqn (51)ii, we get eqn (91)iv.

Analogously, we can study the bifurcation parameter C_1 as a function of the driving potential ϵ defined by

$$\epsilon = \Psi_0 - \Psi_s \tag{92}$$

Theorem 4. *The function $C_1 = C_1(\epsilon)$ satisfies the following properties:*

$$\begin{cases} \text{(i)} \quad C_1 > 2; & \text{(ii)} \quad \dfrac{\partial C_1}{\partial \epsilon} < 0, \ \forall \epsilon > 0; \\ \text{(iii)} \quad \lim\limits_{\epsilon \to 0^+} C_1(\epsilon) = +\infty; & \text{(iv)} \quad \lim\limits_{\epsilon \to +\infty} C_1(\epsilon) = 2. \end{cases} \tag{93}$$

Proof. By taking into account the Lemma 4, and the parameters δ and ϵ are related by the following expression

$$\delta = \sqrt{\mu\epsilon}, \text{ with } \mu = \frac{8K_s(\theta_s - \theta_n)}{a} \tag{94}$$

the results (i)–(iv) hold.

From Theorem 5 we obtain the following conclusions:

(i) It is clear that only the ' + ' branch, in relation eqn (42), occurs when the driving potential $\epsilon = \Psi_0 - \Psi_s$ goes to zero because of eqn (93)iii). Therefore, the '−' branch has not physical meaning.

(ii) For the other cases (the driving potential $\epsilon = \Psi_0 - \Psi_s$ is positive) the two branches ('+' and '−') in relation (42) have a physical meaning.

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APPENDIX A

We shall show a new proof of Theorem 1 by studying a single trascendental equation for the sorptivity S. If we substitute eqn (34) in eqn (41), we obtain for the unknown S the following equation

$$F(S,C)=Q^*(S,C),\ S>0\ (C>1:\ \text{parameter}) \quad (A1)$$

where

$$F(S,C)=\frac{\gamma(S,C)\sqrt{a}}{CS}=\frac{1}{C}+\frac{a\delta^2(C-1)}{4S^2C},\ S>0,\ C>1 \quad (A2)$$

and

$$Q^*(S,C)=Q\left(\frac{\gamma(S,C)}{2}\right),\ S>0,\ C>1 \quad (A3)$$

with

$$\gamma(S,C)=\frac{S}{\sqrt{a}}+\frac{\sqrt{a}\delta^2(C-1)}{4S},\ S>0,\ C>1 \quad (A4)$$

By elementary computations we deduce that the functions Q^* and F satisfy the following properties.

Lemma 5. *We have:*

(i)

$$Q^*(0,C)=Q^*(+\infty,C)=1,\ C>1$$

;

(ii)

$$\frac{\partial Q^*}{\partial S}(S,C)=\begin{cases}<0 \text{ if } S<S^*, & C>1\\ 0 \text{ if } S=S^*, & C>1\\ >0 \text{ if } S>S^*, & C>1\end{cases}$$

where

$$S^*=S^*(C)=\frac{\sqrt{a}}{2}\gamma_0(C)=\frac{\delta\sqrt{a(C-1)}}{2} \quad (A5)$$

is the minimum point of function Q^* with respect to S, for all $C>1$.

(iii) $Q^*(S^*,C)=Q\left(\frac{\gamma_0(C)}{2}\right),\ C>1.$

Lemma 6. *We have:*

$$\text{(i)}F(0,C)=+\infty, C>1;\ \text{(ii)}F(+\infty,C)=\frac{1}{C}, C>1; \quad (A6)$$

$$\text{(iii)}\frac{\partial F}{\partial S}(S,C)<0,\ S>0,\ C>1;\ \text{(iv)}F(S^*,C)=\frac{2}{C},\ C>1.$$

Theorem 7. *The eqn (A1) for the sorptivity S with a parameter $C>1$, admits a unique solution $S_1^*>S^*$ if $1<C<C_1$ or, $S_2^*<S^*$ if $C>C_1$, where C_1 is the unique solution of the eqn (23).*

Proof. *Functions F and Q^* satisfy the following relations:*

$$\text{(a) } F(S^*,C)>Q^*(S^*,C)\Leftrightarrow\frac{2}{C}>Q\left(\frac{\gamma_0(C)}{2}\right)\Leftrightarrow M(C)<2\Leftrightarrow 1<C<C_1.$$

$$\text{(b) } F(S^*,C)<Q^*(S^*,C)\Leftrightarrow\frac{2}{C}<Q\left(\frac{\gamma_0(C)}{2}\right)\Leftrightarrow M(C)>2\Leftrightarrow C>C_1.$$

Therefore, for a fixed C, we have that if $1<C<C_1$ the abscisa S_1^* of the intersection point of the graphs of the functions F and Q^* is to the right of the minimum point $S^*(S_1^*>S^*)$, in other case this point S_2^* is to the left of the minimum point $(S_2^*<S^*)$.

Now, we can relate the solutions S_1^* and S_2^* of the eqn (A1) according to the two cases $1<C<C_1$ and $C>C_1$ respectively, which are given by the above Theorem 8, with the expressions eqns (64) and (67) obtained in Theorem 1.

Theorem 8. *We have*

$$\text{(i) } S_1^*=S_1(C)=\frac{\sqrt{a}}{2}\left(\gamma_1(C)+\sqrt{\gamma_1^2(C)-\gamma_0^2(C)}\right),\ 1<C<C_1$$

$$\text{(ii) } S_2^*=S_2(C)=\frac{\sqrt{a}}{2}\left(\gamma_2(C)-\sqrt{\gamma_2^2(C)-\gamma_0^2(C)}\right),\ C>C_1$$

$$\text{(iii) } S_1^*=S_2^*=S(C_1)=\frac{\sqrt{a}}{2}\gamma_0(C_1)=\frac{\delta}{2}\sqrt{a(C_1-1)}=\sqrt{2(\Psi_0-\Psi_s)(\theta_s-\theta_n)(C_1-1)},\ C=C_1.$$

Proof. S_1^* and S_2^* must satisfy the expression eqn (42). On the other hand, for $1<C<C_1$ we have $S_1^*>S^*=\frac{\sqrt{a}}{2}\gamma_0(C)$. Then S_1^* is given by the sign '+' in eqn (42) (analogously for $C>C_1$ we have S_2^* is given by the sign '−' in eqn (42)) because the functions

$$g_3(x)=x+\sqrt{x^2-\gamma_0^2(C)},\ g_4(x)=x-\sqrt{x^2-\gamma_0^2(C)},\ x>\gamma_0(C)$$

satisfy the following properties:

$$g_3(\gamma_0(C))=g_4(\gamma_0(C))=\gamma_0(C),$$

$$g_3(+\infty)=+\infty,\ g_4(+\infty)=0,$$

$$g_3'(x)>0,\ g_4'(x)<0.$$

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