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DETERMINATION OF UNKNOWN THERMAL COEFFICIENTS THROUGH A FREE BOUNDARY PROBLEM FOR A NON LINEAR HEAT CONDUCTION EQUATION WITH A CONVECTIVE TERM

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ABSTRACT

We determinate unknown thermal coefficients of a semi-infinite material with an overspecified condition on the fixed face following the ideas developed in C. Rogers (ZAMP, **39**, 122-128 (1988)) and in D. A. Tarzia (Adv. Appl. Math, **3**, 74-82 (1982)). We also obtain formulae for the unknown coefficients and, the necessary and sufficient condition for the existence of solution.
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Introduction

The modeling of solidification systems is a problem of a great mathematical and industrial significance. Phase-change problems appear frequently in industrial processes an other problems of technological interest [1,3,4,5,6,9,14]. A large bibliography on the subject was given in [13]. Here, we consider a phase-change process (one-phase Stefan problem) for a non-linear heat conduction equation with a convective term [10] which admits a class of exact solutions analogous to the classical Lamé Clapeyron solution [8].

In this paper we consider an overspecified condition on the fixed face of the type [11] to the phase-change material following the model developed in [10]. This allows us to consider some thermal coefficients as unknowns and to calculate them, under certain specified restrictions upon data. We also obtain formulae for the unknown coefficients and, necessary and

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sufficient conditions for the existence of solution.

We shall consider the following free boundary problem for a semi-infinite region $x > 0$:

$$(P_1)\quad \begin{cases} (1) & \rho c \frac{\partial \theta}{\partial t} = \frac{\partial}{\partial x}\left(k(\theta, x)\frac{\partial \theta}{\partial x}\right) - v(\theta)\frac{\partial \theta}{\partial x}\ ,\ 0 < x < s(t)\ ,\ t > 0 \\ (2) & \theta(0,t) = -\theta_o < 0\ ,\ t > 0 \\ (3) & k\left(\theta(0,t),0\right)\frac{\partial \theta}{\partial x}(0,t) = \frac{q_0}{\sqrt{t}}\ ,\ t > 0\ \ (q_0 > 0) \\ (4) & k\left(\theta(s(t),t), s(t)\right)\frac{\partial \theta}{\partial x}(s(t),t) = \rho l\ \dot{s}\ (t)\ ,\ t > 0 \\ (5) & \theta\left(s(t),t\right) = 0,\ t > 0 \\ (6) & s(0) = 0\ , \end{cases}$$

where (2) and (3) are the boundary and overspecified condition on the fixed face $x = 0$, and the coefficients k and v in the differential equation (1) are given by

$$k\left(\theta, x\right) = \rho c\,\frac{1+dx}{(a+b\theta)^2} \quad , \quad v(\theta) = \rho c\,\frac{d}{2(a+b\theta)^2} \quad , \quad (a, b, d > 0)$$

where

$\theta(x,t)$: distribution of temperature in the semi-infinite material,

θ_0 : oppositive of the temperature on the fixed face,

x : spacial variable,

t : temporal variable,

$s(t)$: free boundary (location of the phase-change interface),

c : specified heat per unit of mass (constant pressure),

ρ : density of mass,

k : thermal conductivity,

l : heat latent of fusion by unit of mass,

v : velocity,

q_0 : coefficient which characterized the heat flux on the fixed face,

a, b, d : positive constants (parameters).

The cases with constant thermal coefficients were considered in [12]. Exact solutions for nonlinear diffusion equation are given in [7].

The goal of this paper is to determinate the temperature $\theta = \theta(x,t)$, the free boundary $x = s(t)$ and an unknown thermal coefficient chosen among ρ, c, l, a , b and d, as a function of data q_0 and θ_0 which must be determined by an experimental phase-change process [2].

In order to improve the lecture of the paper we have written an appendix which contains the definition of the functions used in the text with their corresponding properties. The restrictions upon data which are useful to describe the necessary and sufficient conditions for the existence of a solution are described in the text.

Free boundary problem and the similarity solution

Problem (P_1) is given by the free boundary problem for the heat equation with initial and free boundaries conditions given by (2), (5), (6) and the following equations:

$$(7) \qquad \frac{\partial\theta}{\partial t} = \frac{\partial}{\partial x}\left(\frac{1+dx}{(a+b\theta)^2}\frac{\partial\theta}{\partial x} + \frac{d}{2b(a+b\theta)}\right), \quad 0 < x < s(t), \ t > 0,$$

$$(8) \qquad \frac{1}{(a+b\theta(0,t))^2}\frac{\partial\theta}{\partial x}(0,t) = \frac{q_0^*}{\sqrt{t}}, \ t > 0, \ \left(q_0^* = \frac{q_0}{\rho c}\right),$$

$$(9) \qquad \frac{1+d\,s(t)}{a^2}\frac{\partial\theta}{\partial x}(s(t),t) = \alpha\,\dot{s}(t), \ t > 0 \ \left(\alpha = \frac{l}{c} > 0\right).$$

where $\theta(x,t)$, $s(t)$ and a thermal coefficient, chosen among ρ, c, l, a, b and d, are unknown.

Following [10] we can reduce the problem (2), (5)-(9) by using the transformations (T_1) and (T_2), given respectively by

$$(T_1) \qquad y = \frac{2}{d}\left[(1+dx)^{\frac{1}{2}} - 1\right] \quad , \quad S(t) = \frac{2}{d}\left[(1+ds(t))^{\frac{1}{2}} - 1\right] \quad ,$$

and

$$(T_2) \qquad \left|\begin{array}{l} y^* = y^*(y,t) = \int_{S(t)}^{y} (a+b\theta(\sigma,t))\,d\sigma + (\alpha b + a)S(t), \ t^* = t, \\ \theta^* = \frac{1}{a+b\theta}, \ S^* = y^*\,|_{y=S} = (\alpha b + a)\,S(t), \ \left(\theta \neq -\frac{a}{b}\right). \end{array}\right.$$

Then we get the following free boundary problem for the heat equation with an over-specified condition :

$$(P_2) \qquad \left|\begin{array}{ll} (10) & \frac{\partial\theta^*}{\partial t^*} = \frac{\partial^2\theta^*}{\partial y^{*2}}, \ V(t^*) - V(0) < y^* < S^*(t^*), \ t^* > 0, \\ (11) & \theta^*\left(V(t^*) - V(0), t^*\right) = \theta_0^* > \theta_f^*, \ t^* > 0, \\ (12) & \frac{\partial\theta^*}{\partial y^*} = -\frac{b\,q_0^*\,\theta_0^*}{\sqrt{t^*}}, \ y^* = V(t^*) - V(0), \ t^* > 0, \\ (13) & \frac{\partial\theta^*}{\partial y^*} = \alpha^*\frac{\partial S^*}{\partial t^*}, \ y^* = S^*(t^*), \ t^* > 0, \\ (14) & \theta^*\left(S^*(t^*), t^*\right) = \theta_f^*, \ t^* > 0, \\ (15) & S^*(0) = 0. \end{array}\right.$$

where

$$\left| \begin{array}{lll} (16)\ \theta_f^* = \frac{1}{a} & , \quad \theta_0^* = \frac{1}{a - b\theta_0} & , \quad (a - b\theta_0 \neq 0) \\ (17)\ \alpha^* = -\frac{\alpha b}{(a+\alpha b)a} & , \quad (18)\ \ \dot{V}(t) = b\frac{q_0^*}{\sqrt{t}} & \end{array} \right.$$

and we have employed an alternative expression for y^*, that is

$$y^*(y,t) = \int_0^y (a + b\theta(\sigma,t))\, d\sigma + V(t) - V(0)\ .$$

Taking into account that problem (P_2) is a classic Stefan-like problem with an overspecified condition, the two free boundary conditions imply that necessarily the free boundary $S^*(t^*)$ (that is $S(t)$) is given by

$$(19) \qquad S^*(t^*) = \sqrt{2\gamma t^*}, \qquad \left(S(t) = \sqrt{2\gamma t}\right)$$

where $\gamma > 0$ is an unknown parameter to be found. Therefore, if we propose a similarity solution of type

$$(20) \qquad \theta^* = \Theta^*(\xi^*) \quad , \quad \xi^* = \frac{y^*}{\sqrt{2\gamma t^*}}$$

the problem (P_2) reduces to:

$$(P_3) \left| \begin{array}{ll} (21) & \frac{d^2\Theta^*}{d\xi^{*2}} + \gamma\xi^*\frac{d\Theta^*}{d\xi^*} = 0, \ \ bq_0^*\sqrt{\frac{2}{\gamma}} < \xi^* < \alpha b + a\ , \\ (22) & \Theta^*\left(bq_0^*\sqrt{\frac{2}{\gamma}}\right) = \theta_0^*\ , \\ (23) & \frac{d\Theta^*}{d\xi^*} = -q_0^*\, b\sqrt{2\gamma}\,\theta_0^*\ , \ \ \text{for } \xi^* = bq_0^*\sqrt{\frac{2}{\gamma}}, \\ (24) & \Theta^*_{\xi^*} = -\frac{\alpha b}{a}\gamma\ , \ \ \text{for } \xi^* = \alpha b + a, \\ (25) & \Theta^* = \theta_f^*\ , \ \ \text{for } \xi^* = \alpha b + a\ \ . \end{array} \right.$$

The solution of (21) is given by

$$(26) \qquad \Theta^* = A\,\mathrm{erf}\left[\sqrt{\tfrac{\gamma}{2}}\xi^*\right] + B,$$

where the constants A, B, γ, and the unknown coefficient (chosen among l, c, ρ, a, b and d) are determined by the conditions (22)-(25) which yields

$$(27) \qquad A\exp\left(-b^2 q_0^*\right) = -q_0^* b\sqrt{\pi}\left[A\,\mathrm{erf}\left(bq_0^*\right) + B\right]$$

$$(28) \qquad A\,\mathrm{erf}\left[\sqrt{\tfrac{\gamma}{2}}\left(\alpha b + a\right)\right] + B = \theta_f^*$$

$$(29) \qquad A\exp\left[-\tfrac{\gamma}{2}\left(\alpha b + a\right)^2\right] = -\frac{\alpha b}{a}\sqrt{\frac{\gamma\pi}{2}}$$

$$(30) \qquad A\,\mathrm{erf}\left(bq_0^*\right) + B = \theta_0^*.$$

Finally, we invert the relations (20), (T_1) and (T_2), and we use conditions (27) and (28) to obtain the solution :

$$(31) \qquad \theta(\xi) = \frac{1}{b}\left[\frac{1}{A\,\mathrm{erf}\left[\sqrt{\frac{\gamma}{2}}\xi^*\right]+B} - a\right], \quad \xi = \frac{y}{\sqrt{2\gamma t}} = \frac{\frac{2}{d}\left[(1+dx)^{\frac{1}{2}}-1\right]}{\sqrt{2\gamma t}},$$

where ξ^* and ξ are related by the following expression

$$(32) \qquad \xi = \int_{bq_0\sqrt{\frac{2}{\gamma}}}^{\xi^*} \left[A\,\mathrm{erf}\left(\sqrt{\tfrac{\gamma}{2}}\sigma\right) + B\right] d\sigma$$

where the constants A and B are given by

$$(33) \qquad A = -\frac{bq_0^*\sqrt{\pi}}{a}\left[\exp[-(q_o^*b)^2] + q_o^*b\sqrt{\pi}\,\mathrm{erf}(q_o^*b) - q_o^*b\sqrt{\pi}\,\mathrm{erf}\left(\sqrt{\tfrac{\gamma}{2}}(a+\alpha b)\right)\right]^{-1},$$

$$(34) \qquad B = \frac{1}{a} +$$
$$+\frac{bq_0^*\sqrt{\pi}}{a}\left[\exp[-(q_o^*b)^2] + q_o^*b\sqrt{\pi}\,\mathrm{erf}(q_o^*b) - q_o^*b\sqrt{\pi}\,\mathrm{erf}\left(\sqrt{\tfrac{\gamma}{2}}(a+\alpha b)\right)\right]^{-1}\mathrm{erf}\left(\sqrt{\tfrac{\gamma}{2}}(a+\alpha b)\right)$$

Then, the coefficient γ and the unknown coefficient (chosen among l, c, ρ, a, b and d) must satisfy the following system of equations.

$$(35) \qquad g\left(\sqrt{\tfrac{\gamma}{2}}\left(\tfrac{lb}{c}+a\right), \tfrac{1}{\sqrt{\pi}}\left(1+\tfrac{ac}{lb}\right)\right) = g\left(\tfrac{bq_0}{\rho c}, \tfrac{1}{\sqrt{\pi}}\right),$$

$$(36) \qquad g\left(\tfrac{bq_0}{\rho c}, \tfrac{b\theta_0}{a\sqrt{\pi}}\right) = \mathrm{erf}\left[\sqrt{\tfrac{\gamma}{2}}\left(\tfrac{lb}{c}+a\right)\right].$$

The solution of the original problem (P_1) is given by:

$$(37) \qquad \theta = \theta\left(\frac{\frac{2}{d}(\sqrt{1+dx}-1)}{\sqrt{2\gamma t}}\right)$$

together with the free boundary

$$(38) \qquad x = s(t) = \sqrt{2\gamma t} + \frac{d\gamma t}{2}.$$

Unknown thermal coefficients through a free boundary problem

We shall give conditions to obtain solution to above system (35)-(36) and we also give formulae for the coefficient γ and the unknown thermal coefficient, and the necessary and sufficient condition for the existence of a solution in the following six cases:

Case 1: Determination of the unknown coefficients γ, ρ (c. f. Theorem 1);

Case 2: Determination of the unknown coefficients γ, c (c. f. Theorem 2);

Case 3: Determination of the unknown coefficients γ, l (c. f. Theorem 3);

Case 4: Determination of the unknown coefficients γ, a (c. f. Theorem 4);

Case 5: Determination of the unknown coefficients γ, b (c. f. Theorem 5);

Case 6: Determination of the unknown coefficients γ, d (c. f. Theorem 6).

Now, we shall prove the results for the cases 1, 3, 5 and 6.

Theorem 1.- If data b, θ_0 and a verify condition

$$(R_1) \qquad 0 < \tfrac{b\theta_0}{a} < 1 \quad ,$$

then there exists a unique solution to problem (P_1) which is given by (31)-(34) and the unknown coefficients γ and ρ are given by

$$(39) \qquad \gamma = 2\left(\tfrac{\eta_1}{a}\right)^2\left(1+\tfrac{lb}{ac}\right)^{-2}, \qquad \rho = \tfrac{bq_0}{c\nu_1}$$

where

$$(40) \qquad \eta_1 = \mathrm{erf}^{-1}\left[g\left(\nu_1, \tfrac{b\theta_0}{a\sqrt{\pi}}\right)\right]$$

and ν_1 is the solution of the equation

$$(41) \qquad R(F(\nu)) = \left(\frac{1-\frac{b\theta_0}{a}}{1+\frac{ac}{lb}}\right)R(\nu) \;\text{ with } \nu > \nu_0 = Q^{-1}\left(\tfrac{b\theta_0}{a}\right).$$

Proof.- The system (35)-(36) in the unknowns γ and ρ is equivalent to

$$(42) \qquad g(\eta, p) = g\left(\nu, \tfrac{1}{\sqrt{\pi}}\right) \quad , \qquad (43) \qquad \mathrm{erf}(\eta) = g\left(\nu, \tfrac{b\theta_0}{\sqrt{\pi}a}\right)$$

in the new unknowns η and ν which are defined by

$$(44) \qquad \eta = \left(1+\tfrac{lb}{ac}\right)a\sqrt{\tfrac{\gamma}{2}} \quad , \qquad \nu = \tfrac{bq_0}{\rho c} \quad ,$$

where

$$(45) \qquad p = \tfrac{1}{\sqrt{\pi}}\left(1+\tfrac{ac}{lb}\right),$$

is a positive parameter. Let be $\beta = \frac{b\theta_0}{a\sqrt{\pi}}$. From (43) we deduce $\eta = \mathrm{erf}^{-1}\left[g\left(\nu, \beta\right)\right]$ if and only if (R_1) is satisfied. Taking into account (42) we obtain the following equation in the unknown ν

$$(46) \qquad \mathrm{erf}^{-1}\left(g(\nu, \beta)\right) = R^{-1}\left(\frac{\frac{1}{\sqrt{\pi}}-\beta}{\delta}R(\nu)\right), \quad \nu > \nu_0,$$

with $\delta = \frac{1+\frac{ac}{lb}}{\sqrt{\pi}}$. Equation (46) is equivalent to

$$(47) \qquad F(\nu) = H_1(\nu) \quad , \qquad \nu > \nu_0 \quad .$$

The properties of functions F and H_1(See Appendix) assure the existence of a unique solution ν_1 to equation (47). Thus, there exists a unique solution ν_1, η_1 to (42)-(43). Then from (44) we obtain the expressions for γ and ρ given by (39).

Theorem 2.- If data a, b and θ_0 verify condition (R_1), then there exists a unique solution to problem (P_1) which is given by (31)-(34) and the unknown coefficients γ and c are given by

(48) $$\gamma = 2\left(\frac{\eta_2}{a}\right)^2\left(1+\frac{lb}{ac}\right)^{-2}, \qquad c = \frac{bq_0}{\rho\nu_2}$$

where

(49) $$\eta_2 = \mathrm{erf}^{-1}\left[g\left(\nu_2,\beta\right)\right]$$

and ν_2 is the solution of the equation

(50) $$F(\nu) = H_2(\nu) \text{ with } \nu > \nu_0$$

and β is given as in Case 1.

Theorem 3.- If data a, b, c, ρ, θ_0 and q_0 satisfy (R_1) and

(R_2) $$\frac{bq_0}{\rho c} > \nu_0 = Q^{-1}\left(\frac{b\theta_0}{a}\right) \quad , \quad 0 < \frac{b\theta_0}{a} < 1 \quad ,$$

then there exists a unique solution to problem (P_1) which is given by (31)-(34) and the unknown coefficients γ and l are given by

(51) $$l = \frac{ac}{b(\sqrt{\pi}p_3-1)}\ , \ \gamma = 2\left(\frac{\eta_3}{a}\right)^2\left(1+\frac{lb}{ac}\right)^{-2},$$

where

(52) $$\eta_3 = \mathrm{erf}^{-1}\left[g\left(\frac{bq_0}{\rho c},\beta\right)\right]$$ and (53) $$p_3 = \Psi(\eta_3).$$

Proof.- The system (35)-(36), in the unknowns γ and l, is equivalent to

(54) $$g(\eta,p) = g\left(\frac{bq_0}{\rho c},\frac{1}{\sqrt{\pi}}\right)$$ and (55) $$\mathrm{erf}(\eta) = g\left(\frac{bq_0}{\rho c},\beta\right).$$

in the new unknowns η and p which are defined by (44) and (45). The equation (55) admits a unique solution η_3, given by (52), if and only if (R_1) and (R_2) are satisfied. Then, p satisfies the following equation

(56) $$\mathrm{erf}(\eta_3) + p\,R(\eta_3) = g\left(\frac{bq_0}{\rho c},\frac{1}{\sqrt{\pi}}\right) \quad , \quad p > 0$$

which has a unique solution given by (53). So, if (R_1) and (R_2) are satisfied, then there exists a unique solution l and γ to the system (35)-(36) given by (51).

Theorem 4.- There exists a unique solution to problem (P_1) which is given by (31)-(34) and the unknown coefficients γ and a are given by

(57) $$\gamma = \frac{2}{\pi}\left(\frac{\eta_4 c}{lbp_4}\right)^2 \quad , \quad a = \frac{lb}{c}\left(\sqrt{\pi}p_4 - 1\right),$$

with $p_4 = \Psi\left(\eta_4\right)$ where η_4 is the unique solution of the equation

(58) $$g\left(\frac{bq_0}{\rho c}, \frac{c\theta_0}{\sqrt{\pi}l\left(\sqrt{\pi}\,\Psi(\eta)-1\right)}\right) = \operatorname{erf}(\eta)\ ,\ \text{with } \eta > \frac{bq_0}{\rho c}.$$

Theorem 5.- If data a, c, l, ρ, q_0 and θ_0 verify conditions

$$(R_3) \quad 0 < \frac{\rho c\theta_0}{aq_0\sqrt{\pi}} < 1 \qquad \text{and} \qquad (R_4) \quad 0 < \frac{\rho l}{aq_0} < x_s \quad ,$$

where x_s is the unique solution of the equation

(59) $$x = R\left(\operatorname{erf}^{-1}\left(\frac{xc\theta_0}{l\sqrt{\pi}}\right)\right) \quad , \quad 0 < x < \frac{l\sqrt{\pi}}{c\theta_0},$$

then there exists a unique solution to problem (P_1) which is given by (31)-(34) and the unknown coefficients γ and b are given by

(60) $$b = \frac{\rho c}{q_0}\nu_5 \quad , \quad \gamma = 2\left(\frac{\eta(\nu_5)}{a}\right)^2\left(1+\frac{lb}{ac}\right)^{-2}$$

where $\eta(\nu)$ is defined implicitly by $\Phi(\eta,\nu)=0$ and ν_5 is the unique solution of the equation

(61) $$G(\nu) = \operatorname{erf}(\eta(\nu)) \quad , \quad \text{with } 0 < \nu < \nu^* ,$$

where $\nu^* > 0$ is the unique solution of the equation

(62) $$Q(x) = \frac{\rho c\theta_0}{aq_0}x \quad , \quad x > 0 \quad .$$

Proof.- The system (35)-(36) in the unknowns γ and b is equivalent to

(63) $$g\left(\eta, \frac{1}{\sqrt{\pi}}\left(1+\frac{aq_0}{l\rho\nu}\right)\right) = g\left(\nu, \frac{1}{\sqrt{\pi}}\right)$$ and (64) $$\operatorname{erf}(\eta) = g\left(\nu, \frac{\rho c\theta_0}{\sqrt{\pi}aq_0}\nu\right),$$

in the unknowns η and ν which are defined by (44). The equation (63) is equivalent to $\Phi(\eta,\nu)=0$ which defines implicitly $\eta=\eta(\nu), \forall\nu>0$. Then, we replace η in (64) and we obtain, for the unknown ν, the following equation

(65) $$G(\nu) = g(\nu,\mu\nu) = \operatorname{erf}(\eta(\nu)) \quad , \quad \nu > 0.$$

The equation (62) has a unique solution $\nu^* \in \left(0, \frac{aq_0}{\rho c\theta_0}\right)$ because restriction (R_3). Moreover, we have $G(\nu^*)=1$. The equation (65) has an unique solution $0<\nu_5<\nu^*$ if (R_3) and (R_4) are satisfied. Therefore, there exists a unique solution $\nu_5, \eta_5 = \eta(\nu_5)$ to (63)-(64), then we obtain the coefficients γ and b given by (60).

Theorem 6.- If data a, b, c, l, ρ, q_0 and θ_0 verify the condition (R_2) and

(R_5) $$l = \frac{\frac{ac}{b}}{\left[1-\frac{b\theta_0}{a}\right]\frac{R\left(\frac{bq_0}{\rho c}\right)}{R\left(\operatorname{erf}^{-1}\left(g\left(\frac{bq_0}{\rho c}, \frac{b\theta_0}{a\sqrt{\pi}}\right)\right)\right)} - 1},$$

then there exists infinite solutions to problem (P_1) which is given by (31)-(34) and the unknown coefficient γ is given by

(66) $\gamma = 2\left(\frac{\eta_6}{a}\right)^2\left(1+\frac{lb}{ac}\right)^{-2}$, for any $d>0$

where

(67) $\eta_6 = \mathrm{erf}^{-1}\left[g\left(\frac{bq_0}{\rho c},\beta\right)\right]$.

Proof.- The system (35)-(36) in the unknown γ is equivalent to

(68) $g\left(\eta, \frac{1}{\sqrt{\pi}}\left(1+\frac{ac}{lb}\right)\right) = g\left(\frac{bq_0}{\rho c}, \frac{1}{\sqrt{\pi}}\right)$ and (69) $\mathrm{erf}(\eta) = g\left(\frac{bq_0}{\rho c},\beta\right)$.

in the unknown η which is defined by (44). As in case 1, from (69), the condition (R_2) assures that $\eta_6 = \mathrm{erf}^{-1}\left[g\left(\frac{bq_0}{\rho c},\beta\right)\right]$. Then, η_6 satisfies (68) if and only if

(70) $l = \dfrac{ac}{b(M-1)}$ with $M = \left[1-\frac{b\theta_0}{a}\right]\frac{R\left(\frac{bq_0}{\rho c}\right)}{R(\eta_6)}$,

that is (R_5).

Moreover, $l>0 \Leftrightarrow M>1 \Leftrightarrow \mathrm{erf}\left(\frac{bq_0}{\rho c}\right) > Z_1\left(\frac{bq_0}{\rho c}\right) \Leftrightarrow Z_2\left(\frac{bq_0}{\rho c}\right) > 0$.

The right hand side inequality is always verifies for properties of function Z_2 (see appendix). Then, if (R_2) and (R_5) are satisfied there exists a unique solution η_6 to the system (68)-(69) and then we obtain the coefficient γ by (66). Moreover, the parameter d may assume any positive value.

Conclusion

We have solved six free booundary problems for the heat conduction equation with a convective term and an overspecified condition on the fixed face with an unknown thermal coefficient. Moreover, for each case, we give the necessary and sufficient conditions for the existence of solution and the corresponding formula for the unknown coefficient.

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Appendix

We define the following parameters

$$\beta = \frac{b\theta_0}{a\sqrt{\pi}}\ ,\ \delta = \frac{1+\frac{ac}{lb}}{\sqrt{\pi}}\ ,\ r = 1 - \frac{b\theta_0}{a}\ ,\ u = \frac{aq_0}{l\rho}\ ,\ \mu = \frac{\rho c\theta_0}{q_0 a\sqrt{\pi}}\ ,$$

and the following functions

$$R(x) = \frac{\exp(-x^2)}{x}\ ,\quad Q(x) = \sqrt{\pi}x\exp(x^2)\operatorname{erfc}(x)\ ,\quad x > 0\ ,$$
$$g(x,p) = \operatorname{erf}(x) + pR(x)\ ,\quad p > 0\ ,\quad x > 0\ ,$$
$$F(x) = \operatorname{erf}^{-1}(g(x,\beta))\ ,\quad \text{for } x > Q^{-1}(\beta\sqrt{\pi})\ ,\quad \text{if } \beta\sqrt{\pi} < 1\ ,$$
$$H_1(x) = R^{-1}\left(\frac{\frac{1}{\sqrt{\pi}}-\beta}{\delta}R(x)\right)\ ,\ H_2(x) = R^{-1}\left(\frac{r}{1+ux^{-1}}R(x)\right)\ ,\ x > 0\ ,$$
$$\Psi(x) = \left[g\left(\frac{bq_0}{\rho c},\frac{1}{\sqrt{\pi}}\right) - \operatorname{erf}(x)\right]x\exp(x^2), G(x) = g(x,\mu x) = \operatorname{erf}(x) + \mu\exp(-x^2), x > 0,$$
$$V(x) = g\left(\frac{bq_0}{\rho c},\frac{c\theta_0}{\sqrt{\pi}l(\sqrt{\pi}\Psi(x)-1)}\right)\ ,\quad x \neq \frac{bq_0}{\rho c}\ ,$$
$$\Phi(x,y) = g\left(x,\frac{1}{\sqrt{\pi}}\right) + \frac{aq_0}{\sqrt{\pi}\rho l y}R(x) - g\left(y,\frac{1}{\sqrt{\pi}}\right),\quad x > 0\ ,\quad y > 0,$$
$$Z_1(x) = \operatorname{erf}\left(R^{-1}\left[(1-\beta\sqrt{\pi})R(x)\right]\right) - \beta R(x)\ ,\ Z_2(x) = \operatorname{erf}(x) - Z_1(x)\ ,\beta < \frac{1}{\sqrt{\pi}}\ ,\ x > 0.$$

The above functions have the following properties:

$$R(0^+) = +\infty\ ,\ R(+\infty) = 0\ ,\ R'(x) < 0\ ,\ \forall x > 0\ ,$$
$$Q(0) = 0\ ,\ Q(+\infty) = 1\ ,\ Q'(x) > 0\ ,\ \forall x > 0\ ,$$
$$g(+\infty,p) = \begin{cases} 1^+ & \text{for } p \geq \frac{1}{\sqrt{\pi}}, \\ 1^- & \text{for } 0 < p < \frac{1}{\sqrt{\pi}}, \end{cases}\quad ,\quad g(0^+,p) = +\infty\ ,\ \forall p > 0\ ,$$
$$\frac{\partial g}{\partial x}(x,p) = \begin{cases} < 0\ , & \forall x > 0\ , & \text{for } p \geq \frac{1}{\sqrt{\pi}}, \\ < 0\ , & 0 < x < \sqrt{\frac{p}{2(\frac{1}{\sqrt{\pi}}-p)}}\ , & \text{for } 0 < p < \frac{1}{\sqrt{\pi}}, \\ = 0\ , & x = \sqrt{\frac{p}{2(\frac{1}{\sqrt{\pi}}-p)}}\ , & \text{for } 0 < p < \frac{1}{\sqrt{\pi}}, \\ > 0\ , & x > \sqrt{\frac{p}{2(\frac{1}{\sqrt{\pi}}-p)}}\ , & \text{for } 0 < p < \frac{1}{\sqrt{\pi}}, \end{cases}$$
$$g(x_0,p) = 1 \text{ with } x_0 = Q^{-1}(p\sqrt{\pi}) \text{ for } 0 < p < \frac{1}{\sqrt{\pi}}\ ,$$
$$F\left(Q^{-1}(\beta\sqrt{\pi})\right) = +\infty\ ,\quad F(+\infty) = +\infty\ ,$$
$$F'(x) = \begin{cases} < 0 & \text{if } Q^{-1}(\beta\sqrt{\pi}) < x < \sqrt{\frac{\beta}{2(\frac{1}{\sqrt{\pi}}-\beta)}}, \\ = 0 & \text{if } x = \sqrt{\frac{\beta}{2(\frac{1}{\sqrt{\pi}}-\beta)}}, \\ > 0 & \text{if } x > \sqrt{\frac{\beta}{2(\frac{1}{\sqrt{\pi}}-\beta)}}, \end{cases}$$

$$F(x) \cong \sqrt{\log\left(\frac{x\exp(x^2)}{\sqrt{\pi}(\frac{1}{\sqrt{\pi}}-\beta)}\right)}\ ,\ H_1(x) \cong \sqrt{\log\left(\frac{\delta\, x\exp(x^2)}{\frac{1}{\sqrt{\pi}}-\beta}\right)}\ ,\ x \to +\infty\ ,$$

$H_1(0) = 0 \quad , \quad H_1(+\infty) = +\infty \quad , \quad H_1'(x) > 0\ ,$

$H_2(0) = R^{-1}\left(\frac{r}{u}\right) \quad , \quad H_2(+\infty) = +\infty \quad , \quad H_2'(x) > 0\ ,$

$$H_2(x) \cong \sqrt{\log\left(\frac{x\exp(x^2)}{\sqrt{\pi}(\frac{1}{\sqrt{\pi}}-\beta)} + u\frac{\exp(x^2)}{\sqrt{\pi}(\frac{1}{\sqrt{\pi}}-\beta)}\right)} \quad , \quad x \to +\infty\ ,$$

$\Psi(0) = 0\ ,\ \Psi(+\infty) = +\infty\ ,\ \Psi'(x) > 0 \quad \forall x > 0\ ,\ \Psi\left(\frac{bq_0}{\rho c}\right) = \frac{1}{\sqrt{\pi}}\ ,$

$V(0) = \mathrm{erf}\left(\frac{bq_0}{\rho c}\right) - \frac{c\rho T_0}{\sqrt{\pi} l b q_0}\exp\left(-\left(\frac{bq_0}{\rho c}\right)^2\right) \quad , \quad V(+\infty) = \mathrm{erf}\left(\frac{bq_0}{\rho c}\right) \quad ,$

$V\left(\left(\frac{bq_0}{\rho c}\right)^+\right) = +\infty\ ,\ V\left(\left(\frac{bq_0}{\rho c}\right)^-\right) = -\infty\ ,\ V'(x) < 0 \quad \forall \quad x \neq \frac{bq_0}{\rho c}\ ,$

$G(0) = \mu \quad , \quad G(+\infty) = 1 \quad ,$

$G(\nu^*) = 1$ with ν^* is the solution of $Q(x) = \mu x\sqrt{\pi}\ ,\ x > 0$, for $\mu < 1$.

$$G'(x) = \begin{cases} < 0 & \text{if } x > \frac{1}{\sqrt{\pi}\mu} \\ = 0 & \text{if } x = \frac{1}{\sqrt{\pi}\mu} \\ > 0 & \text{if } 0 < x < \frac{1}{\sqrt{\pi}\mu} \end{cases} \quad ,$$

$\Phi_x(x,y) < 0 \quad x > 0 \quad , \quad y > 0 \quad ; \quad \Phi_y(x,y) > 0 \quad x > 0 \quad , \quad y > 0 \quad ,$

$\Phi(x,y) = 0$ define implicitely $x = x(y)$ which verifies :

$x(0) = R^{-1}\left(\frac{\rho l}{a q_0}\right) \quad , \quad x(+\infty) = +\infty \quad , \quad x'(y) > 0 \quad , \quad \forall\, y > 0,$

$Z_1(0^+) = -\infty\ ,\ Z_1(+\infty) = 1\ ,\ Z_2(0^+) = +\infty\ ,\ Z_2(+\infty) = 0,$

$Z_1'(x) > 0\ ,\ Z_2'(x) < 0\ ,\ \forall x > 0\ .$

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